Quantum feedback with weak measurements

Seth Lloyd[1] and Jean-Jacques E. Slotine[2]
[1]*d'Arbeloff Laboratory for Information Systems and Technology, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
[2]*Nonlinear Systems Laboratory, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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The problem of feedback control of quantum systems by means of weak measurements is investigated in detail. When weak measurements are made on a set of identical quantum systems, the single-system density matrix can be determined to a high degree of accuracy while affecting each system only slightly. If this information is fed back into the systems by coherent operations, the single-system density matrix can be made to undergo an arbitrary nonlinear dynamics, including, for example, a dynamics governed by a nonlinear Schrödinger equation. We investigate the implications of such nonlinear quantum dynamics for various problems in quantum control and quantum information theory. The nonlinear dynamics induced by weak quantum feedback could be used to create a novel form of quantum chaos in which the time evolution of the single-system wave function depends sensitively on initial conditions.

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The conventional theory of quantum feedback control assumes the use of strong or projective measurements to acquire information about the quantum system under control [1–10]. Such measurements typically disturb the quantum system, destroying quantum coherence and giving a stochastic character to quantum feedback control. But strong measurements are not the only tool available for acquiring information about quantum systems [11,12]. In nuclear magnetic resonance (NMR), for example, one makes collective measurements on a set of effectively identical systems: by monitoring the induction field produced by a large number of precessing spins, one can obtain the average value of their magnetization along a given axis while only slightly perturbing the states of the individual spins [13]. We will call such measurements ''weak measurements'' since they only weakly perturb the individual systems in the set. (Such weak measurements on large sets of identical systems should not be confused with the *single-system* weak measurements debated in [14–16].) The information acquired by weak measurement can then be fed back to the spins, for example to suppress super-radiant decay [17,18]. NMR is not the only system in which weak measurement is possible: one can perform weak measurements on essentially any set of quantum systems that can be coupled weakly to an external apparatus. This paper provides a general theory of quantum feedback control using weak measurements. Since weak measurements allow the accurate determination of the complete single-system density matrix of each member of a set of identical quantum systems, while affecting each system in the set arbitrarily weakly, quantum feedback by weak measurement will be shown to be capable of accomplishing tasks that are not possible using conventional, strong measurements. A model of quantum feedback using weak measurements is given and applications are proposed. In addition to NMR, quantum feedback by weak measurements could be used in quantum optics and atomic and molecular systems to effect arbitrary nonlinear Schrödinger equations, to create solitons and Schrödinger cat states, to perform quantum computations, and to institute novel forms of quantum chaos.

Quantum feedback via weak measurement represents a novel paradigm for coherent control of quantum systems. It enables the performance of operations that are impossible in the normal, strong measurement paradigm for quantum control. For example, suppose that each of the systems in the set is in the same unknown pure state. Then feedback with weak measurement can be used to drive them to any desired pure state *reversibly*, while preserving quantum coherence. This contrasts markedly to quantum feedback using strong measurements, where a system in an unknown quantum state can be driven to any desired quantum state, but only at the cost of disturbing the system's state irreversibly and stochastically, destroying quantum coherence in the process.

The general picture of quantum feedback control using weak measurements is as follows. Suppose that we have a set of N identical noninteracting quantum systems, each with density matrix ρ. (Of course, no set of quantum systems is perfectly noninteracting, but in many situations — e.g., liquid-state NMR, quantum optics — the noninteracting approximation holds to a high degree of accuracy.) Assume that the system is coherently open-loop controllable, so that we can perform arbitrary unitary transformations U on the system (necessary and sufficient conditions for open-loop coherent control of quantum systems are well known [1–6]). Now assume that we are able to make a sequence of collective weak measurements on these systems that allow us to determine the single-system reduced density matrix ρ to some degree of accuracy δ, while disturbing this density matrix by an amount ϵ. As will be seen below, both δ and ϵ can go to zero in the limit that the number of systems goes to infinity. If the systems are individual nuclear spins, for example, the single-spin density matrix can be determined by measuring the induction signal produced about two different axes: this allows one to determine the expectation of the magnetization along the x, y, and z axes, which is in turn sufficient information to determine the single-spin density matrix. Now feed that information back into the set by applying to each system a unitary transformation $U_\delta(\rho)$, where U_δ is some potentially nonlinear function of ρ, and the subscript δ indicates that U_δ discriminates between different ρ to an accuracy δ. The time evolution of the system with feedback by weak measurement is accordingly given by

$$\rho' = U_\delta(\rho)(\rho + \Delta\rho)U_\delta^\dagger(\rho), \tag{1a}$$

where $\Delta\rho$ is the perturbation to the single-system reduced density matrix induced by the weak measurement, with $\|\Delta\rho\| \leqslant \epsilon$ for a suitable norm $\| \ \|$ such as the sup norm. As will be shown below, in the limit $N\to\infty$, the collective measurement can be performed in such a way that both δ and $\epsilon \to 0$, and the time evolution of the single-system density matrix is governed by the equation

$$\rho' = U(\rho)\rho U^\dagger(\rho). \tag{1b}$$

The remainder of this paper will be devoted to exploring the implications of Eqs. (1a) and (1b). These equations have a variety of interesting features. The first, perhaps most obvious, is that they can be nonlinear as a function of ρ: if $\rho = \alpha\rho_1 + \beta\rho_2$, it need not be the case that $U(\rho)\rho U^\dagger(\rho) = \alpha U(\rho_1)\rho_1 U^\dagger(\rho_1) + \beta U(\rho_2)\rho_2 U^\dagger(\rho_2)$. (It is important to note that although the single-system reduced density matrix undergoes a nonlinear evolution, the density matrix for the set of systems taken collectively undergoes a conventional linear time evolution: no laws of quantum mechanics are broken in constructing this nonlinearity.) If the weak measurement is made continuously in time, then in the limit $N \to\infty$, $\delta\to 0$, $\epsilon\to 0$, feedback causes the single-system density matrix to obey a nonlinear Schrödinger equation

$$\partial\rho/\partial t = -i[H(\rho),\rho], \tag{2}$$

where $H(\rho)$ is the Hamiltonian corresponding to $U(\rho)$. Such nonlinearities in the case of sets of nuclear spins are well-known: for example, if each nuclear spin in the set interacts with the mean field generated by the spins taken together, then the single-spin density matrix obeys a nonlinear Bloch equation [19]. Feedback by weak measurement allows one to impose an *arbitrary* nonlinear Hamiltonian dynamics on the single-system density matrices: if in the open-loop case, without feedback, one can apply any conventional linear time evolution, then in the closed-loop case, with feedback of the results of weak measurements, one can apply any desired nonlinear dynamics that preserves the eigenvalues of the density matrix. That is, one can take $\rho\to f(\rho)$, where $f(\rho)$ has the same eigenvalues as ρ. If one can apply open-system operations [20,21] as well as closed-system, unitary transformations, then one can alter the eigenvalues of the density matrix as well as take $\rho\to g(\rho)$, where $g(\rho)$ can be an arbitrary density matrix.

Now let us look more closely at the dynamics of the weak measurement process, in order to determine how accurately the single-system density matrix can be measured and at what cost. There are two measures of the cost of weak measurement: first, the size N of the set required to attain a given accuracy δ, and second, the amount ϵ by which the individual systems are perturbed by the weak interaction with the measuring apparatus. Here we construct a specific model of weak measurement applicable to a wide range of physical systems.

The general picture of measurement on N identical systems is as follows. The density matrix for the systems is $\rho_{\rm tot} = \rho\otimes\rho\otimes\cdots\otimes\rho$. A positive-operator-valued measure (POVM) on this system corresponds to a set of operators $\{A_{\mu i}\}$ such that $\Sigma_{\mu i} A_{\mu i}^\dagger A_{\mu i} = \mathbf{I}$, where $\mathbf{I}$ is the identity operator; the measurement corresponding to the POVM gives the result μ with probability $p_\mu = {\rm tr}\Sigma_i A_{\mu i}\rho_{\rm tot}A_{\mu i}^\dagger$, in which case the system is left in the state $\rho_{{\rm tot}\mu} = (1/p_\mu)\Sigma_i A_{\mu i}\rho_{\rm tot}A_{\mu i}^\dagger$ and the density matrix for the ℓth subsystem goes to $\rho_{\ell\mu} = {\rm tr}_{\ell'\neq\ell}\rho_{{\rm tot}\mu}$.

We will define a weak measurement to be one that leaves the single-system density matrices unchanged to within a small accuracy ϵ: $\Sigma_\mu p_\mu \|\rho - \rho_{\ell\mu}\| \leqslant \epsilon$. For example, a useful POVM is the set of Gaussian quasiprojections: $A_\mu = [1/(2\pi)^{1/4}\Delta^{1/2}]\int_{-\infty}^{\infty} e^{-(a-\mu)^2/4\Delta^2}|a\rangle\langle a| da$, where the normalization is chosen so that ${\rm tr}A_\mu^\dagger A_\mu = 1$ and $\int_{-\infty}^{\infty} A_\mu^\dagger A_\mu d\mu = \mathbf{I}$ (here there is no need for the auxiliary index i). If we write the single-system density matrix in the a basis as $\Sigma_{aa'} e_{aa'}|a\rangle\langle a'|$, then the measurement corresponding to the A_μ determines the value of $\overline{a} = {\rm tr}\rho A$ to an accuracy Δ, where $A = \int a|a\rangle\langle a| da$. In addition, the measurement has the effect of reducing the off-diagonal terms of ρ by a factor $e^{-(a-a')^2/2\Delta^2}$, corresponding to a perturbation of size $\epsilon \approx \Delta A^2/2\Delta^2$, where $\Delta A = \sqrt{{\rm tr}\rho A^2 - \overline{a}^2}$. If $\Delta \gg \Delta A$, the measurement perturbs the system only weakly. Of course, the more weakly the measurement perturbs the system, the less information it acquires. By making a weak measurement on all the systems in the set simultaneously, however, one can obtain very precise information about the single-system density matrix while perturbing it only slightly. Consider the N-system POVM given by

$$A_{N\mu} = [1/(2\pi)^{1/4}\Delta^{1/2}]^N \int_{-\infty}^{\infty} e^{-(\Sigma_\ell a_\ell - N\mu)^2/4\Delta^2}|a_1\rangle \times\langle a_1|\cdots|a_N\rangle\langle a_N| da_1\cdots da_N.$$

If a collective measurement corresponding to this POVM is performed on the systems in the set, one obtains the value of $\overline{a}$ to an accuracy $\sqrt{\Delta^2/N^2 + \Delta A^2/N}$, while still perturbing the single-system density matrix by the amount $\epsilon \approx \Delta A^2/2\Delta^2$. It can be clearly seen that in the limit $N\to\infty$ we can take $\Delta \propto \sqrt{N}$, giving an arbitrarily accurate determination of $\overline{a}$ together with an arbitrarily small perturbation of the single-system density matrix. After the measurement, the overall density matrix is in the form $\rho\otimes\cdots\otimes\rho + O(\epsilon)$, so that the assumption of no correlation between the systems is only true to order ϵ. In the limit $N\to\infty$, $\epsilon\to 0$, however, the no-correlation assumption still holds.

Now we construct a model of how such a weak measurement might be performed. Our model is analogous to weak measurements in NMR, in which each system in the set is coupled weakly to the electromagnetic field in the measurement coil. Couple each system to the measurement apparatus via a single continuous quantum variable (''pointer position'') [22], described by an operator $Q=\int q|q\rangle\langle q|dq$, via a Hamiltonian coupling γAP, where P is the momentum corresponding to Q; $[P,Q]=i$. This gives a dynamics for the system and pointer: $|a\rangle|q\rangle\rightarrow|a\rangle|q+a\gamma t\rangle$ over time t. Now suppose that all N systems are coupled symmetrically to the pointer by an interaction $(A_1+\cdots+A_N)P$. If the systems are all originally in the state $\rho_i=\rho$ as above, and the pointer is originally in the state $|\psi\rangle=\int\psi(q)|q\rangle dq$, then the interaction between the systems and the measurement apparatus gives

$$\begin{aligned}\rho_1\otimes\cdots\otimes\rho_N\otimes|\psi\rangle\langle\psi|\rightarrow\rho_{SM}(t)\\ =\sum_{a_1a_1'\cdots a_Na_N'}\int dq\,dq'\,\psi(q)\overline{\psi}(q')\\ \times\rho_{a_1a_1'}\cdots\rho_{a_Na_N'}|a_1\rangle\langle a_1'|\otimes\cdots\otimes|a_N\rangle\langle a_N'|\\ \times\otimes|q+(a_1+\cdots+a_N)\gamma t\rangle\langle q'+(a_1'+\cdots+a_N')\gamma t|.\end{aligned}\tag{3}$$

One can then find the state of the apparatus at time t by taking

$$\begin{aligned}\rho_M(t)&=\mathrm{tr}_S\rho_{SM}(t)\\ &=\sum_{a_1\cdots a_N}\int dq\,dq'\,\psi(q)\overline{\psi}(q')\rho_{a_1a_1}\cdots\rho_{a_Na_N}\\ &\quad\times|q+(a_1+\cdots+a_N)\gamma t\rangle\langle q'+(a_1+\cdots+a_N)\gamma t|.\end{aligned}\tag{4}$$

That is, after the measurement the pointer registers the sum of N-independent samples of A, where each result a occurs with probability $p_a=\rho_{aa}$. The sum is registered to an accuracy $\Delta Q=\sqrt{\langle\psi|Q^2|\psi\rangle-\langle\psi|Q|\psi\rangle^2}$. ΔQ measures the initial spread of the pointer wave function $|\psi\rangle$. Accordingly, after the coupling of the pointer to the systems, the pointer registers the result $\langle A\rangle=\mathrm{tr}\rho A\pm\Delta A\sqrt{1/\epsilon N^2+1/N}$, where $\epsilon=(\gamma t/\Delta Q)^2$ will be seen to be a measure of the degree of perturbation of each individual system, and $\Delta A=\sqrt{\mathrm{tr}\rho A^2-(\mathrm{tr}\rho A)^2}$ is the standard deviation of A.

Now determine the amount of disturbance induced on the systems in the set. The state of any one of the systems in the set after the coupling with the pointer is given by tracing over all the other systems and the pointer state. Since the systems are identical by symmetry, just look at the first:

$$\begin{aligned}\rho(t)&=\sum_{aa'}\sum_{a_2\cdots a_N}\int dq\,dq'\,\psi(q)\overline{\psi}(q')\rho_{aa'}|a\rangle\\ &\quad\times\langle a'|\rho_{a_2a_2}\cdots\rho_{a_Na_N}\langle q'+(a'+a_2+\cdots+a_N)\gamma t\\ &\quad\times|q+(a+a_2+\cdots+a_N)\gamma t\rangle\\ &=\sum_{aa'}\int dq\,\psi(q)\overline{\psi}[q+(a-a')\gamma t]\rho_{aa'}|a\rangle\langle a'|.\end{aligned}\tag{5}$$

That is, the off-diagonal parts of ρ_1 are reduced by an amount $1-\int dq\,\psi(q)\overline{\psi}(q+(a-a')\gamma t)$. A convenient initial pointer state $|\psi\rangle$ is a Gaussian wave packet centered at 0 with standard deviation ΔQ (analogous to a coherent state of the electromagnetic field). In this case, it is easily seen that the effect of the coupling to the pointer is to multiply the aa' off-diagonal terms of ρ by a factor $e^{-[\gamma t(a-a')]^2/2\Delta Q^2}$. That is, when $(\gamma t\Delta A)^2/2\Delta Q^2\approx\epsilon\ll 1$, the effect of coupling each member of the set to the same pointer is essentially the same as the effect of coupling each member to a different measuring apparatus, with a perturbation of size $\epsilon=(\gamma t\Delta A/\Delta Q)^2$. This model of measurement can be seen to be equivalent to the abstract POVM given above.

It is interesting to note that the ''weakness'' of this model of measurement can be tuned by adjusting the spread ΔQ of the initial pointer wave packet. As ΔQ becomes small, the measurement becomes stronger and stronger, revealing more information about an individual system while perturbing its wave function more and more. In the limit that $\Delta Q\rightarrow 0$, this model reduces to von Neumann's original model of strong measurement. ΔQ acts as a knob that allows us to tune continuously from weak to strong measurement.

We can weakly measure several observables $\mathcal{O}_\ell$ simultaneously by adjoining several pointer variables Q_ℓ and coupling $\Sigma_\ell\gamma_\ell\mathcal{O}_\ell Q_\ell$. In the limit $\gamma\rightarrow 0$, $N\rightarrow\infty$, the ℓth pointer provides an accurate assessment of $\langle O_\ell\rangle$ while perturbing each system by as small an amount as desired. Note that $\mathcal{O}_\ell$ need not commute with each other: in the weak measurement limit where the ΔQ_ℓ are large and γ_ℓ are small, the measurements do not interfere with each other. By monitoring d^2-1 observables, one can obtain an assessment of all terms in the density matrix simultaneously.

This concludes the detailed discussion of weak measurement. To summarize: by adjoining a suitable measuring apparatus and making the number of systems N in the set large, one can obtain the density matrix to a precision $\delta=\sqrt{1/\epsilon N^2+1/N}$ while perturbing each system by an amount $\epsilon(d^2-1)$. By making N sufficiently large, the single-system reduced density matrix can be determined to arbitrary precision while perturbing each system by an arbitrarily small amount. It can be seen that the detailed model gives the same results as the abstract model of weak measurement given above.

It is interesting to investigate whether the underlying statistics (fermionic or bosonic) of the systems in the set affect the results above. Since both wave functions and interactions are assumed to be symmetric, the results derived above hold equally well for bosonic systems. If the systems are fermionic, in contrast, they cannot be in completely identical states. However, if each system possesses additional degrees of freedom (position, for example, in the case of nuclear spins) that do not figure in the interaction with the measuring apparatus, then the discussion above applies to fermions as well.

In fact, although we have assumed a symmetric situation in which the systems are described by identical density matrices ρ, this restriction is not necessary. If the systems are prepared in the uncorrelated state $\rho_1\otimes\cdots\otimes\rho_N$, where in general $\rho_i \neq \rho_j$, then the entire set of results derived here applies to the determination and control of the *average* single-system density matrix $\overline{\rho} = (1/N)\Sigma_i \rho_i$.

Let us now assume that we can perform arbitrary weak measurements on a set of quantum systems, and feed the results of those measurements back continuously and coherently using the well-known techniques of coherent control. That is, assume that we can implement arbitrary nonlinear unitary transformations as in Eq. (1b) and nonlinear Schrödinger equations as in Eq. (2). How might this technique be applied?

The first potential use of this technique is simulation: a weak feedback controller could be used to simulate the dynamics of a variety of systems that obey a nonlinear Schrödinger equation. Nonlinear Schrödinger equations tend to arise in sets of weakly coupled quantum systems: as noted above in the context of the nonlinear Bloch equation, such coupled sets can be thought of as naturally occurring examples of weak feedback. For example, weak feedback can be used to simulate any set of systems that can be adequately described by a mean-field theory, in which each system is coupled weakly to the expectation value of some operators on the ensemble as a whole.

The use of a nonlinear Schrödinger equation is common in quantum optics to describe the evolution of photons that are weakly interacting with matter, as in a nonlinear optical fiber [23,24]. As just noted, such an effect can be thought of as a naturally occurring example of quantum feedback by weak measurements: the atoms in the fiber weakly monitor and act on the photons. The use of weak feedback to create nonlinearities has the advantage that the form and strength of the nonlinearity induced by the quantum controller can be varied at will. For example, an optical weak feedback apparatus could be constructed by instrumenting a fiber with photodetectors and feeding back their signals to the fiber via electro-optic modulators [25]. Such an optical controller could be used as a quantum analog computer to investigate the effect of time and spatially varying nonlinearities on the propagation of light down the fiber. It is important to note that such a fiber need not itself be nonlinear: all the nonlinearity could be supplied by the controller. In addition, weak feedback could be used to create and investigate the properties of optical solitons in a variety of nonlinear media.

As noted by Haus [23], systems that obey nonlinear Schrödinger equations can be used to create Schrödinger's cats—quantum systems that exist in superpositions of two widely differing quasiclassical states. Although optical fibers are lossy and tend to introduce decoherence, single-mode optical cavities of the sort constructed by Kimble *et al.* are good candidates for control by weak feedback [26]. The very high Q of such cavities implies that the mode in the cavity is only weakly coupled to modes outside the cavity. Heterodyne monitoring of the cavity field therefore constitutes a weak measurement on the photons in the cavity. Nonlinearities induced by weak measurement could be used to create Schrödinger's cat states in the manner of Haus. Wiseman and Milburn have also proposed a cavity quantum electrodynamics enactment of feedback via weak measurements to perform optical squeezing [7].

Finally, quantum feedback via weak measurement could be used to create a novel form of quantum chaos. The usual linear Schrödinger equation does not exhibit sensitive dependence to initial conditions: the ''distance'' between any two states $|\phi\rangle$ and $|\phi'\rangle$, as measured by their inner product $\langle\phi|\phi'\rangle$, remains constant [27,28]. (Traditionally, quantum chaos is not the study of sensitive dependence of quantum trajectories on initial conditions, but rather the study of quantized versions of classical chaotic systems.) The nonlinear Schrödinger equation, in contrast, need not preserve distances between quantum states, and *can* exhibit sensitive dependence on initial conditions [29,30]. Quantum feedback via weak measurement, because it can be used to effect arbitrary nonlinear Schrödinger equations, offers unique opportunities for investigating the sensitive dependence of quantum trajectories on initial conditions. Such ensemble quantum chaos could be used, for example, to construct a ''Schrödinger microscope'' to detect and amplify small differences in quantum wave functions.

We close by examining the relationship between nonlinearity induced by feedback of weak measurements and intrinsically nonlinear quantum mechanics. (Once again, the nonlinearity discussed in this paper is an *effective* nonlinearity: the underlying quantum dynamics of weak feedback is linear.) Nonlinear Schrödinger equations of the type found in Eq. (2) above are common in nonlinear quantum mechanics [31]. Nonlinear quantum mechanics is known to exhibit a number of pathologies, including superluminal communication [32], violations of the second law of thermodynamics [33], and the ability to solve hard computational problems [34]. Since nonlinearity induced by weak feedback occurs entirely within the conventional framework of quantum mechanics, it cannot exhibit the first two of these pathologies. It might allow the solution of hard computational problems by the mechanism of the previous paragraph, viz., constructing a ''Schrödinger microscope'' to detect small perturbations in the wave function of a quantum computer (see also [35]); however, to obtain an exponential speed-up over classical computations, one needs to amplify exponentially small differences in the wave function, which in turn requires an exponentially large number of systems in the set. Nonetheless, it may be the case that weak feedback can be used to provide polynomial acceleration for computational problems.

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