Quantum properties of classical Fisher information

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The Fisher information of a quantum observable is shown to be proportional to both (i) the difference of a quantum and a classical variance, thus providing a measure of nonclassicality; and (ii) the rate of entropy increase under Gaussian diffusion, thus providing a measure of robustness. The joint nonclassicality of position and momentum observables is shown to be complementary to their joint robustness in an exact sense.

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I. INTRODUCTION

Fisher information was originally introduced by Fisher in 1925 [1], as a measure of ''intrinsic accuracy'' in statistical estimation theory. It provides in particular a bound on the degree to which members of a family of probability distributions can be distinguished [2]. Quantum generalizations of Fisher information may be given, providing corresponding bounds on the degree to which members of a family of quantum states can be distinguished by measurement [3]. However, in both the classical and quantum contexts, the bounds are typically not achievable. Hence the primary application of the Fisher information has been in providing unsharp statistical inequalities.

In this paper two curious connections between *classical* Fisher information and *quantum* systems will be pointed out, which involve exact *equalities*. First, it is shown that the classical Fisher information of a quantum observable is proportional to the difference between the quantum variance and the classical variance of the conjugate observable. Thus it is a measure of the *nonclassicality* of the conjugate observable. Second, it is shown that the classical Fisher information is proportional to the rate of entropy increase of the observable when the quantum system is subjected to Gaussian diffusion, i.e., Brownian motion. Hence it is also a measure of the *robustness* of the observable with respect to noise. The results further lead to natural measures of joint nonclassicality and joint robustness for quantum states, which are inversely related to each other.

Fisher information is defined in the following section, and its relation to statistical measures of uncertainty briefly reviewed. In Sec. III the role of Fisher information as a measure of nonclassicality is developed and explored, based on a natural decomposition of each quantum observable into a ''classical'' and a ''nonclassical'' component. The ''joint nonclassicality'' of a quantum system is defined, and it is conjectured that it has a nontrivial lower bound for pure quantum states, i.e., such states are inherently nonclassical. In Sec. IV the connection between Fisher information and quantum diffusion is demonstrated, essentially generalizing de Bruijn's identity for classical systems [4–6]. It follows that the robustness of a quantum system with respect to noise is inversely proportional to its degree of nonclassicality, i.e., the more robust the state is with respect to noise, the more classical it is. Generalizations to higher dimensions are briefly discussed in Sec. V, and conclusions given in Sec. VI.

II. FISHER INFORMATION AND FISHER LENGTH

The classical Fisher information associated with translations of a one-dimensional observable X with corresponding probability density $p(x)$ is given by [1,2]

$$F_X = \int dx\, p(x)[d \ln p(x)/dx]^2 > 0. \tag{1}$$

The primary application of this quantity in classical estimation theory is the lower bound

$$\mathrm{Var}\, X \geq F_X^{-1} \tag{2}$$

for the variance of X, known as the Cramer-Rao inequality [2].

One may also define a corresponding Fisher *length* for X, by

$$\delta X = F_X^{-1/2}. \tag{3}$$

From Eq. (1) δX is seen to quantify the length scale over which $p(x)$ [or more precisely $\ln p(x)$] varies appreciably. The Cramer-Rao inequality Eq. (2) may then be rewritten as the simple length inequality

$$\Delta X \geq \delta X \tag{4}$$

for the root mean square deviation ΔX of X.

It is worth noting that Eq. (4) can be derived via the properties of a length measure of fundamental geometric significance, the *ensemble length* of X, given by the exponential of the entropy of $p(x)$:

$$L_X = \exp\left[- \int dx\, p(x) \ln p(x) \right]. \tag{5}$$

L_X is the unique measure of uncertainty that satisfies several basic geometric properties expected of a ''length'' [7], and one has

$$\sqrt{2\pi e}\, \Delta X \geq L_X \geq \sqrt{2\pi e}\, \delta X. \tag{6}$$

The first inequality in Eq. (6) corresponds to the well known property that entropy is maximized for a fixed value of ΔX by a Gaussian distribution. The second inequality may be derived from either an identity of de Bruijn [5,6] (see also Sec. IV), or from a logarithmic Sobelov inequality [8], and is also saturated by Gaussian distributions. The Cramer-Rao inequality Eq. (4) immediately follows from Eq. (6).

III. MEASURE OF NONCLASSICALITY

A. Position

Consider a quantum system described by wave function $\psi(x)$. The position probability density is then $p(x) = |\psi(x)|^2$, and hence from Eq. (1) the corresponding Fisher information is

$$F_X = \int dx\, |\psi(x)|^2 [\psi'(x)/\psi(x) + \psi^{*\prime}(x)/\psi^*(x)]^2$$

$$= 4 \int dx\, \psi^{*\prime}(x)\psi'(x) + \int dx\, |\psi(x)|^2 [\psi'(x)/\psi(x)$$

$$- \psi^{*\prime}(x)/\psi^*(x)]^2$$

$$= (4/\hbar)^2 [\langle P^2 \rangle_\psi - \langle P_{cl}^2 \rangle_\psi], \tag{7}$$

where P denotes the momentum observable conjugate to X, and P_{cl} is a *classical* momentum observable corresponding to the state ψ, given by the function

$$P_{cl}(x) = (\hbar/2i)[\psi'(x)/\psi(x) - \psi^{*\prime}(x)/\psi^*(x)]$$

$$= \hbar[\arg\psi(x)]'. \tag{8}$$

The identification of the observable P_{cl} with a classical momentum is strongly supported on two grounds. First, the probability density $|\psi(x)|^2$ is well known to satisfy the classical continuity equation [9]

$$\partial|\psi(x)|^2/\partial t + (\partial/\partial x)[|\psi(x)|^2 m^{-1} P_{cl}(x)] = 0, \tag{9}$$

as a direct consequence of the Schrödinger equation. Thus $m^{-1}P_{cl}(x)$ is the local velocity of probability flow in position space, implying $P_{cl}(x)$ may be interpreted as a classical momentum of a particle at position x, where the probability of finding the particle at x is $|\psi(x)|^2$. Second, one has the identity

$$\langle P \rangle_\psi = \langle P_{cl} \rangle_\psi \tag{10}$$

following from Eq. (8) (using integration by parts). Hence the expectation values of the observables P and P_{cl} are equal for all wave functions.

Now, given the quantum and classical momentum observables P and P_{cl}, it is natural to define the *nonclassical* momentum of the system by $P_{nc} = P - P_{cl}$. Thus the momentum P separates into a classical and a nonclassical contribution. From Eq. (8) one has

$$\langle PP_{cl} + P_{cl}P \rangle_\psi = \int dx\, (P\psi)^*(P_{cl}\psi) + \int dx\, (P_{cl}\psi)^*(P\psi)$$

$$= (\hbar/i) \int dx\, [\psi'(x)\psi^*(x)$$

$$- \psi^{*\prime}(x)\psi(x)]P_{cl}(x)$$

$$= 2\langle P_{cl}^2 \rangle_\psi,$$

and so from Eq. (10)

$$\mathrm{Var}_\psi P = \mathrm{Var}_\psi P_{cl} + \mathrm{Var}_\psi P_{nc}. \tag{11}$$

Hence the classical and nonclassical contributions are uncorrelated in variance.

The main result of this section is a simple relationship between nonclassicality and Fisher information. In particular, from Eqs. (7), (10), and (11) one has

$$F_X = (4/\hbar^2)(\Delta P_{nc})^2. \tag{12}$$

The position Fisher information is therefore proportional to the nonclassical variance of the conjugate momentum.

A *direct* measure of the nonclassicality of the momentum, representing the size of nonclassical momentum fluctuations, is given by the root mean square deviation ΔP_{nc}. From Eq. (3) one may equivalently write Eq. (12) as

$$\delta X \Delta P_{nc} = \hbar/2. \tag{13}$$

Thus *the Fisher length of position is inversely proportional to the nonclassicality of momentum*. Equation (13) is rather similar in form to the Heisenberg uncertainty relation, and indeed the latter may be immediately derived from it. In particular, one has

$$\Delta X \Delta P \geq \delta X \Delta P \geq \delta X \Delta P_{nc} = \hbar/2, \tag{14}$$

where the first inequality follows from Eq. (4), and the second from Eq. (11). The inequality $\delta X \Delta P \geq \hbar/2$ implicit in Eq. (14) was previously proved by Stam [4,6].

Note that the Fisher length is always finite from Eqs. (1) and (3), and hence the momentum nonclassicality is never zero. Further, from Eq. (11), the momentum nonclassicality is maximum, for a fixed value of ΔP, when the variance of P_{cl} vanishes, i.e., when P_{cl} is a constant. From Eq. (8) this occurs when the phase of $\psi(x)$ is linear in x. Thus

$$\Delta P_{nc} = \Delta P \quad \text{iff } \arg\psi(x) = \alpha + k_0 x, \tag{15}$$

for constants α and k_0.

B. Momentum

One may, in direct analogy with Eqs. (11), (12), and (13), obtain the conjugate equalities

$$F_P = (4/\hbar^2)(\Delta X_{nc})^2 = (4/\hbar^2)[\mathrm{Var}_\psi X - \mathrm{Var}_\psi X_{cl}], \tag{16}$$

$$\Delta X_{nc} \delta P = \hbar/2, \tag{17}$$

for the Fisher information F_P and the Fisher length δP of the momentum observable P conjugate to X. Here $X_{nc} = X - X_{cl}$, and

$$X_{cl}(p) = (i\hbar/2)[\phi'(p)/\phi(p) - \phi^{*\prime}(p)/\phi^*(p)] \tag{18}$$

is a classical position observable corresponding to state ψ, where $\phi(p)$ denotes the momentum wave function of the system. Thus F_P and δP are related to the nonclassicality of the position.

The identification of X_{cl} as a classical position observable has a similar justification to the analogous interpretation of P_{cl}. In particular, conservation of momentum probability $|\phi(p)|^2$ implies a continuity equation of the form

$$\partial|\phi(p)|^2/\partial t+(\partial/\partial p)[|\phi(p)|^2 F(p)]=0, \qquad (19)$$

where $F(p)$ is the momentum flow, i.e., *force*, associated with momentum p. If the system is subject to a potential energy $V(x)$, then multiplying the Schrödinger equation in the momentum representation by $\phi^*(p)$, taking the imaginary part, and expanding $V(x)$ in a Taylor series, one finds

$$F(p)=-(\partial/\partial x)V[X_{cl}(p)]+O(\hbar^2). \qquad (20)$$

Thus the observable X_{cl} corresponds to the classical force $-V'(x)$ associated with the system, to first order in $\hbar$. One has also an equality analogous to Eq. (10), i.e.,

$$\langle X\rangle_\psi=\langle X_{cl}\rangle_\psi. \qquad (21)$$

C. Joint nonclassicality

A natural (dimensionless) measure of *joint* nonclassicality for a quantum state ψ may now be defined, as

$$J_{nc}=\Delta X_{nc}\Delta P_{nc}/(\hbar/2). \qquad (22)$$

From Eqs. (13) and (17) one then has

$$J_{nc}=(\hbar/2)(\delta X\delta P)^{-1}, \qquad (23)$$

i.e., *the joint nonclassicality is inversely proportional to the product of the position and momentum Fisher lengths*. Recalling that equality holds throughout Eq. (6) for Gaussian distributions, it follows that $J_{nc}=1$ for minimum uncertainty states.

It is of interest to ask whether there is some maximum *upper* bound for joint nonclassicality set by quantum theory, corresponding to a lower bound for the product $\delta X\delta P$. The answer is in the negative; in particular, there is no direct analog of the Heisenberg uncertainty relation Eq. (14) for Fisher lengths. As an example, consider the nth energy eigenstate of a one-dimensional harmonic oscillator. For this case the momentum and position wave functions are both real up to a constant phase factor. Hence from Eq. (15) and its analog for the momentum wave function,

$$\delta X\delta P=(\hbar^2/4)(\Delta X\Delta P)^{-1}=\hbar/(4n+2), \qquad (24)$$

which becomes arbitrarily small as $n\rightarrow\infty$. Thus the joint nonclassicality becomes arbitrarily large with increasing n. Note that the first equality in Eq. (24) holds whenever the position wave function is (up to a linear phase factor and a translation) a symmetric or antisymmetric real function.

It is seen that the coherent states of a harmonic oscillator, being minimum uncertainty states, have a relatively small

nonclassicality $J_{nc}=1$, in comparison to $J_{nc}=2n+1$ for the number states. This is in accordance with the picture that the relatively localized coherent states correspond to the classical harmonic motion of a particle [10]. It should be noted that the Wentzel-Kramers-Brillouin (WKB) approximation, with its semiclassical interpretation that high energy quantum states correspond to classical regions of motion, does not imply that such states (e.g., number states) are themselves classical in any sense. In particular, all quantum transition probabilities are correctly asymptotically reproduced for such states [11], and hence the WKB approximation contains all quantum properties in the high energy limit.

The definition of joint nonclassicality may be extended to mixed states, represented by density operators, via Eqs. (1), (3), and (23). For such states J_{nc} can be arbitrarily small (e.g., consider thermal states of the harmonic oscillator, which have Gaussian position and momentum distributions, in the high temperature limit). This is reasonable, as one expects certain mixed states, such as thermal states, to be equivalent to classical states in appropriate limits. However, it would be of interest to determine whether there is a *nonzero* minimum value for the joint nonclassicality of *pure* states. This would correspond to the idea that there is necessarily something inherently nonclassical about a pure quantum state. The general uncertainty relation $\Delta A\Delta B\geq|\langle[A,B]\rangle_\psi|/2$ (Ref. [9]) implies via Eq. (22) that

$$J_{nc}\geq|1+(i/\hbar)\langle[P_{cl},X_{cl}]\rangle_\psi|, \qquad (25)$$

suggesting the conjecture $J_{nc}\geq1$ for pure states.

D. Kinetic energies and quantum potentials

From Eq. (7) it is seen that the position Fisher information F_X is proportional to the difference of a quantum and a classical kinetic energy. Thus the average energy of a quantum particle of mass m is increased relative to the corresponding average classical energy by the additional amount

$$E_F=\hbar^2 F_X/(8m). \qquad (26)$$

Now, it is known from the de Broglie–Bohm approach to quantum mechanics that there is an *exact* correspondence between a quantum particle and an ensemble of classical particles, where the latter has probability density $p(x)=|\psi(x)|^2$, momentum $P_{cl}(x)$ associated with position x, and is subjected to a quantum potential $Q(x)$ in addition to the classical potential $V(x)$, where [12]

$$Q(x)=\hbar^2/(8m)[p'(x)^2/p(x)^2-2p''(x)/p(x)]. \qquad (27)$$

The average energy increase due to $Q(x)$ is therefore $\langle Q(x)\rangle_\psi$, and hence from Eq. (26) one has

$$\langle Q(x)\rangle_\psi=\hbar^2 F_X/(8m). \qquad (28)$$

Thus F_X is proportional to the average value of the quantum potential, providing another link between Fisher information and nonclassicality. It is tempting, on the basis of this result,

to regard the first term in Eq. (27) as a nonclassical kinetic energy, and the second term as a nonclassical dispersive potential.

Equation (28) was recently derived by Reginatto [13] based on an even stronger connection between the quantum potential and Fisher information. In particular, consider the variation of F_X in Eq. (1) with respect to the probability density $p(x)$. One then finds, using integration by parts, the remarkable relation

$$\delta F_X = (8m/\hbar^2) \int dx\, Q(x)\, \delta p. \quad (29)$$

This result is the basis of a new approach to quantum mechanics, where a Fisher information term and a classical hydrodynamical action term are added, representing "epistemological" and "ontological" contributions, respectively, to the total action [13,14]. This approach is to be distinguished from that of Frieden [15], where essentially a *generalized* Fisher information is defined for wave functions, proportional to the quantum kinetic energy $\langle P^2/(2m)\rangle_\psi$.

IV. MEASURE OF ROBUSTNESS

A. de Bruijn's identity

Consider the entropy increase of an observable X subjected to Gaussian diffusion, i.e., Brownian motion. The probability density $p_t(x)$ satisfies the diffusion equation

$$\dot{p}_t = \gamma p_t'' \quad (30)$$

for some diffusion rate constant γ, with solution

$$p_t(x) = (\pi \gamma t)^{-1/2} \int dy\, p_0(x-y) \exp[-y^2/(\gamma t)], \quad (31)$$

and hence the initial density is convolved with a Gaussian of variance $\gamma t/2$. The rate of entropy increase at time t follows as

$$\dot{S}_X(t) = -\int dx [1 + \ln p_t(x)] \dot{p}_t(x) = \gamma F_X(t), \quad (32)$$

where $F_X(t)$ is the Fisher information at time t and the second equality follows from Eqs. (1) and (30), using integration by parts.

This link between Fisher information and entropy increase is known as *de Bruijn's identity* [4–6]. Since an observable that is robust to noise will have a small rate of entropy increase, and vice versa, it follows that $F_X = F_X(0)$ is inversely related to the *robustness* of X with respect to the onset of Gaussian noise.

The application of de Bruijn's identity to quantum systems is straightforward. Moreover, even though the position and momentum observables are complementary, and hence cannot be specified simultaneously, it turns out that these observables behave independently when the system is subjected to simultaneous position and momentum diffusion.

Hence the quantum analogs of Eq. (32) for position and momentum can be derived from a *single* quantum diffusion process.

In particular, the diffusion equation for a classical phase space ensemble $\rho(x,p)$ is

$$\dot{\rho} = \gamma(\partial^2/\partial x^2)\rho + \sigma(\partial^2/\partial p^2)\rho = \gamma\{p,\{p,\rho\}\} + \sigma\{x,\{x,\rho\}\},$$

where γ and σ are rate constants and $\{\}$ is the Poisson bracket. Under the Dirac correspondence $\{\} \rightarrow (i\hbar)^{-1}[\ ]$ one thus obtains the quantum diffusion equation

$$\dot{\rho} = -(\gamma/\hbar^2)[P,[P,\rho]] - (\sigma/\hbar^2)[X,[X,\rho]], \quad (33)$$

where ρ is the density operator describing the system [16].

The evolution of the position probability density $p_t(x) = \langle x|\rho|x\rangle$ is therefore given by

$$\dot{p}_t(x) = -(\gamma/\hbar^2)\langle x|[P,[P,\rho]]|x\rangle - (\sigma/\hbar^2)\langle x|[X,[X,\rho]]|x\rangle. \quad (34)$$

The second term on the right vanishes since $X|x\rangle = x|x\rangle$ by definition, while the first term reduces to $\gamma p_t''(x)$ using the relation $\langle x|[P,A]|x\rangle = (\hbar/i)d\langle x|A|x\rangle/dx$ (derived by expanding $|x\rangle$ in momentum eigenstates). Thus $p_t(x)$ satisfies the diffusion equation Eq. (30). A similar result obtains for the evolution of the momentum density, and hence from de Bruijn's identity Eq. (32) one has

$$F_X = \gamma \dot{S}_X(0), \quad F_P = \sigma \dot{S}_P(0). \quad (35)$$

Thus, the position and momentum Fisher information of a quantum system are inversely related to the robustness of the corresponding observables, with respect to the onset of quantum phase space diffusion as per Eq. (33).

Noting Eqs. (12), (16), and (35), the robustness of the position is high (small F_X) when the nonclassicality of the momentum is low, and vice versa. Thus *the more classical an observable is, the more robust the conjugate observable is with respect to noise.*

B. Joint robustness

From Eqs. (3) and (35) a natural (dimensionless) measure of *joint* robustness for a quantum system is given by

$$J_r = \delta X\, \delta P/(\hbar/2). \quad (36)$$

In particular, J_r is relatively *large* when the position and momentum entropies increase *slowly* under the onset of phase space diffusion, and vice versa. Note from Eq. (24) that the joint robustness can be arbitrarily small. Conversely, thermal states of the harmonic oscillator have arbitrarily large robustness in the high temperature limit.

Comparison of Eqs. (23) and (36) shows that

$$J_{nc} J_r = 1, \quad (37)$$

i.e., *the nonclassicality and robustness of a quantum state are inversely proportional.* This is in accord with other re-

sults in the literature suggesting that classical behavior is associated with robustness to noise [17].

V. HIGHER DIMENSIONS

In more than one dimension the Fisher information generalizes to a matrix. However, the results of the previous sections can and do generalize in different ways, to relations involving either the matrix, its trace, or its determinant (which are all equivalent in one dimension). It is therefore useful to indicate explicitly the higher-dimensional analogs of various results.

First, for an n-dimensional observable $\mathbf{X}$ with probability density $p(\mathbf{x})$, the analog of the Fisher information in Eq. (1) is the (positive definite) Fisher matrix [2]

$$F_{\mathbf{X}} = \int d^n \mathbf{x} \, p(\mathbf{x}) [\nabla \ln p(\mathbf{x})][\nabla \ln p(\mathbf{x})]^T, \qquad (38)$$

where ∇ is the gradient operator and T denotes the vector transpose. The Cramer-Rao inequality Eq. (2) then generalizes to [2]

$$\text{Cov}(\mathbf{X}) = \langle \mathbf{X}\mathbf{X}^T \rangle - \langle \mathbf{X} \rangle \langle \mathbf{X}^T \rangle \geq F_{\mathbf{X}}^{-1} \qquad (39)$$

for the covariance matrix of $\mathbf{X}$, and the inequality chain in Eq. (6) becomes

$$(\Delta R/\sqrt{n})^n \geq \Delta V \geq (2\pi e)^{-n/2} V_X \geq \delta V \geq (\sqrt{n}\,\delta R)^n, \quad (40)$$

where ΔR and δR are the covariance radius $[\text{tr}\,\text{Cov}(\mathbf{X})]^{1/2}$ and Fisher radius $[\text{tr}\,F_{\mathbf{X}}]^{-1/2}$ respectively; ΔV and δV are the covariance volume $[\det\text{Cov}(\mathbf{X})]^{1/2}$ and Fisher volume $[\det F_{\mathbf{X}}]^{-1/2}$ respectively; and V_X denotes the ensemble volume, given by the exponential of the ensemble entropy [7]. The first and last inequalities are immediate consequences of the geometric mean of (positive) matrix eigenvalues being no greater than their arithmetic mean; the second inequality corresponds to the variational property that entropy is maximized for a given covariance by a Gaussian distribution, and the third inequality is given by Dembo *et al.* (Sec. IV C of [6]).

For a quantum system described by wave function $\psi(\mathbf{x})$ one finds, in analogy to Eqs. (11) and (12),

$$F_{\mathbf{X}} = (4/\hbar^2)[\text{Cov}_\psi(\mathbf{P}) - \text{Cov}_\psi(\mathbf{P}_{cl})] = (4/\hbar^2)\text{Cov}_\psi(\mathbf{P}_{nc}), \qquad (41)$$

where $\mathbf{P}_{cl}$ is a classical momentum vector defined by replacing $\psi'(x)$ by $\nabla \psi(\mathbf{x})$ in Eq. (8), and $\mathbf{P} = \mathbf{P}_{cl} + \mathbf{P}_{nc}$. Hence *the position Fisher matrix is proportional to the covariance matrix of the nonclassical momentum.* A conjugate relation holds for $F_{\mathbf{P}}$.

There are two natural scalar measures of joint nonclassicality, which reduce to the measure in Eq. (22) for $n = 1$. The first is

$$J_{nc}^{(1)} = (\hbar/2)^{-n}[\det\text{Cov}_\psi(\mathbf{X}_{nc})\det\text{Cov}_\psi(\mathbf{P}_{nc})]^{1/2}$$

$$= (\hbar/2)^n[\det F_{\mathbf{X}} \det F_{\mathbf{P}}]^{1/2}, \qquad (42)$$

which may be interpreted as a (dimensionless) nonclassical phase space volume. The second is the (dimensionless) nonclassical *area*

$$J_{nc}^{(2)} = (\Delta R)_{\mathbf{X}_{nc}}(\Delta R)_{\mathbf{P}_{nc}}/(\hbar/2) = (\hbar/2)/[(\delta R)_{\mathbf{X}}(\delta R)_{\mathbf{P}}], \qquad (43)$$

where ΔR and δR are the covariance and Fisher radiuses defined following Eq. (40).

The analog of Eq. (7) is

$$\text{tr}\,F_{\mathbf{X}} = (4/\hbar^2)[\langle \mathbf{P} \cdot \mathbf{P} \rangle_\psi - \langle \mathbf{P}_{cl} \cdot \mathbf{P}_{cl} \rangle_\psi], \qquad (44)$$

and thus the trace of the Fisher matrix is proportional to the difference of a quantum and a classical kinetic energy (and hence to the average of the quantum potential energy as per Sec. III D). Also, from Eq. (41) one has the Stam inequality $\text{Cov}(\mathbf{P}) \geq (\hbar^2/4)F_{\mathbf{X}}$ (Refs. [4,6]), which multiplied by Eq. (39), yields the Heisenberg uncertainty relation

$$\text{Cov}(\mathbf{X})\text{Cov}(\mathbf{P}) \geq (\hbar^2/4)I_n, \qquad (45)$$

where I_n denotes the $n \times n$ identity matrix.

Finally, the de Bruijn identities in Eq. (35) generalize to give a somewhat less direct connection between the Fisher matrix and entropy increase in higher dimensions. In particular consider the n-dimensional analog of the diffusion equation Eq. (30),

$$\dot{p}_t = (\nabla^T \Gamma \nabla)p_t, \qquad (46)$$

where Γ denotes a real symmetric positive diffusion matrix with constant coefficients. Under the coordinate transformation $\mathbf{y} = \Gamma^{-1/2}\mathbf{x}$ this reduces to the canonical form $\dot{p}_t = \nabla^2 p_t$, and in exact analogy to the derivation of Eq. (32) one finds that the *trace* of the Fisher matrix for $\mathbf{Y}$ is equal to the rate of entropy increase of $\mathbf{Y}$. But it follows directly from the coordinate transformation that $F_{\mathbf{Y}} = \Gamma^{1/2}F_{\mathbf{X}}\Gamma^{1/2}$ and $S_{\mathbf{Y}} = S_{\mathbf{X}} - (1/2)\ln\det\Gamma$, and hence one has the generalization

$$\dot{S}_{\mathbf{X}}(t) = \text{tr}[\Gamma F_{\mathbf{X}}(t)] \qquad (47)$$

of Eq. (32).

For the case of isotropic diffusion, $\Gamma = \gamma I_n$, it follows that $\text{tr}\,F_{\mathbf{X}}$ is inversely related to the robustness of $\mathbf{X}$ with respect to the onset of diffusion noise. In particular, for the *quantum isotropic diffusion equation*

$$\dot{\rho} = -(\gamma/\hbar^2)\delta_{ij}[P_i,[P_j,\rho]] - (\sigma/\hbar^2)\delta_{ij}[X_i,[X_j,\rho]] \qquad (48)$$

(with summation over repeated indices), one finds the analog

$$\dot{S}_{\mathbf{X}}(0) = \gamma\,\text{tr}\,F_{\mathbf{X}}, \qquad \dot{S}_{\mathbf{P}}(0) = \sigma\,\text{tr}\,F_{\mathbf{P}} \qquad (49)$$

of Eq. (35), leading to the natural generalization of Eq. (36)

$$J_r = (\hbar/2)^{-1}[\text{tr}\,F_{\mathbf{X}}\,\text{tr}\,F_{\mathbf{P}}]^{-1/2} = \delta R_{\mathbf{X}}\delta R_{\mathbf{P}}/(\hbar/2) \qquad (50)$$

for the joint robustness of a quantum state. Comparison of Eqs. (43) and (50) shows that joint nonclassicality and joint robustness are inversely related as before.

VI. CONCLUSIONS

It is seen that the Fisher information of a quantum observable is essentially the variance of the nonclassical component of the conjugate observable, as per Eqs. (12) and (16). Thus F_X is a measure of the nonclassicality of P, and vice versa. Moreover, a measure of joint nonclassicality for two conjugate observables may be naturally defined to be inversely proportional to the product of their Fisher lengths, as per Eq. (23). It would be of interest to determine whether pure states have a nontrivial minimum joint nonclassicality.

Application of de Bruijn's identity to quantum diffusion processes shows that the Fisher information of an observable is also essentially the rate of entropy increase of the observable at the onset of phase space diffusion, and hence is inversely related to the robustness of the observable with respect to noise. The Fisher length in particular provides a *direct* measure of robustness, being large when the entropy increase is small and vice versa. The joint robustness of two conjugate obervables is therefore defined as being proportional to the product of their Fisher lengths, as per Eq. (36). Joint robustness is simply related to joint nonclassicality as per Eq. (37): the more nonclassical a state is, the less robust it is, and vice versa.

These results can be generalized to vector observables as indicated in Sec. V, where the Fisher matrix is essentially the nonclassical covariance matrix for the conjugate observable, and its trace is essentially the rate of entropy increase under the onset of isotropic diffusion. Joint nonclassicality and joint robustness are again simply related, and indeed are inversely proportional if defined via Fisher radiuses as per Eqs. (43) and (50).

Finally, it is interesting to note that the Fisher length appears naturally in a bound for the average information, which may be obtained per measurement of X on members of an ensemble $\mathcal{E}$. In particular, this average information, $I(X|\mathcal{E})$, is given by the entropy corresponding to the average distribution of X over the ensemble minus the average entropy of X over the members of the ensemble [18]. From Eq. (6) one then immediately has the bound

$$I(X|\mathcal{E}) \leqslant \ln[(\Delta X)_{\mathcal{E}}/(\delta X)_{\min}] \tag{51}$$

i.e., the information is bounded by the logarithm of the ratio of two lengths (the ensemble root mean square deviation of X, and the minimum Fisher length of X over the ensemble). For higher dimensions corresponding inequalities may be obtained via Eq. (40). Other information-theoretic inequalities involving Fisher information are given by Dembo *et al.* [6], and further inequalities are discussed by Romera *et al.* in the context of obtaining bounds on radial expectation values [19].

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