Temperature dependence of the Casimir effect between metallic mirrors

Cyriaque Genet, Astrid Lambrecht, and Serge Reynaud

Laboratoire Kastler Brossel, Unité de l'Ecole Normale Supérieure, de l'Université Pierre et Marie Curie, et du Centre National de la Recherche Scientifique, Campus Jussieu, case 74, 75252 Paris Cedex 05, France

We calculate the Casimir force and free energy for plane metallic mirrors at nonzero temperature. Numerical evaluations are given with temperature and conductivity effects treated simultaneously. The results are compared with the approximation where both effects are treated independently and the corrections simply multiplied. The deviation between the exact and approximated results takes the form of a temperature dependent function for which an analytical expression is given. The knowledge of this function allows simple estimations that are accurate below the 1% level.

I. INTRODUCTION

The Casimir force [1] has been observed in a number of ''historic'' experiments [2–5]. It has been measured recently with an improved experimental precision [6–8]. This should allow for an accurate comparison with the predictions of quantum field theory, provided that these predictions account for the differences between real experiments and the idealized Casimir situation. In particular, experiments are performed at room temperature between metallic mirrors and not at zero temperature between perfect reflectors. The theoretical expectations should be computed with a high accuracy if the aim is to test agreement between theory and experiment at, say, the 1% level. The efforts for accuracy are also worth for making it possible to control the effect of Casimir force when studying small short range forces [9–11].

The influence of thermal field fluctuations on the Casimir force are known to become important for distances of the order of a typical length [12–15]

$$\lambda_T = \frac{2\pi c}{\omega_T} = \frac{\hbar c}{k_B T}. \tag{1}$$

When evaluated at room temperature, this length λ_T is approximately 7 μm. In contrast, the finite conductivity of metals has an appreciable effect for distances smaller than or of the order of the plasma wavelength λ_P determined by the plasma frequency ω_P of the metal (see Ref. [16], and references therein)

$$\lambda_P = \frac{2\pi c}{\omega_P}. \tag{2}$$

For metals used in the recent experiments, this wavelength lies in the range 0.1–0.2 μm. This means that conductivity and thermal corrections to the Casimir force are important in quite different distance ranges. Thermal corrections are usually ignored in the sub-μm range where the effect of imperfect reflection is significant whereas the conductivity correction is unimportant above a few μm where the effect of temperature becomes appreciable. This explains why the two corrections are usually treated independently from each other. When an accurate comparison between experimental and theoretical values of the Casimir force is aimed at, the error induced by this approximation has, however, to be precisely evaluated. Furthermore, the region of overlap of the two corrections is precisely in the μm range, which is a crucial distance range for the comparison between experiment and theory.

The purpose of this paper is to give an accurate evaluation of the Casimir force F taking into account finite conductivity and temperature corrections at the same time. To characterize the whole correction, we will compute the factor η_F describing the combined effect of conductivity and temperature

$$\eta_F = \frac{F}{F_{\text{Cas}}},$$

$$F_{\text{Cas}} = \frac{\hbar c A \pi^2}{240 L^4}. \tag{3}$$

F_{Cas} is the ideal Casimir force corresponding to perfect mirrors in vacuum. L is the distance between the mirrors, A their surface, and $\hbar$ and c, respectively, the Planck constant and the speed of light. We will also evaluate the factors associated with each effect taken separately from each other

$$\eta_F^P = \frac{F^P}{F_{\text{Cas}}}, \qquad \eta_F^T = \frac{F^T}{F_{\text{Cas}}}. \tag{4}$$

F^P is the Casimir force evaluated by accounting for finite conductivity of the metals but assuming zero temperature and F^T is the Casimir force evaluated at temperature T for perfect reflectors. Of course η_F^P depends on the ratio L/λ_P and η_F^T on the ratio L/λ_T.

Now the question raised in the previous paragraphs may be stated precisely: to which level of accuracy can the complete correction factor η_F be approximated as the product of the factors η_F^P and η_F^T? To answer this question we will evaluate the quantity

$$\delta_F = \frac{\eta_F}{\eta_F^P \eta_F^T} - 1 \tag{5}$$

which measures the degree of validity of the approximation where both effects are evaluated independently from each other. We will give an analytical estimation of this deviation which may thus be taken into account without any difficulty.

We will also give the same results for the Casimir energy by defining a factor η_E measuring the whole correction of Casimir energy due to conductivity and temperature and then discussing the factors η_E^P and η_E^T and the deviation δ_E in the same manner as for the force.

Some additional remarks have to be made at this point. First, recent experiments are not performed in the plane-plane but in the plane-sphere configuration. The Casimir force in this geometry is usually estimated from the proximity theorem [17–21]. Basically this amounts to evaluating the force by adding the contributions of various distances as if they were independent. In the plane-sphere geometry the force evaluated in this manner turns out to be given by the Casimir energy evaluated in the plane-plane configuration for the distance L being defined as the distance of closest approach in the plane-sphere geometry. Hence, the factor η_E evaluated in this paper for energy can be used to infer the factor for the force measured in the plane-sphere geometry. Then, surface roughness corrections will not be considered in the present paper. Finally the dielectric response of the metallic mirrors will be described by a plasma model. This model is known to describe correctly the Casimir force in the long distance range which is relevant for the study of temperature effects. Keeping these remarks in mind, our results will provide one with an accurate evaluation of the Casimir force in the whole range of experimentally explored distances.

II. CASIMIR FORCE AND FREE ENERGY

When real mirrors are characterized by frequency-dependent reflection coefficients, the Casimir force is obtained as an integral over frequencies and wavevectors associated with vacuum and thermal fluctuations [22]. The Casimir force is a sum of two parts corresponding to the two field polarizations with the two parts having the same form in terms of the corresponding reflection coefficients

$$F = \sum_{k=-\infty}^{\infty} \frac{\omega_T}{2} \mathbb{F}[k\omega_T],$$

$$\mathbb{F}[\omega \geq 0] = \frac{\hbar A}{2\pi^2} \int_{\frac{\omega}{c}}^{+\infty} d\kappa \kappa^2 f,$$

$$(6)$$

$$f = \frac{r_\perp^2(i\omega, i\kappa)}{e^{2\kappa L} - r_\perp^2(i\omega, i\kappa)} + \frac{r_\parallel^2(i\omega, i\kappa)}{e^{2\kappa L} - r_\parallel^2(i\omega, i\kappa)},$$

$$\mathbb{F}[-\omega] = \mathbb{F}[\omega].$$

$r_\perp(r_\parallel)$ denotes the amplitude reflection coefficient for the orthogonal (parallel) polarization of one of the two mirrors. The mirrors are here supposed to be identical, otherwise $r_\perp^2$ should be replaced by the product of the two coefficients. ω is the frequency and κ the wave vector along the longitudinal direction of the cavity formed by the two mirrors. $\mathbb{F}[\omega]$ is defined for positive frequencies and extended to negative ones by parity.

The Casimir force (6) may also be rewritten after a Fourier transformation, as a consequence of Poisson formula [15]

$$F = \sum_{m=-\infty}^{\infty} \widetilde{\mathbb{F}}(m\lambda_T),$$

$$\widetilde{\mathbb{F}}(x) = \int_0^\infty d\omega \cos\left(\frac{\omega x}{c}\right) \mathbb{F}[\omega]. \tag{7}$$

The contribution of vacuum fluctuations, that is also the limit of a null temperature ($\omega_T \rightarrow 0$) in Eq. (6), corresponds to the contribution $m=0$ in Eq. (7)

$$F^P = \widetilde{\mathbb{F}}(0) = \int_0^\infty d\omega \mathbb{F}[\omega]. \tag{8}$$

Hence, the whole force (7) is the sum of this vacuum contribution $m=0$ and of thermal contributions $m \neq 0$.

We will consider metallic mirrors with the dielectric function $\varepsilon(i\omega)$ for imaginary frequencies given by the plasma model

$$\varepsilon(i\omega) = 1 + \frac{\omega_P^2}{\omega^2}. \tag{9}$$

ω_P is the plasma frequency related to the plasma wavelength λ_P by Eq. (2). For the metals used in recent experiments, the values chosen for the plasma wavelength λ_P will be 107 nm for Al and 136 nm for Cu and Au. These values are in agreement with knowledge from solid state physics [23,24] as well as with the integration of optical data described in detail in Ref. [16]. As already known, the results obtained from the plasma model departs from the more accurate integration of optical data for small distances. In this limit, however, the thermal corrections do not play a significant role. In the present paper we will restrict our attention to the plasma model and discuss the validity of the results obtained in this manner at the end of the next section.

We will also focus the attention on mirrors with a large optical thickness for which the reflection coefficients $r_\perp(i\omega, i\kappa)$ and $r_\parallel(i\omega, i\kappa)$ correspond to a simple vacuum-metal interface. With the plasma model, these coefficients are read as

$$r_\perp = -\frac{\sqrt{\omega_P^2 + c^2\kappa^2} - c\kappa}{\sqrt{\omega_P^2 + c^2\kappa^2} + c\kappa},$$

$$r_\parallel = \frac{\sqrt{\omega_P^2 + c^2\kappa^2} - c\kappa(1 + \omega_P^2/\omega^2)}{\sqrt{\omega_P^2 + c^2\kappa^2} + c\kappa(1 + \omega_P^2/\omega^2)}. \tag{10}$$

For wave vectors $c\kappa$ smaller than ω_P, mirrors may be considered as perfectly reflecting. When converted to the distance domain, this entails that the force approaches the ideal Casimir expression when evaluated at large distances $L \gg \lambda_P$.

The Casimir energy will be obtained from the force by integration over the mirrors relative distance

$$E = \int_L^\infty F(x)\,dx. \qquad (11)$$

As this procedure is performed at constant temperature, the energy thus obtained corresponds to the thermodynamical definition of a free energy. For simplicity we will often use the denomination of an energy. We will define a factor η_E measuring the whole correction of energy due to conductivity and temperature effects with respect to the ideal Casimir energy

$$\eta_E = \frac{E}{E_{\mathrm{Cas}}},$$

$$E_{\mathrm{Cas}} = \frac{\hbar c A \pi^2}{720 L^3}. \qquad (12)$$

The positive value of the energy here means that the Casimir energy is a binding energy while the positive value of the force is associated with an attractive character. We will then define two factors η_E^P and η_E^T associated with each effect taken separately from each other, as in Eq. (4). As already done for the force correction factors in Eq. (5), we will finally evaluate the quantity δ_E which characterizes the degree of validity of the approximation where both effects are evaluated independently from each other. As mentioned in the Introduction, the results obtained for energy allows one to deal with the Casimir force in the plane-sphere geometry when trusting the proximity force theorem.

III. NUMERICAL EVALUATIONS

In the following we present the numerical evaluation of the correction factors of the Casimir force and energy using equations written in the former section. The force correction factor was evaluated for the experimentally relevant distance range of 0.1–10 μm with the help of Eq. (7), supposing explicitly a plasma model for the dielectric function, and the result was normalized by the ideal Casimir force. A double integration over frequencies and wavevectors had to be performed. Due to the cosine dependence in Eq. (7), the integrand turned out to be a highly oscillating function. Hence, the integration required care although it was performed with standard numerical routines. The energy correction factor was then calculated by numerically integrating the force and normalizing by the ideal Casimir energy [see Eq. (12)]. Integration was restricted to a finite interval, the upper limit exceeding at least by a factor of 10^4 the distance at which the energy value was calculated. Extending the integration range by a factor of 100 changed the numerical result by less than 10^{-7}.

The results of the numerical evaluation of η_F are shown as the solid lines in Fig. 1 for Al and for Cu-Au assuming a temperature of $T = 300$ K. They are compared with the force reduction factor η_F^P due to finite conductivity (dashed lines) and the force enhancement factor η_F^T calculated for perfect mirrors at 300 K (dashed-dotted lines).

Figure 2 shows similar results for the factor η_E obtained through numerical evaluation of the Casimir free energy. The
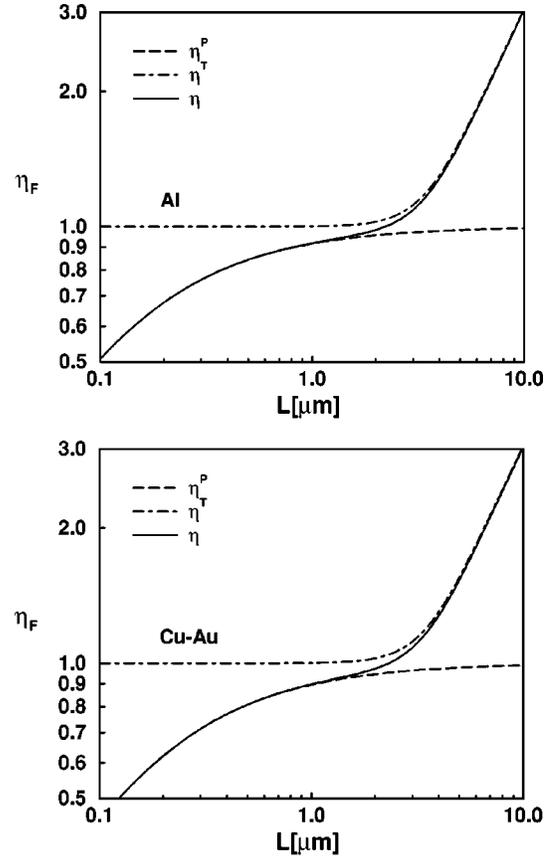

FIG. 1. Force correction factor for Al (upper figure) and Cu and Au (lower graph) as function of the mirrors distance at $T = 300$ K.

shape of the graphs is similar to the ones of the force. However, while finite conductivity corrections are more important for the force, thermal effects have a larger influence on energy.

For the force as well as for the energy, temperature corrections are negligible in the short distance limit while conductivity corrections may be ignored at large distances. The whole correction factor η behaves roughly as the product $\eta^P \eta^T$ of the two correction factors evaluated separately. However, both correction factors are appreciable in the distance range $1-4$ μm in between the two limiting cases. Since this range is important for the comparison between experiments and theory, it is necessary to discuss in a more precise manner how good is the often used approximation which identifies η to the product $\eta^P \eta^T$. In order to assess the quality of this approximation, we have plotted in Fig. 3 the quantities δ_F and δ_E as a function of the distance for Al, Cu-Au, and two additional plasma wavelengths. A value of $\delta = 0$ would signify that the approximation gives an exact estimation of the whole correction. An important outcome of our calculation is that the errors δ_F and δ_E are of the order of 1% for Al and Cu-Au at a temperature of 300 K.

For estimations at the 5% level, the separate calculation of η^P and η^T and the evaluation of η as the product $\eta^P \eta^T$ can therefore be used. However, if a 1% level or a better accuracy is aimed at, this approximation is not sufficient. It should be noticed furthermore that the error increases when
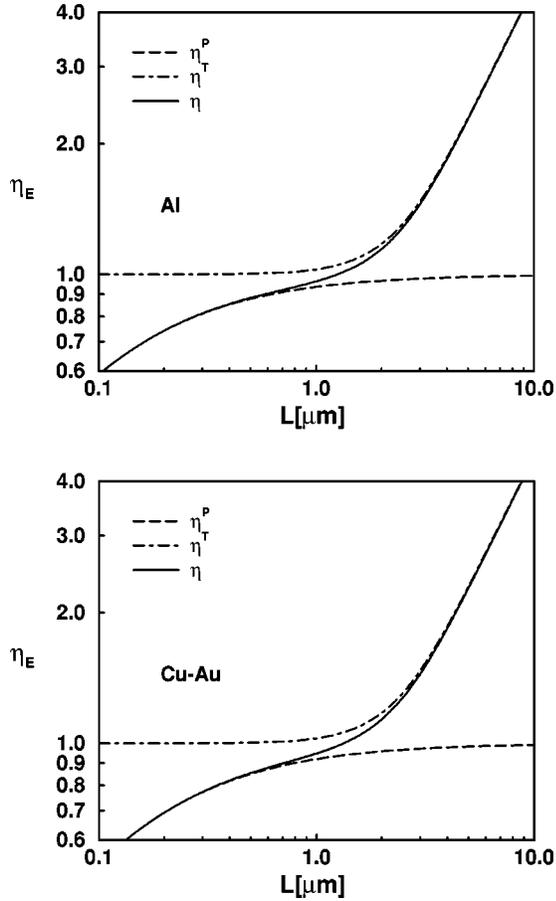
FIG. 2. Energy correction factor for Al (upper figure) and Cu and Au (lower graph) as function of the mirrors distance at $T = 300$ K.

FIG. 3. δ_F (upper graph) and δ_E (lower graph) as a function of the mirrors distance. The results are given for the three metals Al, Cu-Au, and two larger plasma wavelengths.

the temperature or the plasma wavelength are increased. It becomes of the order of 4% for a plasma wavelength of 0.5 μm at 300 K. The sign obtained for δ means that the approximation gives too small values of force and energy.

We want now to emphasize a few points. In order to make the discussion precise, we give numerical values of the correction factors for two experimentally relevant distances, namely, 0.5 and 3 μm. The first distance corresponds to the smallest distance for which the plasma model gives results in correct agreement with the integration of optical data [16]. For this distance, $L = 0.5$ μm, the thermal corrections do not play a significant role ($\eta_F^T = 1.000$; $\eta_E^T = 1.004$).

	Al	Cu$-$Au	
η_F^P	0.843	0.808	
η_F	0.843	0.808	
η_E^P	0.879	0.851	(13)
η_E	0.883	0.855	

At shorter distances the results obtained with the plasma model depart from the values calculated from the integration of optical data by more than 1%. Hence, the values of η_F^P and η_E^P used for distances smaller than 0.5 μm have to take into account the more accurate dielectric function obtained through an integration of optical data [16]. In this short dis-
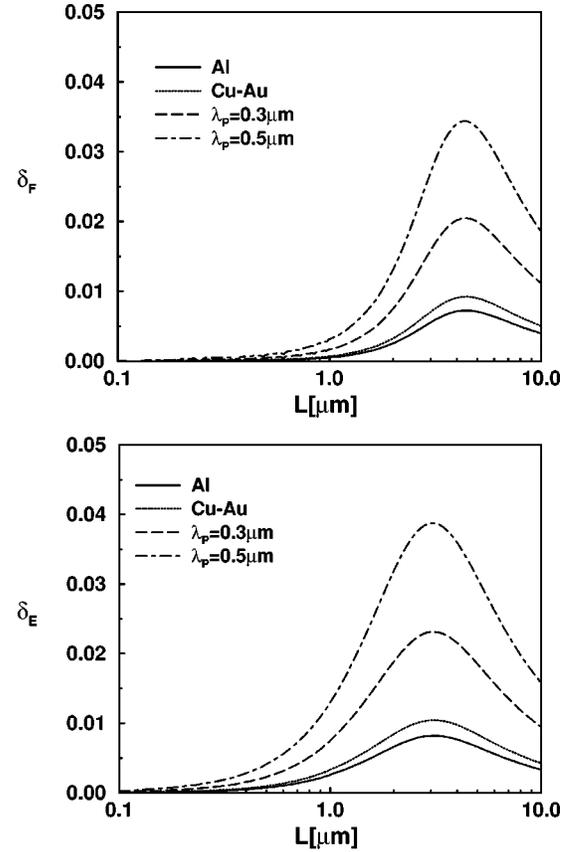
tance range however, the whole correction factors η_F and η_E may be obtained as the products $\eta_F^P \eta_F^T$ and $\eta_E^P \eta_E^T$.

In the long distance range in contrast, the temperature correction becomes predominant. The conductivity correction has still to be accounted for but it may be calculated by using the plasma model. This is illustrated by the correction factors obtained for a distance of $L = 3$ μm ($\eta_F^T = 1.117$; $\eta_E^T = 1.470$):

	Al	Cu$-$Au	
η_F^P	0.971	0.963	
$\eta_F^P \eta_F^T$	1.084	1.076	
η_F	1.090	1.083	(14)
η_E^P	0.978	0.972	
$\eta_E^P \eta_E^T$	1.437	1.429	
η_E	1.449	1.444	

For this distance, all corrections have to be taken into account. The metals cannot be considered as perfect reflectors yet, the temperature corrections are significant and the deviation between the exact correction and the mere product has to be included if a high accuracy is aimed at. This is especially true in the case of Casimir energy.

IV. SCALING LAWS FOR THE DEVIATIONS

An inspection of Fig. 3 shows that the curves corresponding to different plasma wavelengths λ_P have similar shapes
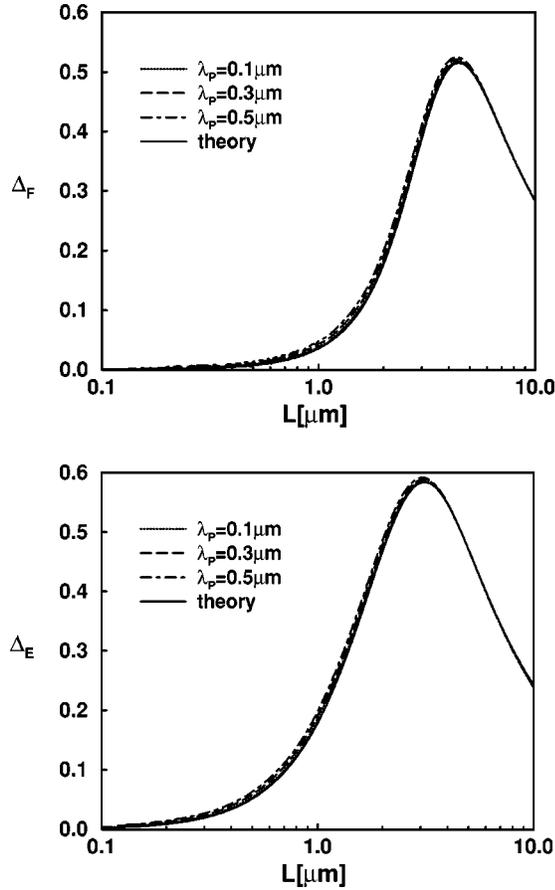
FIG. 4. The deviations are represented for the force (upper graph) and the free energy (lower graph) after the rescaling described by Eq. (15). Different plasma wavelengths lead to nearly identical functions, drawn as dotted, dashed, and dotted-dashed lines. These functions are hardly distinguishable from the solid lines which represent the analytical expressions derived in the next section.

with a maximum which is practically attained for the same distance between the mirrors. The amplitude of the deviations, which is larger for the energy than for the force, is found to vary linearly as a function of the plasma wavelength λ_P.

This scaling property is confirmed by Fig. 4 where we have drawn the deviations after an appropriate rescaling

$$\Delta = \frac{\lambda_T}{\lambda_P}\delta. \qquad (15)$$

The curves obtained for Δ_F and Δ_E for different plasma wavelengths at temperature $T=300$ K are nearly perfectly identical to each other. These curves correspond to values of the plasma wavelength small compared to the thermal wavelength and the scaling law would not be obeyed so well otherwise.

In other words, the deviations δ_F and δ_E are proportional to the factor λ_P/λ_T on one hand, and to the functions Δ_F and Δ_E on the other hand. The latter functions, which no longer

depend on λ_P, provide a simple method for reaching a good accuracy in the theoretical estimation of the whole correction factor

$$\eta = \eta^P \eta^T \left(1 + \frac{\lambda_P}{\lambda_T}\Delta \right). \qquad (16)$$

This method is less direct than the complete numerical integration of the forces which has been performed for obtaining the curves presented in the previous section. But it requires easier computations while nevertheless giving accurate estimations of the correction factors. Typically, the deviation δ with a magnitude of the order of the % may be estimated with a much better precision through the mere inspection of Fig. 4. Alternatively, one may use the analytical expression of the functions Δ presented in the next section and drawn as the solid lines on Fig. 4.

V. ANALYTICAL EXPRESSIONS OF THE DEVIATIONS

The results of numerical integrations presented in the foregoing section have shown that the deviations δ_F and δ_E are proportional to the plasma wavelength λ_P, for plasma wavelengths small compared to the thermal wavelength. In this final section, we explain this scaling law by using a partial analytical integration of the whole correction factors.

To this aim, we write the force correction factor by dividing Eq. (7) by the value of the ideal Casimir force

$$\eta_F = \eta_F^P + (\eta_F^T - 1) + \Delta\eta_F. \qquad (17)$$

The first term in Eq. (17) corresponds to the vacuum contribution (8)

$$\eta_F^P = \frac{120L^4}{\pi^4}\int_0^\infty d\kappa\,\kappa^3 \int_0^1 dy\,f \qquad (18)$$

with f still given by Eq. (6). A dimensionless frequency $y = \omega/c\kappa$ measured with respect to the wave vector has been introduced. Note also that the wavevector κ is involved through the dimensionless quantity κL, except in the expressions of reflection coefficients. In Eq. (18), the integration over y may be performed analytically (see Appendix). At long distances, η_F^P tends to the limit of perfect reflection with a known correction [15]

$$L \gg \lambda_P \rightarrow \eta_F^P = 1 - \frac{8}{3\pi}\frac{\lambda_P}{L} + \cdots. \qquad (19)$$

This expansion has been the subject of a number of papers and it has been used to propose interpolation formulas [20,21]. However such a series expansion can hardly reproduce the behavior at small distances where η_F^P varies as L/λ_P, which just means that the conductivity effect is not a small perturbation at short distances (see the Appendix).

Coming back to the whole expression (17) of the force correction factor, it remains to discuss the thermal contributions, that is, the second and third terms. These two terms come from the contributions $m \neq 0$ to Eq. (7). The opposite values of m give equal contributions and they have been

gathered. The thermal contributions have been split in to two parts, the second and third terms in Eq. (17), which correspond respectively to the limit of perfect mirrors on one hand,

$$\eta_F^T - 1 = \frac{240L^4}{\pi^4} \sum_{m=1}^{\infty} \int_0^{\infty} d\kappa \, \kappa^3 \int_0^1 dy \, \cos(my\kappa\lambda_T) f_1,$$

(20)

$$f_1 = \frac{2}{e^{2\kappa L} - 1}$$

and the remainder, on the other hand,

$$\Delta \eta_F = \frac{240L^4}{\pi^4} \sum_{m=1}^{\infty} \int_{\omega/c}^{\infty} d\kappa \, \kappa^3 \int_0^1 dy \, \cos(my\kappa\lambda_T) \Delta f,$$

$$\Delta f = f - f_1$$

(21)

$$= -\frac{e^{2\kappa L}}{e^{2\kappa L} - 1} \left(\frac{1 - r_\perp^2}{e^{2\kappa L} - r_\perp^2} + \frac{1 - r_\parallel^2}{e^{2\kappa L} - r_\parallel^2} \right).$$

The contribution (20) has been denoted ($\eta_F^T - 1$) with η_F^T the correction factor obtained for perfect mirrors at a nonzero temperature. For this term the integration over y is trivial and the integration over κ may be performed analytically, leading to the known expression [13–15]

$$\eta_F^T - 1 = \frac{480L^4}{\pi^4} \sum_{m=1}^{\infty} \int_0^{\infty} d\kappa \, \frac{\kappa^2}{e^{2\kappa L} - 1} \frac{\sin(m\kappa\lambda_T)}{m\lambda_T}$$

$$= 30 \sum_{m=1}^{\infty} \left(\frac{1}{(\alpha m)^4} - \frac{\cosh(\alpha m)}{\alpha m \sinh^3(\alpha m)} \right),$$

(22)

$$\alpha = \frac{\pi\lambda_T}{2L}.$$

To obtain the overall correction factor (17) it now remains to evaluate the last expression (21). This can be done numerically, thus leading to the same results as in the previous section since no approximation has been performed up to now. But the results of the previous section suggest that we may obtain an accurate estimation of this term through an expansion in powers of λ_P. The plasma wavelength λ_P is indeed much smaller than the thermal wavelength λ_T in all experimental situations studied up to now. Also, the deviation studied in the foregoing section is appreciable only for distances L much larger than λ_P. Hence an accurate description of the deviation factor should be obtained by evaluating $\Delta \eta_F$ at the first order in λ_P.

This first order term is easily deduced from Eqs. (10), (21)

$$\Delta f \approx -\frac{e^{2\kappa L}}{(e^{2\kappa L} - 1)^2} (1 - r_\perp^2 + 1 - r_\parallel^2),$$

(23)

$$\approx -\frac{e^{2\kappa L}}{(e^{2\kappa L} - 1)^2} \frac{2\kappa\lambda_P}{\pi} (1 + y^2).$$

It is proportional to λ_P and to a function ϕ_F which does no longer depend on λ_P

$$\Delta \eta_F \simeq \frac{\lambda_P}{L} \phi_F,$$

$$\phi_F = \frac{15}{\pi} \sum_{m=1}^{\infty} \left(\frac{\cosh(\alpha m)}{(\alpha m)^3 \sinh(\alpha m)} + \frac{1}{(\alpha m)^2 \sinh^2(\alpha m)} \right.$$

$$\left. + \frac{4 \cosh(\alpha m)}{\alpha m \sinh^3(\alpha m)} - \frac{2 + 4 \cosh^2(\alpha m)}{\sinh^4(\alpha m)} \right).$$

(24)

Collecting the results obtained up to now, we get an estimation of the force correction factor η_F valid in the long distance range $L \gg \lambda_P$

$$\eta_F = \eta_F^P \eta_F^T + (1 - \eta_F^P)(\eta_F^T - 1) + \Delta \eta_F,$$

$$\simeq \eta_F^P \eta_F^T + \frac{8}{3\pi} \frac{\lambda_P}{L} (\eta_F^T - 1) + \frac{\lambda_P}{L} \phi_F.$$

(25)

Coming back to the notations of the previous section, this result is equivalent to the following expression for the function Δ_F:

$$\Delta_F = \frac{8}{3\pi} \frac{\lambda_T}{L} \frac{\eta_F^T - 1}{\eta_F^T} + \frac{\lambda_T}{L} \frac{\phi_F}{\eta_F^T}.$$

(26)

This function is plotted as the solid line on Fig. 4 and it is found to fit well the results of the complete numerical integration presented in the previous section.

Similar manipulations can be done for evaluating correction factors for the Casimir free energy. We give below the main results, that is the thermal correction factor evaluated for perfect mirrors

$$\eta_E^T - 1 = 45 \sum_{m=1}^{\infty} \left(-\frac{2}{(\alpha m)^4} + \frac{1}{(\alpha m)^3 \tanh(\alpha m)} \right.$$

$$\left. + \frac{1}{(\alpha m)^2 \sinh^2(\alpha m)} \right)$$

(27)

and the first order correction

$$\Delta \eta_E \simeq \frac{\lambda_P}{L} \phi_E,$$

$$\phi_E = \frac{45}{\pi} \sum_{m=1}^{\infty} \left(-\frac{4}{(\alpha m)^4} + \frac{1}{(\alpha m)^3 \tanh(\alpha m)} \right.$$

$$\left. + \frac{1}{(\alpha m)^2 \sinh^2(\alpha m)} + \frac{2\cosh(\alpha m)}{\alpha m \sinh^3(\alpha m)} \right).$$

(28)

Since the long distance expansion of η_E^P up to first order in the plasma wavelength is given by

$$L \gg \lambda_P \rightarrow \eta_E^P = 1 - \frac{2}{\pi} \frac{\lambda_P}{L} + \cdots$$

(29)

we deduce the function Δ_E

$$\Delta_E = \frac{2}{\pi} \frac{\lambda_T}{L} \frac{\eta_E^T - 1}{\eta_E^T} + \frac{\lambda_T}{L} \frac{\phi_E}{\eta_E^T}. \tag{30}$$

This function is plotted as the solid line on the second graph of Fig. 4 and also found to fit well the results of the numerical integration.

VI. SUMMARY

In the present paper, we have given an accurate evaluation of the Casimir force and Casimir free energy between two plane metallic mirrors, taking into account conductivity and temperature corrections at the same time. The whole corrections with respect to the ideal Casimir formulas, corresponding to perfect mirrors in vacuum, have been characterized by factors η_F for the force and η_E for the energy. These factors have been computed through a numerical evaluation of the integral formulas. They have also been given a simplified form as a product of three terms, namely, the reduction factor associated with conductivity at null temperature, the increase factor associated with temperature for perfect mirrors, and a further deviation factor measuring a kind of interplay between the two effects. This last factor turns out to lie in the 1% range for metals used in the recent experiments performed at ambient temperature. Hence the conductivity and temperature corrections may be treated independently from each other and simply multiplied for theoretical estimations above this accuracy level.

However, when accurate comparisons between experimental and theoretical values of the Casimir force are aimed at, the deviation factor has to be taken into account in theoretical estimations. The deviation factor is appreciable for distances greater than the plasma wavelength λ_P but smaller or of the order of the thermal wavelength λ_T. We have used this property to derive a scaling law of the deviation factor. This law allows one to obtain a simple but accurate estimation of the Casimir force and free energy through a mere inspection of Fig. 4. Alternatively one can use analytical expressions which have been obtained through a first order expansion in λ_P of the thermal contributions to Casimir forces and fit well the results of complete numerical integration.

We have represented the optical properties of metals by the plasma model. This model does not lead to reliable estimations of the forces at small distances but this deficiency may be corrected by using the real dielectric function of the metals. This does not affect the discussion of the present paper, except for the fact that the pure conductivity effect has to be computed through an integration of optical data for distances smaller than 0.5 μm. Finally surface roughness corrections, which have not been considered in the present paper, are expected to play a significant role in theory-experiment comparisons in the short distance range.

APPENDIX: THE VACUUM CONTRIBUTION

In the present appendix, we give further analytical expressions for the correction factor η_F^P due to conductivity, calculated with the plasma model for a null temperature. Introducing the notations

$$\rho = \frac{\sqrt{\omega_P^2 + c^2 \kappa^2} - c\kappa}{\sqrt{\omega_P^2 + c^2 \kappa^2} + c\kappa}, \qquad y = \frac{\omega}{c\kappa}$$

we rewrite the reflection coefficients (10)

$$r_\perp = -\rho, \qquad r_\| = \rho \frac{y^2(1-\rho) - 2}{y^2(1-\rho) + 2\rho}.$$

In this case one integration may be performed analytically in Eq. (18)

$$\eta_F^P = \frac{120 L^4}{\pi^4} \int_0^\infty d\kappa \kappa^3 \frac{2\rho^2 + \rho e^{\kappa L} g}{e^{2\kappa L} - \rho^2},$$

$$g = \frac{1 + a_-^2}{a_-} \arctan \frac{1}{a_-} - \frac{1 + a_+^2}{a_+} \arctan \frac{1}{a_+},$$

$$a_\pm = \sqrt{\frac{e^{\kappa L} \pm \rho}{e^{\kappa L} \mp \rho} \frac{1 + \rho}{1 - \rho} - 1}.$$

At the large distance limit, η_F^P tends to unity, that is the value obtained for perfect reflectors. At the small distance limit, η_F^P is found to vary as [16]

$$\eta_F^P \simeq \alpha \frac{L}{\lambda_P}, \qquad L \ll \lambda_P$$

$$\alpha = \frac{30}{\pi^2} \int_0^\infty dK e^{-3K/4} \left(\frac{K^2}{\sqrt{\sinh(K/2)}} \right.$$

$$\left. - \frac{K^2}{\sqrt{\cosh(K/2)}} \right)$$

$$\simeq 1.193.$$

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[2] B.V. Deriagin and I.I. Abrikosova, Sov. Phys. JETP **3**, 819 (1957).
[3] M.J. Sparnaay, Physica (Amsterdam) **XXIV**, 751 (1958); W. Black, J.G.V. De Jongh, J.Th.G. Overbeek, and M.J. Sparnaay, Trans. Faraday Soc. **56**, 1597 (1960).
[4] D. Tabor and R.H.S. Winterton, Nature (London) **219**, 1120 (1968).

[5] E.S. Sabisky and C.H. Anderson, Phys. Rev. A **7**, 790 (1973).

[6] S.K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997); **81**, 5475 (E) (1998).

[7] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998).

[8] A. Roy, C.Y. Lin, and U. Mohideen, Phys. Rev. D **60**, 111101 (1999).

[9] E. Fischbach and C. Talmadge, *The Search for Non-Newtonian Gravity* (AIP, New York, 1998), and references therein; E. Fischbach and D.E. Krause, Phys. Rev. Lett. **82**, 4753 (1999).

[10] G. Carugno, Z. Fontana, R. Onofrio, and C. Rizzo, Phys. Rev. D **55**, 6591 (1997).

[11] M. Bordag, B. Geyer, G.L. Klimchitskaya, and V.M. Mostepanenko, Phys. Rev. D **60**, 055004 (1999).

[12] E.M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956); E.M. Lifshitz and L.P. Pitaevskii, *Landau and Lifshitz Course of Theoretical Physics: Statistical Physics* (Butterworth-Heinemann, Oxford, 1980), Pt. 2, Chap. VIII.

[13] J. Mehra, Physica (Amsterdam) **57**, 147 (1967).

[14] L.S. Brown and G.J. Maclay, Phys. Rev. **184**, 1272 (1969).

[15] J. Schwinger, L.L. de RaadJr., and K.A. Milton, Ann. Phys. (N.Y.) **115**, 1 (1978).

[16] A. Lambrecht and S. Reynaud, Eur. Phys. J. **D8**, 309 (2000).

[17] B.V. Deriagin, I.I. Abrikosova, and E.M. Lifshitz, Q. Rev. Chem. Soc. **10**, 295 (1968).

[18] J. Blocki, J. Randrup, W.J. Swiatecki, and C.F. Tsang, Ann. Phys. (N.Y.) **105**, 427 (1977).

[19] V.M. Mostepanenko and N.N. Trunov, Sov. J. Nucl. Phys. **42**, 812 (1985).

[20] V.B. Bezerra, G.L. Klimchitskaya, and C. Romero, Mod. Phys. Lett. A **12**, 2613 (1997).

[21] G.L. Klimchitskaya, A. Roy, U. Mohideen and V.M. Mostepanenko, Phys. Rev. A **60**, 3487 (1999).

[22] M.T. Jaekel and S. Reynaud, J. Phys. I **1**, 1395 (1991).

[23] L.G. Schulz, Philos. Mag. Suppl. **6**, 102 (1957).

[24] H. Ehrenreich and H.R. Philipp, Phys. Rev. **128**, 1622 (1962).