# Space-time geometry of quantum dielectrics

Ulf Leonhardt
*School of Physics and Astronomy, University of St. Andrews, North Haugh, St. Andrews, Fife KY16 9SS, Scotland*
*and Physics Department, Royal Institute of Technology (KTH), Lindstedtsvägen 24, S-10044 Stockholm, Sweden*
(Received 27 January 2000; published 16 June 2000)

Light experiences dielectric matter as an effective gravitational field and matter experiences light as a form of gravity as well. Light and matter waves experience each other as dual space-time metrics, thus establishing a unique model in field theory. *Actio et reactio* are governed by Abraham's energy-momentum tensor and equations of state for quantum dielectrics.

PACS number(s): 03.65.Bz, 03.75.−b, 03.50.−z, 04.20.−q

## I. INTRODUCTION

A moving dielectric medium appears to light as an effective gravitational field [1–4]. The medium alters the way in which an electromagnetic field perceives space and time, formulated most concisely in Gordon's effective space-time metric [1]

$$g^F_{\alpha\beta}=g_{\alpha\beta}+\left(\frac{1}{\varepsilon\mu}-1\right)u_\alpha u_\beta. \tag{1}$$

We allow for a background metric $g_{\alpha\beta}$, mostly to have the convenience of choosing arbitrary coordinates, but also for the possible inclusion of a genuine gravitational field. Gordon's metric (1) depends on the dielectric properties of the medium, on the permittivity $\varepsilon$, and on the magnetic permeability $\mu$, as well as on the four-dimensional flow $u^\alpha$ of the medium (the local four-velocity). The product $\varepsilon\mu$ is the square of the refractive index and the prefactor $1-(\varepsilon\mu)^{-1}$ is known as Fresnel's dragging coefficient [5–7] (in Fresnel's days, the part of the ether that the moving medium is able to drag [6]). In the limit of geometrical optics [8], light rays are zero-geodesic lines with respect to Gordon's metric [1–4]. In the special case of a medium at rest, this result is equivalent to Fermat's principle [8] and to the formulation of geometrical optics as a non-Euclidean geometry in space [9].

Light experiences dielectric matter as an effective space-time metric. How does matter experience light? In atom optics [10], the traditional role of light and matter is reversed: Atomic de Broglie waves are subject to atom-optical instruments made of light. Light acts on matter waves in a similar way to the way matter acts on light. This paper indicates that an atomic matter wave experiences an electromagnetic field as the effective metric

$$g^A_{\alpha\beta}=(1-a\mathscr{L}_F)g_{\alpha\beta}-bT^F_{\alpha\beta} \tag{2}$$

with

$$a=\frac{1}{mc^2\rho}\left(\varepsilon+\frac{1}{\mu}-2\right),\quad b=\frac{1}{mc^2\rho}\left(\varepsilon-\frac{1}{\mu}\right). \tag{3}$$

Here $\mathscr{L}_F$ is the Lagrangian of the free electromagnetic field, defined in Eq. (8) below, and $T^F_{\alpha\beta}$ is the free-field energy-momentum tensor (10). As usual, $c$ denotes the speed of light in vacuum and $m$ is the mass of a single dielectric atom. In the definition (3), $\rho$ can be regarded as the probability density of the atomic de Broglie wave, for most practical purposes. (Strictly speaking, $mc^2\rho$ describes the total enthalpy density of the matter wave, including the rest energy as the lion's share.) Throughout this paper we employ SI units and use the Landau-Lifshitz convention [11] of general relativity (with the exception of using greek space-time and latin space indices). To derive the result (2) with the dielectric parameters (3) we postulate that the interaction between light and matter takes on the general form of a metric. Then we demonstrate the consistency of this idea with previous knowledge, and in particular with Gordon's metric (1).

The metric (2) indicates that the energy-momentum of light curves the space-time of a dielectric matter wave directly. Under normal circumstances the deviation from the background geometry is very small [see Eqs. (2) and (3)], because the ratio between the electromagnetic energy and the atomic rest energy $mc^2$ is typically an extremely small number. In the Newtonian limit of general relativity [11], the gravitational correction to a flat Minkowski space-time is tiny as well, because the correction is proportional to the ratio between the potential energy and $mc^2$ of a test particle. For weak gravitational fields and low test-particle velocities, general relativity is an equivalent formulation of Newtonian physics that agrees in all predicted effects and yet establishes a radically different physical interpretation. Similarly, given the current state of the art in atom optics, the idea that light curves the space-time for matter waves is an equivalent formulation of the known light forces, i.e., of the dipole force and of the recently investigated Röntgen interaction [12]. However, one can conceive of significantly enhancing the dielectric properties of matter waves [4] using similar methods as in the spectacular demonstrations of slow light [13]. Loosely speaking, a large effective dielectric constant $\varepsilon$ could counteract the rest energy $mc^2$ in the relations (3). In this way one could use light to build atom-optical analogs of astronomical objects on Earth, for example, a black hole made of light.

## II. ELECTROMAGNETIC FIELDS

### A. Field tensors

Let us first agree on the definitions of the principal electromagnetic quantities in SI units in general relativity. We employ the space-time coordinates $x^\mu=(ct,\mathbf{x})$. The electromagnetic four-potential is

 

$$A_\nu = (U, -c\mathbf{A}). \tag{4}$$

The electromagnetic-field-strength tensor is constructed as

$$F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5}$$

using the covariant derivatives $D_\mu$ with respect to the background metric $g_{\mu\nu}$. As is well known [11], in the definition (5) of $F_{\mu\nu}$ on a possibly curved space-time, we have been able to replace the $D_\mu$ by ordinary partial derivatives $\partial_\mu \equiv \partial/\partial x^\mu$. The field-strength tensor reads in local Galilean coordinates (in a local Minkowski frame)

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}. \tag{6}$$

It will become useful at a later stage of this enterprise to introduce a four-dimensional formulation $H^{\mu\nu}$ of the dielectric **D** and **H** fields,

$$H^{\mu\nu} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z/c & H_y/c \\ D_y & H_z/c & 0 & -H_x/c \\ D_z & -H_y/c & H_x/c & 0 \end{pmatrix}, \tag{7}$$

here defined in local Galilean coordinates.

### B. Quadratic field tensors

In dielectric media, induced atomic dipoles constitute an interaction between light and matter that is quadratic in the electromagnetic-field-strength tensor [14]. Let us therefore list a set of linearly independent second-rank tensors that are quadratic in $F_{\mu\nu}$. The most elementary one is the product of the metric tensor $g_{\mu\nu}$ with the scalar Lagrangian $\mathscr{L}_F$ of the free electromagnetic field [11]. This Lagrangian is

$$\mathscr{L}_F = -\frac{\varepsilon_0}{4} F_{\alpha\beta} F^{\alpha\beta} = -\frac{\varepsilon_0}{4} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'}, \tag{8}$$

or, in local Galilean coordinates,

$$\mathscr{L}_F = \frac{\varepsilon_0}{2}(E^2 - c^2 B^2). \tag{9}$$

Another quadratic second-rank tensor is the free electromagnetic energy-momentum tensor [11]

$$T^F_{\mu\nu} = \varepsilon_0 F_{\mu\alpha} g^{\alpha\beta} F_{\beta\nu} - \mathscr{L}_F g_{\mu\nu}, \tag{10}$$

or, in local Galilean coordinates,

$$T^F_{\mu\nu} = \begin{pmatrix} I & -\mathbf{S}/c \\ -\mathbf{S}/c & \sigma \end{pmatrix}, \quad T^{\mu\nu}_F = \begin{pmatrix} I & \mathbf{S}/c \\ \mathbf{S}/c & \sigma \end{pmatrix} \tag{11}$$

with

$$I = \frac{\varepsilon_0}{2}(E^2 + c^2 B^2), \quad \mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \wedge \mathbf{B}, \tag{12}$$

$$\sigma = \varepsilon_0 \left[ \left( \frac{E^2}{2} + \frac{c^2 B^2}{2} \right) \mathbf{1} - \mathbf{E} \otimes \mathbf{E} - c^2 \mathbf{B} \otimes \mathbf{B} \right].$$

Here $I$ denotes the intensity, **S** is the Poynting vector, and $\sigma$ is Maxwell's stress tensor. The symbols $\wedge$ and $\otimes$ denote the three-dimensional vector and tensor product, respectively.

We can form second-rank tensors from $F_{\alpha\beta} F^{\alpha'\beta'}$ only by some contraction. Consequently, the linear combinations of the two elementary tensors $\mathscr{L}_F g_{\mu\nu}$ and $T^F_{\mu\nu}$ form the complete class of second-rank tensors that are quadratic in the field strengths $F_{\mu\nu}$.

## III. CLASSICAL ATOMS

### A. Postulates

Consider a classical atom in an electromagnetic field. The atom is pointlike, has a mass $m$, and can sustain induced electric and magnetic dipoles. In the rest frame of the atom the dipoles respond to the square of the electric field strength, $E^2$, and the magnetic field strength, $B^2$, respectively. How does a dielectric atom experience the electromagnetic field when the atom is moving?

Let us postulate that the atom experiences the field as an effective metric. Consequently, according to general relativity [11], the action $S_0$ of the atom is

$$S_0 = -mc \int ds, \quad ds^2 = g^A_{\mu\nu} dx^\mu dx^\nu. \tag{13}$$

Let us further postulate that the metric of the atom, $g^A_{\mu\nu}$, is quadratic in the electromagnetic field strengths. Any metric is a second-rank tensor. Hence, we obtain from Sec. II B the general form (2) mentioned in the Introduction.

### B. Properties

A metric of the structure (2) has nice mathematical properties. In particular, the contravariant metric tensor $g^{\mu\nu}_A$ (the inverse of $g^A_{\mu\nu}$) takes on a simple analytic form,

$$\sqrt{-g_A}\, g^{\mu\nu}_A = \sqrt{-g}\,[(1 - a\mathscr{L}_F) g^{\mu\nu} + b T^{\mu\nu}_F] \tag{14}$$

with

$$g_A \equiv \det(g^A_{\mu\nu}) \tag{15}$$

and

$$\sqrt{-g_A}=\sqrt{-g}\left((1-a\mathscr{L}_F)^2-\frac{b^2}{4}T^F_{\alpha\beta}T_F^{\alpha\beta}\right), \tag{16}$$

as one verifies in local Galilean coordinates, with the relation $T^F_{\alpha\beta}T_F^{\alpha\beta}=\varepsilon_0^2[(E^2-c^2B^2)^2+4c^2(\mathbf{E}\cdot\mathbf{B})^2]$.

### C. Nonrelativistic limit

So far, we have not seen how the metric theory (2) and (13) is related to the model of a moving induced dipole. Let us consider the nonrelativistic limit of velocities low compared with the speed of light. This limit corresponds to a motion in an inertial frame close to a rest frame comoving with the atom. We also regard the electromagnetic field energy to be weak compared with the atomic rest energy $mc^2$. We neglect any genuine gravitational field, and obtain in Cartesian coordinates

$$\begin{aligned} ds&=\sqrt{(1-a\mathscr{L}_F)(c^2dt^2-d\mathbf{x}^2)-bT^F_{\mu\nu}dx^\mu dx^\nu}\\ &\approx\sqrt{c^2dt^2-d\mathbf{x}^2-(a\mathscr{L}_F+bT^F_{00})c^2dt^2}\\ &\approx\left(1-\frac{v^2}{2c^2}-\frac{a\mathscr{L}_F+bT^F_{00}}{2}\right)c\,dt \end{aligned} \tag{17}$$

with $\mathbf{v}=d\mathbf{x}/dt$. Consequently, we can write the action $S_0$ as

$$S_0=-mc\int ds\approx\int(-mc^2+L_0)dt \tag{18}$$

with the nonrelativistic Lagrangian

$$L_0=\frac{m}{2}v^2+\frac{\alpha_E}{2}E^2+\frac{\alpha_B}{2}c^2B^2 \tag{19}$$

and

$$\begin{gathered} \alpha_E=\frac{a+b}{2}\varepsilon_0mc^2,\quad \alpha_B=\frac{b-a}{2}\varepsilon_0mc^2,\\ a=\frac{\alpha_E-\alpha_B}{\varepsilon_0mc^2},\quad b=\frac{\alpha_E+\alpha_B}{\varepsilon_0mc^2}. \end{gathered} \tag{20}$$

The Lagrangian $L_0$ describes indeed a nonrelativistic atom with electric and magnetic polarizibilities $\alpha_E$ and $\alpha_B$, respectively. In this way we have verified that the metric theory (2) and (13) agrees with the physical picture of traveling dipoles and, simultaneously, we have been able to express the coefficients $a$ and $b$ of the metric (2) in terms of atomic quantities.

## IV. MATTER WAVES

### A. Postulate

Gordon has shown [1] that an electromagnetic field experiences dielectric matter as the effective metric (1). Here we postulate that the opposite is also true: A dielectric matter wave experiences the electromagnetic field as a metric, and in particular as the metric (2) that we have motivated for traveling dipoles in Sec. III. We demonstrate the consistency of this idea with Gordon's theory in Sec. V. Let us model the matter wave as, fittingly, a complex Klein-Gordon scalar $\psi$ in an effectively curved space-time. The action $S_A$ of the atom wave $\psi$ is

$$S_A=\int\mathscr{L}_A\sqrt{-g}d^4x \tag{21}$$

in terms of the Klein-Gordon Lagrangian [15]

$$\begin{aligned} \mathscr{L}_A&=\sqrt{\frac{g_A}{g}}\left(\frac{1}{2m}g_A^{\mu\nu}(-i\hbar\partial_\mu\psi^*)(i\hbar\partial_\nu\psi)-\frac{mc^2}{2}\psi^*\psi\right)\\ &=\sqrt{\frac{g_A}{g}}\left(\frac{\hbar^2}{2m}(D^\mu_A\psi^*)(D^A_\mu\psi)-\frac{mc^2}{2}\psi^*\psi\right), \end{aligned} \tag{22}$$

where we have employed the covariant derivatives $D^A_\mu$ with respect to the effective metric (2). The action (21) is minimal if the matter wave $\psi$ obeys the Klein-Gordon equation

$$D^A_\mu D^\mu_A\psi+\frac{m^2c^2}{\hbar^2}\psi=0, \tag{23}$$

or, written explicitly [11],

$$\frac{1}{\sqrt{-g_A}}\partial_\mu(\sqrt{-g_A}g_A^{\mu\nu}\partial_\nu\psi)+\frac{m^2c^2}{\hbar^2}\psi=0. \tag{24}$$

Equation (24) together with the functions (14) and (16) and the parameters (20) describes how atomic matter waves respond to electromagnetic fields.

### B. Röntgen limit

Let us prove explicitly that the Klein-Gordon Lagrangian (22) contains the known light forces in the limit of relatively low velocities (compared with $c$) and of weak fields (compared with $mc^2$). We separate from the atomic wave function $\psi$ the notorious rapid oscillations due to the rest energy $mc^2$ by defining

$$\varphi\equiv\psi\exp\left(i\frac{mc^2}{\hbar}t\right). \tag{25}$$

We neglect gravity and obtain in Cartesian coordinates

$$\mathscr{L}_A \approx \sqrt{-g_A}\left(\frac{1}{2}g_A^{00}(mc^2\varphi^*\varphi + i\hbar\varphi^*\dot{\varphi} - i\hbar\dot{\varphi}^*\varphi) + \frac{i\hbar c}{2}g_A^{0k}(\varphi^*\partial_k\varphi - \varphi\partial_k\varphi^*) - \frac{\hbar^2}{2m}g_A^{kl}(\partial_k\varphi^*)(\partial_l\varphi) - \frac{mc^2}{2}\varphi^*\varphi\right)$$

$$\approx \frac{mc^2}{2}\varphi^*\varphi(1 - a\mathscr{L}_F + bT_F^{00}) + \frac{i\hbar}{2}(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) + \frac{i\hbar c}{2}bT_F^{0k}(\varphi^*\partial_k\varphi - \varphi\partial_k\varphi^*) - \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\varphi^*)\cdot(\boldsymbol{\nabla}\varphi)$$

$$- \frac{mc^2}{2}(1 - 2a\mathscr{L}_F)\varphi^*\varphi$$

$$= \frac{i\hbar}{2}(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) - \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\varphi^*)\cdot(\boldsymbol{\nabla}\varphi) + \left(\frac{\alpha_E}{2}E^2 + \frac{\alpha_B}{2}c^2B^2\right)\varphi^*\varphi + \frac{\alpha_E + \alpha_B}{2m}(\mathbf{E}\wedge\mathbf{B})\cdot i\hbar(\varphi^*\boldsymbol{\nabla}\varphi - \varphi\boldsymbol{\nabla}\varphi^*). \quad (26)$$

This result agrees with the Röntgen Lagrangian of Ref. [16] in the limit of weak fields and, consequently, indeed describes the known nonresonant light forces including the Röntgen interaction [12].

### C. Dielectric flow

Accelerated by light forces, an atomic matter wave will form a probability current that appears as a dielectric flow. Let us calculate the flow from the phase $S$ of the wave function,

$$\psi = |\psi|e^{iS}. \quad (27)$$

We introduce

$$w^\mu \equiv -\frac{\hbar}{mc}g_A^{\mu\nu}\partial_\nu S, \quad (28)$$

and obtain from the Klein-Gordon equation (24) the conservation law of the four-dimensional probability current,

$$D_\mu^A(|\psi|^2 w^\mu) = \frac{1}{\sqrt{-g_A}}\partial_\mu(\sqrt{-g_A}|\psi|^2 w^\mu) = 0. \quad (29)$$

In the absence of electromagnetic forces, $w^\mu$ describes the local four-velocity of a free matter wave. In the presence of a field, we introduce the dielectric flow $u^\mu$ by normalizing $w^\mu$ to unity with respect to the background metric $g_{\mu\nu}$,

$$u^\mu \equiv \frac{w^\mu}{w}, \quad w \equiv \sqrt{g_{\mu\nu}w^\mu w^\nu}. \quad (30)$$

We define two densities $\varrho$ and $\rho$ as

$$\varrho \equiv |\psi|^2 w\sqrt{\frac{g_A}{g}}, \quad \rho \equiv \varrho w. \quad (31)$$

We obtain from the conservation law (29)

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\varrho u^\mu) = D_\mu(\varrho u^\mu) = 0. \quad (32)$$

Consequently, $\varrho$ is the scalar probability density of the atomic de Broglie wave. For most practical purposes the two densities $\varrho$ and $\rho$ are identical, because $w$ is unity to a very good approximation. The difference between $\varrho$ and $\rho$ is subtle: In Sec. V E we show that $mc^2\rho$ is the total enthalpy density of the dielectric matter wave, with the rest-energy density $mc^2\varrho$ as the lion's share.

### D. Hydrodynamic limit

As has been mentioned, the objective of this paper is the proof that the metric interaction (2) between matter waves and light is compatible with the known theory of dielectrics [1,14]. When a matter wave or, more likely, a macroscopic condensate of identical matter waves reaches the status of a dielectric it behaves like a quantum fluid. In this macroscopic limit the de Broglie density varies over significantly larger ranges than the de Broglie wavelength (the same applies to frequencies), and a hydrodynamic approach has become extremely successful [17]. Let us approximate

$$i\hbar\partial_\nu\psi \approx -\psi\hbar\partial_\nu S. \quad (33)$$

We obtain from the Klein-Gordon Lagrangian (22) the hydrodynamic approximation

$$\mathscr{L}_A = \sqrt{\frac{g_A}{g}}|\psi|^2\left(\frac{\hbar^2}{2m}g_A^{\mu\nu}(\partial_\mu S)(\partial_\nu S) - \frac{mc^2}{2}\right). \quad (34)$$

Let us consider the Euler-Lagrange equations derived from the hydrodynamic Lagrangian (34). We obtain from the $\partial_\mu S$ dependence of $\mathscr{L}_A$ the dielectric flow (32) and from a variation with respect to $|\psi|^2$ the dielectric Hamilton-Jacobi equation

$$g_A^{\mu\nu}(\partial_\mu S)(\partial_\nu S) = \frac{m^2c^2}{\hbar^2}, \quad (35)$$

or, in terms of the four-vector $w^\mu$ of Eq. (28),

$$g_{\mu\nu}^A w^\mu w^\nu = 1. \quad (36)$$

In the hydrodynamic limit the $w^\mu$ vector represents a four-velocity that is normalized with respect to the effective metric (2). We also see that the hydrodynamic Lagrangian (34) vanishes at the actual minimum that corresponds to the physical behavior of a dielectric matter wave.

## V. QUANTUM DIELECTRICS

### A. Actio et reactio

In the previous section we considered a dielectric matter wave in a given electromagnetic field. Gordon [1] studied the opposite extreme—an electromagnetic field in a given dielectric medium. Let us address here an intermediate regime of *actio et reactio* where light acts on matter and matter also acts on light. Such a physical regime, characterizing a quantum dielectric, occurs, for example, when a Bose-Einstein condensate of an alkali-metal vapor [17] interacts nonresonantly with light [18]. If we were able to arrive at Gordon's metric (1) from our starting point (2) we would be inclined to take this as evidence that our approach is right.

To include the dynamics of the electromagnetic field we add the free-field Lagrangian $\mathscr{L}_F$ to the atomic $\mathscr{L}_A$ in the hydrodynamic approximation (34),

$$\mathscr{L}=\mathscr{L}_F+\mathscr{L}_A, \tag{37}$$

and regard the electromagnetic field as a dynamic object that is subject to the principle of least action. We could also easily include other interactions by additional terms in $\mathscr{L}_A$ such as the atomic collisions within a Bose-Einstein condensate [17] by a Gross-Pitaevskii term. Let us consider the field variation

$$\delta_F\mathscr{L}=\delta_F\mathscr{L}_F+\sqrt{\frac{g_A}{g}}|\psi|^2\frac{\hbar^2}{2m}(\partial_\mu S)(\partial_\nu S)\,\delta_F g_A^{\mu\nu} + \sqrt{\frac{g}{g_A}}\mathscr{L}_A\delta_F\sqrt{\frac{g_A}{g}}. \tag{38}$$

As has been mentioned in Sec. IV D, the atomic Lagrangian $\mathscr{L}_A$ vanishes at the minimum of the action, in the hydrodynamic limit. We utilize the fact that

$$\delta_F g_A^{\mu\nu}=-g_A^{\mu\alpha}g_A^{\nu\beta}\delta_F g^A_{\alpha\beta}, \tag{39}$$

and obtain, using Eqs. (28)–(31),

$$\delta_F\mathscr{L}=\delta_F\mathscr{L}_F-\frac{mc^2}{2}\rho u^\alpha u^\beta\delta_F g^A_{\alpha\beta}. \tag{40}$$

The variation of the Lagrangian with respect to the field determines the field dynamics via the Euler-Lagrange equations. Can we cast $\delta_F\mathscr{L}$ in the role of a dielectric?

### B. Effective Lagrangian

The principal mathematical artifice of this paper is an effective Lagrangian that is designed to agree with $\mathscr{L}$ under field variations, and that describes a dielectric medium,

$$\mathscr{L}_{\rm eff}\equiv\mathscr{L}_F+\frac{mc^2}{2}\rho(g_{\alpha\beta}-g^A_{\alpha\beta})u^\alpha u^\beta \tag{41}$$

with

$$\delta_F\mathscr{L}=\delta_F\mathscr{L}_{\rm eff}. \tag{42}$$

Note that the two field variations in the relation (42) differ in a subtle way: On the left-hand side, $\delta_F$ abbreviates the total variation with respect to the electromagnetic field, whereas on the right-hand side of Eq. (42) we treat $\varepsilon$, $\mu$, and $u^\alpha$ as being fixed, despite their hidden dependence on the field due to the relations (28)–(31).

We show explicitly in Sec. V D that $\mathscr{L}_{\rm eff}$ is indeed the desired Lagrangian of light in a dielectric medium. Here we note that $\mathscr{L}_{\rm eff}$ may metamorphose into a multitude of forms. For example, we introduce the permittivity $\varepsilon$ and the magnetic permeability $\mu$ in terms of elementary atomic quantities and in accordance with the parameters (3) mentioned in the Introduction,

$$\varepsilon=1+\frac{\alpha_E}{\varepsilon_0}\rho,\quad \frac{1}{\mu}=1-\frac{\alpha_B}{\varepsilon_0}\rho. \tag{43}$$

In this way we obtain directly from Eqs. (2) and (3) that

$$\mathscr{L}_{\rm eff}=\frac{1}{2}\left[\left(\varepsilon+\frac{1}{\mu}\right)\mathscr{L}_F+\left(\varepsilon-\frac{1}{\mu}\right)u^\alpha u^\beta T^F_{\alpha\beta}\right]. \tag{44}$$

We can also express the effective Lagrangian as

$$\mathscr{L}_{\rm eff}=\frac{1}{\mu}\mathscr{L}_F+\varepsilon_0\frac{\varepsilon\mu-1}{2\mu}F_{\alpha'\beta'}F_{\alpha\beta}u^\alpha u^{\alpha'}g^{\beta\beta'}, \tag{45}$$

due to the definition (10) of the free-field energy-momentum tensor, or we may perform further manipulations, utilizing the relations

$$F_{\alpha'\beta'}F_{\alpha\beta}u^\alpha u^{\alpha'}g^{\beta\beta'}=F_{\alpha'\beta'}F_{\alpha\beta}g^{\alpha\alpha'}u^\beta u^{\beta'},\qquad F_{\alpha'\beta'}F_{\alpha\beta}u^\alpha u^{\alpha'}u^\beta u^{\beta'}=0, \tag{46}$$

due to the symmetry of the background metric $g^{\alpha\beta}$ and the antisymmetry of the field-strength tensor $F_{\alpha\beta}$.

### C. Gordon's metric

Quite remarkably, one can express the effective Lagrangian in the form [1]

$$\mathscr{L}_{\rm eff}=-\frac{\varepsilon_0}{4\mu}F_{\alpha\beta}F^{(\alpha)(\beta)} \tag{47}$$

with

$$F^{(\alpha)(\beta)}\equiv g_F^{\alpha\alpha'}g_F^{\beta\beta'}F_{\alpha'\beta'} \tag{48}$$

and

$$g_F^{\alpha\beta}=g^{\alpha\beta}+(\varepsilon\mu-1)u^\alpha u^\beta. \tag{49}$$

The effective Lagrangian appears as the free electromagnetic Lagrangian in a curved space-time with metric (49). A short exercise proves that $g_F^{\alpha\beta}$ is the inverse of $g^F_{\alpha\beta}$, i.e., as the notation suggests, the contravariant metric tensor with respect to the covariant $g^F_{\alpha\beta}$. Consequently, we have indeed

arrived at Gordon's space-time geometry of light in moving media, starting from our metric (2), which supports the validity of our postulates.

Note that Gordon's space-time geometry is not completely perfect [1]. The metrics (1) and (49) depend only on the square of the refractive index, $\varepsilon\mu$, whereas a dielectric medium is characterized by two dielectric constants $\varepsilon$ and $\mu$, in general. What is the imperfection in the Lagrangian (47)? In order to describe a density in general relativity, and in particular a Lagrangian density, we must consider the determinant of the metric that describes the scaling of space and time. Gordon [1] calculated the determinant by employing comoving medium coordinates, with the result

$$g_F \equiv \det(g^F_{\alpha\beta}) = \frac{g}{\varepsilon\mu}. \tag{50}$$

Hence we obtain the effective action

$$S_{\text{eff}} = \int \mathscr{L}_{\text{eff}} \sqrt{-g}\, d^4x = -\frac{\varepsilon_0}{4} \int \sqrt{\frac{\varepsilon}{\mu}} F_{\alpha\beta} F^{(\alpha)(\beta)} \sqrt{-g_F}\, d^4x \tag{51}$$

which may deviate from the perfect

$$S_F = -\frac{\varepsilon_0}{4} \int F_{\alpha\beta} F^{(\alpha)(\beta)} \sqrt{-g_F}\, d^4x \tag{52}$$

when $\varepsilon/\mu$ varies significantly. However, when the density profile of the quantum liquid varies smoothly compared with the wavelength of light, we can neglect the variation of $\varepsilon/\mu$. Ultracold atoms or Bose-Einstein condensates [17] are usually in this regime, which is also compatible with the hydrodynamic behavior of the quantum liquid.

### D. Maxwell's equations

The first group of Maxwell's equations follows from the structure (5) of the field-strength tensor $F_{\mu\nu}$. The Euler-Lagrange equations of the effective Lagrangian (47) yield the second group [1,14],

$$D_\alpha H^{\alpha\beta} = 0 \quad \text{or} \quad \partial_\alpha(\sqrt{-g} H^{\alpha\beta}) = 0 \tag{53}$$

with the constitutive equations

$$H^{\alpha\beta} = \frac{\epsilon_0}{\mu} F^{(\alpha)(\beta)}. \tag{54}$$

In local Galilean coordinates we can represent $H^{\alpha\beta}$ in terms (7) of the dielectric **D** and **H** fields in SI units. In this way we find yet another physically meaningful expression for the effective Lagrangian,

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4} F_{\alpha\beta} H^{\alpha\beta} = \frac{\mathbf{E}\cdot\mathbf{D}}{2} - \frac{\mathbf{B}\cdot\mathbf{H}}{2}, \tag{55}$$

which is indeed the explicit form of the Lagrangian for the electromagnetic field in a linear dielectric.

Equation (54) is equivalent [1] to Minkowski's constitutive equations in a moving medium [14,19]. In the limit of low velocities we recover the familiar relations $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$ and $\mu \mathbf{H} = \varepsilon_0 c^2 \mathbf{B}$, and, via Eq. (43),

$$\mathbf{D} \approx (\varepsilon_0 + \alpha_E \varrho)\mathbf{E}, \quad \mathbf{H} \approx (\varepsilon_0 - \alpha_B \varrho) c^2 \mathbf{B}, \tag{56}$$

assuming a weak field when $\rho \approx \varrho$. Relativistic first-order corrections lead to the constitutive equations derived in Ref. [16] which describe, for example, the Röntgen effect [20] or lead to Fresnel's light drag [6] measured in Fizeau's experiment [7].

In case of a smooth dielectric density we can regard $\varepsilon/\mu$ as a constant, and obtain from Maxwell's equations

$$\partial_\alpha(\sqrt{-g_F} F^{(\alpha)(\beta)}) = 0 \quad \text{or} \quad D^F_\alpha F^{(\alpha)(\beta)} = 0. \tag{57}$$

Light experiences the quantum dielectric as the space-time metric (1), i.e., as an effective gravitational field.

### E. Energy-momentum tensor

According to Antoci and Mihich [21], Gordon [1] has already settled the notorious debate about Minkowski's [19] versus Abraham's [22] energy-momentum tensor in Abraham's favor. However, in his paper [1], Gordon assumed the dielectric properties of the medium $\varepsilon$, $\mu$, and $u^\alpha$ as preassigned quantities. Having done so, the derived energy-momentum tensor is valid if and only if the dielectric quantities are constants, i.e., in the case of a uniform medium, because the conservation of energy and momentum presupposes the homogeneity of space-time, according to Noether's theorem. If one tries to determine the energy and momentum of the electromagnetic field in an inhomogeneous medium one must not consider the dielectric properties as given functions, but rather as being generated by a physical object, such as the quantum dielectric studied in this paper. In short, one should take into account *actio et reactio*, and in particular the back action of the medium (an effect seen experimentally [23]). Does Abraham's tensor have significance beyond uniform media?

Let us determine the energy-momentum tensor via general relativity, as a variation of the Lagrangian with respect to the background metric [11],

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathscr{L})}{\delta g_{\mu\nu}} = -2\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} - \mathscr{L} g^{\mu\nu}. \tag{58}$$

A metric variation $\delta_g$ of the Lagrangian gives, in analogy with Eq. (40) and the considerations in Sec. V B,

$$\delta_g \mathscr{L} = \delta_g \mathscr{L}_F - \frac{mc^2}{2} \rho u^\alpha u^\beta \delta_g g^A_{\alpha\beta} = \delta_g \mathscr{L}_{\text{eff}} - \frac{mc^2}{2} \rho u^\alpha u^\beta \delta_g g_{\alpha\beta}. \tag{59}$$

We recall that $\mathscr{L}_A$ vanishes in the hydrodynamic limit. Consequently, we arrive at the total energy-momentum tensor in the form

$$T^{\mu\nu}=-2\frac{\delta\mathscr{L}_{\rm eff}}{\delta g_{\mu\nu}}-\mathscr{L}_F g^{\mu\nu}+mc^2\rho u^\mu u^\nu. \tag{60}$$

We represent this expression as the sum

$$T^{\mu\nu}=T_A^{\mu\nu}+T_{\rm eff}^{\mu\nu} \tag{61}$$

with the atomic component

$$T_A^{\mu\nu}=mc^2\rho u^\mu u^\nu-pg^{\mu\nu}, \tag{62}$$

$$p=\mathscr{L}_F-\mathscr{L}_{\rm eff}=\tfrac{1}{4}F_{\alpha\beta}(H^{\alpha\beta}-\varepsilon_0F^{\alpha\beta}), \tag{63}$$

and

$$T_{\rm eff}^{\mu\nu}=-2\frac{\delta\mathscr{L}_{\rm eff}}{\delta g_{\mu\nu}}-\mathscr{L}_{\rm eff}g^{\mu\nu}. \tag{64}$$

We are inclined to interpret the tensor (64) as the effective energy-momentum tensor of the electromagnetic field in the presence of a dielectric medium.

The atomic tensor (62) appears as the energy-momentum of a fluid under the dielectric pressure (63). In the limit of low flow velocities the pressure approaches $-\varepsilon_0(\alpha_E E^2+\alpha_B c^2B^2)\varrho/2$, according to Eqs. (55) and (56). In this limit, atomic dipoles with positive $\alpha_E$ and $\alpha_B$ are attracted toward increasing field intensities. We also see from the atomic energy-momentum tensor (62) that a dielectric fluid possesses the total enthalpy density $mc^2\rho=mc^2w\varrho$, including the relativistic rest energy. In this way we have found an interpretation for the density $\rho$ that appears at a prominent place in Eq. (3). To calculate the enthalpy, we express the effective Lagrangian (41) in terms of the norm $w$. We use the definition (30) of the four-velocity $u^\alpha$ and the normalization (36) of the $w^\alpha$, and obtain

$$p=\mathscr{L}_F-\mathscr{L}_{\rm eff}=\frac{mc^2\varrho}{2}\left(\frac{1}{w}-w\right), \tag{65}$$

or, by inversion,

$$mc^2\rho=mc^2w\varrho=\sqrt{m^2c^4\varrho^2+p^2}-p. \tag{66}$$

This equation describes how the enthalpy density depends on the pressure and on the dielectric density. On the other hand, Eq. (63) quantifies the pressure, which depends on the dielectric density and flow, and on the electromagnetic field as an external quantity. We may interpret the two formulas (63) and (66) as the equations of state for the quantum dielectric. The density of the fluid's internal energy is the difference between enthalpy density and pressure [24],

$$\epsilon=\sqrt{m^2c^4\varrho^2+p^2}-2p. \tag{67}$$

We see that the internal energy approaches $mc^2+\varepsilon_0(\alpha_E E^2+\alpha_B c^2B^2)$ in the limit of a slow flow and a low dielectric pressure. Atomic dipoles with positive $\alpha_E$ and $\alpha_B$ seem to gain internal energy in the presence of an electromagnetic field.

Let us turn to the energy-momentum tensor of the field. The effective Lagrangian $\mathscr{L}_{\rm eff}$ characterizes a medium with preassigned dielectric functions $\varepsilon$ and $\mu$, i.e., Gordon's case [1]. Consequently [1], the effective energy-momentum tensor of the electromagnetic field is Abraham's [22],

$$T_{\rm eff}^{\mu\nu}=T_{\rm Ab}^{\mu\nu}=T_{\rm Mk}^{\mu\nu}-(\varepsilon\mu-1)u^\mu\Omega^\nu, \tag{68}$$

with Minkowski's tensor [19]

$$T_{\rm Mk}^{\mu\nu}=H^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu}+\tfrac{1}{4}H^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu} \tag{69}$$

corrected by the *Ruhstrahl* [22]

$$\Omega^\nu=F_{\alpha\alpha'}u^{\alpha'}u_\beta(H^{\alpha\beta}u^\nu+H^{\beta\nu}u^\alpha+H^{\nu\alpha}u^\beta). \tag{70}$$

In locally comoving Galilean coordinates or in a medium at rest, the spatial component of the *Ruhstrahl* is proportional to the Poynting vector (hence the name),

$$\Omega^\nu=\left(0,\frac{\mathbf{E}\wedge\mathbf{H}}{c}\right). \tag{71}$$

In this case the effective energy-momentum tensor of the field takes the form

$$T_{\rm Ab}^{\mu\nu}=\begin{pmatrix} I & \mathbf{S}/c \\ \mathbf{S}/c & \sigma \end{pmatrix} \tag{72}$$

with intensity $I$, Poynting vector $\mathbf{S}$, and stress tensor $\sigma$,

$$\begin{aligned} I&=\frac{\mathbf{E}\cdot\mathbf{D}}{2}+\frac{\mathbf{B}\cdot\mathbf{H}}{2},\quad \mathbf{S}=\mathbf{E}\wedge\mathbf{H},\\ \sigma&=\left(\frac{\mathbf{E}\cdot\mathbf{D}}{2}+\frac{\mathbf{B}\cdot\mathbf{H}}{2}\right)\mathbf{1}-\mathbf{E}\otimes\mathbf{D}-\mathbf{B}\otimes\mathbf{H}. \end{aligned} \tag{73}$$

We see that Abraham's tensor indeed describes the effective energy-momentum of the electromagnetic field, even in the general case of a nonuniform medium that is able to move under the pressure of light forces.

## VI. CREDO

Light experiences dielectric matter as an effective gravitational field [1–4] and matter experiences light as a form of gravity as well. Light and matter see each other as dual space-time metrics, a unique model in field theory, to the knowledge of the author. We have solidified this mental picture by postulating the idea and demonstrating its striking consistency with the theory of dielectrics [1,14]. It would be interesting to see whether our model can be derived directly from first principles. In passing, we have determined the energy-momentum tensor that governs *actio et reactio* of electromagnetic fields in quantum dielectrics. The tensor is Abraham's [22] plus the energy-momentum of the medium characterized by a dielectric pressure and an enthalpy density.

Our idea may serve as a guiding line for understanding the effects of slow light [13] on matter waves. Here one can conceive of creating light fields that appear to atoms as quasiastronomical objects. The ultimate result in this field would

be the creation of a black hole made of light.

Light and matter interact with each other as if both were gravitational fields, and light and matter are genuine quantum fields in nature. A distinct quantum regime of dielectrics has been prepared in laboratories where Bose-Einstein condensates of alkali-metal vapors [17] interact nonresonantly with light quanta, but has never been viewed as an analog of quantum gravity, to the knowledge of the author. Sound in superfluids [25] and in alkali-metal Bose-Einstein condensates [26] has been considered as a quantum field in a curved space-time, as being able to emit the acoustic analog of Hawking radiation [27]. However, the quantum sound still propagates in a classical medium, in contrast to light quanta in a quantum dielectric. In many respects, we have reasons to hope that Bose-Einstein condensates may serve as testable prototype models for quantum gravity.

## ACKNOWLEDGMENTS

I am very grateful to Sir Michael Berry, Ignacio Cirac, Carsten Henkel, Susanne Klein, Rodney Loudon, Paul Piwnicki, Stig Stenholm, and Martin Wilkens for conversations on moving media. I acknowledge the generous support of the Alexander von Humboldt Foundation and of the Göran Gustafsson Stiftelse.

[1] W. Gordon, Ann. Phys. (Leipzig) **72**, 421 (1923).
[2] Pham Mau Quan, C.R. Hebd. Seances Acad. Sci. **242**, 465 (1956); Arch. Ration. Mech. Anal. **1**, 54 (1957/58).
[3] U. Leonhardt and P. Piwnicki, Phys. Rev. A **60**, 4301 (1999).
[4] U. Leonhardt and P. Piwnicki, Phys. Rev. Lett. **84**, 822 (2000).
[5] C. Møller, *The Theory of Relativity*, 2nd ed. (Clarendon, Oxford, 1972).
[6] A.J. Fresnel, Ann. Chim. Phys. **9**, 57 (1818).
[7] H. Fizeau, C.R. Hebd. Seances Acad. Sci. **33**, 349 (1851).
[8] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, England, 1999).
[9] E. Bortolotti, Rend. R. Acc. Naz. Linc. 6a **4**, 552 (1926).
[10] See, e.g., *Atom Interferometry*, edited by P. Berman (Academic, San Diego, 1997).
[11] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).
[12] H. Wei, R. Han, and X. Wei, Phys. Rev. Lett. **75**, 2071 (1995). See also Refs. [16,20]; M. Babiker, E.A. Power, and T. Thirunamachandran, Proc. R. Soc. London, Ser. A **332**, 187 (1973); M. Babiker, J. Phys. B **17**, 4877 (1984); C. Baxter, M. Babiker, and R. Loudon, Phys. Rev. A **47**, 1278 (1993); V. Lembessis, M. Babiker, C. Baxter, and R. Loudon, *ibid.* **48**, 1594 (1993); M. Wilkens, *ibid.* **49**, 570 (1994); Phys. Rev. Lett. **72**, 5 (1994); **81**, 1533 (1998); G. Spavieri, *ibid.* **81**, 1533 (1998); **82**, 3932 (1999); Phys. Rev. A **59**, 3194 (1999).
[13] L.V. Hau, S.E. Harris, Z. Dutton, and C.H. Behroozi, Nature (London) **397**, 594 (1999); M.M. Kash, V.A. Sautenkov, A.S. Zibrov, L. Hollberg, G.R. Welch, M.D. Lukin, Y. Rostovsev, E.S. Fry, and M.O. Scully, Phys. Rev. Lett. **82**, 5229 (1999); D. Budiker, D.F. Kimball, S.M. Rochester, and V.V. Yashchuk, *ibid.* **83**, 1767 (1999).
[14] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1984).
[15] L.D. Landau and E.M. Lifshitz, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).
[16] U. Leonhardt and P. Piwnicki, Phys. Rev. Lett. **82**, 2426 (1999).
[17] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[18] U. Leonhardt, T. Kiss, and P. Piwnicki, Eur. Phys. J. D **7**, 413 (1999).
[19] H. Minkowski, Nachr. K. Ges. Wiss. Gett., Math.-Phys. Kl. **1908**, 53.
[20] W.C. Röntgen, Ann. Phys. Chem. **35**, 264 (1888).
[21] S. Antoci and L. Mihich, Nuovo Cimento Soc. Ital. Fis., B **112**, 991 (1997); Eur. Phys. J. D **3**, 205 (1998).
[22] M. Abraham, Rend. Circ. Mat. Palermo **28**, 1 (1909); **30**, 33 (1910).
[23] A. Ashkin and J.M. Dziedzic, Phys. Rev. Lett. **30**, 139 (1973).
[24] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
[25] W.G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981); Phys. Rev. D **51**, 2827 (1995); T.A. Jacobson, *ibid.* **44**, 1731 (1991); T.A. Jacobson and G.E. Volovik, *ibid.* **58**, 064021 (1998); N.B. Kopnin and G.E. Volovik, Pis'ma Zh. Éksp. Teor. Fiz. **67**, 124 (1998) [JETP Lett. **67**, 140 (1998)]; T.A. Jacobson and G.E. Volovik, *ibid.* **68**, 833 (1998); [*ibid.* **68**, 874 (1998)]; G.E. Volovik, *ibid.* **69**, 662 (1999); [*ibid.* **69**, 705 (1999)]; M. Visser, Class. Quantum Grav. **15**, 1767 (1998).
[26] L.J. Garay, J.R. Anglin, J.I. Cirac, and P. Zoller (unpublished).
[27] S.M. Hawking, Nature (London) **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).