Temporally stable coherent states in energy-degenerate systems: The hydrogen atom

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Klauder's recent generalization of the harmonic oscillator coherent states [J. Phys A **29**, L293 (1996)], is applicable only in nondegenerate systems, requiring some additional structure if applied to systems with degeneracies. The author suggests how this structure could be added, and applies the complete method to the hydrogen atom problem. To illustrate how a certain degree of freedom in the construction may be exercised, states are constructed, which are initially localized and evolve semiclassically, and whose long time evolution exhibits "fractional revivals."

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I. INTRODUCTION

Since their early introduction to quantum mechanics, the harmonic oscillator coherent states [1] have served many purposes. In Schrödinger's conception [2], they were viewed as quasiclassical objects, purely quantum in definition though remarkably classical in behavior. From this perspective, some authors have used these states with a classical limit to study correspondence between quantum and classical perturbation series [3]. Stemming from Glauber's study [4], coherent states have a wide application in quantum optics [5], primarily in representations of the electromagnetic field. In fact, being continuously parametrized, coherent states figure prominently in the theory of continuous representations [6,7].

Many generalizations of coherent states appear in the literature. Each generalization tends to preserve a small number of the properties of the harmonic oscillator coherent states in the general scheme at the expense of the remaining properties. A recent generalization due to Klauder [8] preserves many, at the expense of few. Klauder's generalization gives states that (a) evolve among themselves in time (temporally stable), (b) are continuously parametrized, and (c) admit a resolution of the identity. As such, no reservations are made for "semiclassical" properties such as minimum uncertainty, though a certain degree of freedom to be discussed below remains within the construction, which may be optimized according to additional concerns. Two studies have since appeared [9,10] proposing forth conditions that eliminate this degree of freedom, which will be discussed in Sec. IV.

As initially presented, Klauder's construction is appropriate for systems without energy degeneracies. With no additional structure, the resolution of the identity fails for degenerate systems. Energy degeneracies arise when independent operators commute with the Hamiltonian, suggesting a Lie algebraic approach to impose the additional structure. Thus in the presence of degeneracies, excepting those few cases of truly "accidental" degeneracies, the Perelomov approach to constructing coherent states for the degeneracy group is an obvious and general path to take [11].

The attempt to generalize harmonic oscillator coherent states for the hydrogen atom problem is not new. Many involve the construction of the complete Perelomov states via the dynamical group SO(4,2) [12–15]. Others, including the current approach, make some use of SO(4) coherent states [16,17]. Others involve the construction of "temporally stable" coherent states: Klauder's original paper [8], a study by Majumdar and Sharatchandra [9], and another by Fox [18]. The current construction differs from these in some key respects, which will be pointed out in the course of the paper.

Section II discusses the construction due to Klauder, and develops the extension of this construction to degenerate states. In Sec. III, the construction is applied to the hydrogen atom problem. Section IV is a discussion regarding the role that these states and other generalized coherent states hold in physical theory, namely, as representations rather than as physical states. The dynamics of an individual state are explored in Sec. V, in which hydrogenic states are constructed that exhibit fractional and full revivals, and the main points of the paper are summarized in Sec. VI.

II. CONSTRUCTION

The generalization due to Klauder [8] is applicable to the discrete portion of the spectrum of a Hamiltonian $\hat{H}$. (A continuum generalization has also appeared [10].) So, for a nondegenerate $\hat{H}$ with eigenstates $|n\rangle$ and eigenstate energies e_n, the coherent states are given by (using atomic units)

$$|s,\gamma\rangle = M(s^2) \sum_{n=0}^{\infty} \frac{s^n \exp(-i\gamma e_n)}{\sqrt{\rho_n}} |n\rangle, \qquad (2.1)$$

where $s \geq 0$ and γ is real. The factors ρ_n are the moments of a function $\rho(u) > 0, u \geq 0$,

$$\rho_n = \int_0^{\infty} u^n \rho(u) du, \qquad (2.2)$$

and the normalizing function $M(s^2)$ is chosen such that

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$$1 = \langle s, \gamma | s, \gamma \rangle = M^2(s^2) \sum_{n=0}^{\infty} \frac{s^{2n}}{\rho_n}. \quad (2.3)$$

If the discrete portion of the spectrum is finite, the upper limits of the above sums (and the appropriate expressions hereafter) may be replaced with n_{max}.

The choice of $\rho(u)$ is the remaining degree of freedom up to the following restrictions: All the moments ρ_n exist (up to n_{max} if finite), and the sum in Eq. (2.3) exists for all $s \geq 0$. Such functions $\rho(u)$ are known to exist: In the harmonic oscillator, $\rho(u) = e^{-u}$ leads to the standard harmonic oscillator coherent states.

The states $|s, \gamma\rangle$ clearly are continuously parametrized. The states exhibit temporal stability: $\exp(-i\hat{H}t)|s,\gamma\rangle = |s,\gamma + t\rangle$. Further, given $\rho(u)$ and $M^2(u)$, let $k(u)$ be defined by

$$k(u)M^2(u) = \rho(u). \quad (2.4)$$

Then, the coherent states satisfy the resolution of the identity,

$$\hat{I} = \int d\mu(s,\gamma)|s,\gamma\rangle\langle s,\gamma|, \quad (2.5)$$

in which the integration is given by

$$\int d\mu(s,\gamma) = \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_0^\infty ds^2 k(s^2) \int_{-\Gamma}^{\Gamma} d\gamma. \quad (2.6)$$

The limit is necessary to accommodate possible incommensurabilities of energy levels. In proving Eq. (2.5), integration over γ yields the Kronecker delta $\delta_{e_n e_m}$, which may be identified with δ_{nm} only in the nondegenerate case. Also, the identity of Eq. (2.5) should be regarded as a projection operator onto the states contributing to $|s,\gamma\rangle$, i.e., the discrete portion of the spectrum. These properties in concert make these states most useful in the representation of arbitrary, bound, time evolved states.

To extend the construction to energy degenerate states, replace Eq. (2.1) with

$$|s,\gamma,x\rangle = N(s^2) \sum_{n=0}^{\infty} \frac{s^n \exp(-i\gamma e_n)}{\sqrt{\rho_n}} \sqrt{d_n}|n,x\rangle, \quad (2.7)$$

where d_n is the degeneracy of the nth energy level, $|n, x\rangle$ are the Perelomov coherent states [11] for the degeneracy group G, and the normalizing factor $N(s^2)$ is given by

$$1 = \langle s,\gamma,x|s,\gamma,x\rangle = N^2(s^2) \sum_{n=0}^{\infty} \frac{s^{2n}d_n}{\rho_n}. \quad (2.8)$$

In each energy degenerate subspace of the Hilbert space, the Perelomov coherent states satisfy the resolution of the identity

$$\hat{I}_n = d_n V(H) \int_X d\eta(x)|n,x\rangle\langle n,x|, \quad (2.9)$$

in which $V(H)$ is the volume of the isotropy subgroup relative to the fiducial vector in the construction of the Perelomov coherent states, $X = G/H$ is the quotient space formed by the degeneracy group with the isotropy subgroup, and the measure $d\eta$ is induced from the Haar measure on the degeneracy group. The states Eq. (2.7) therefore satisfy the resolution of the identity

$$\hat{I} = \int d\mu(s,\gamma,x)|s,\gamma,x\rangle\langle s,\gamma,x|, \quad (2.10)$$

with

$$\int d\mu(s,\gamma,x)$$
$$= \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_0^\infty ds^2 k(s^2) \int_{-\Gamma}^{\Gamma} d\gamma \, \text{vol}(H) \int_X d\eta(x). \quad (2.11)$$

Since the states $|n,x\rangle$ are formed by superpositions over states that share a common energy eigenvalue e_n, they are also eigenstates of the Hamiltonian and so evolve simply in time. Accordingly, the states $|s,\gamma,x\rangle$ preserve the temporal stability property of the nondegenerate construction.

Majumdar and Sharatchandra's construction [9] also makes explicit use of the Perelomov construction of coherent states for the degeneracy group, and Klauder's construction [8] is less explicit in this regard. A significant difference between this and these other extensions of Klauder's construction is the treatment of the factor d_n. Among other problems [19] Majumdar and Sharatchandra incorporate d_n into the measure after the summation of the state, an operation of questionable justifiability. Klauder, using an adaptation of SO(3) coherent states, incorporates d_n into the states after, and therefore affecting, normalization. Fox [18] constructs temporally stable coherent states, but with a Gaussian taking the place of $s^n \exp(-i\gamma e_n)$ in Eq. (2.1). As such, a similar adaptation of his states may be effected by including a factor of $\sqrt{d_n}$ as in Eq. (2.7).

III. SPECIAL CASE: THE HYDROGEN ATOM

The group theoretical treatment of the hydrogen atom is standard in the literature [20–22]. For the hydrogen atom problem, there are two realizations of the degeneracy group SO(4). One uses the elements of the angular momentum vector, $\hat{L}_j$, and a scaled quantum Runge-Lenz vector, $\hat{A}_j$, as generators of the group, whereas the other decouples these six generators into two sets, $\hat{M}_j = \frac{1}{2}(\hat{L}_j + \hat{A}_j)$, and $\hat{N}_j = \frac{1}{2}(\hat{L}_j - \hat{A}_j)$. In the second representation, one finds that SO(4) = SO(3) ⊗ SO(3), so that, loosely speaking, a Perelomov coherent state for SO(4) may be given by the direct product of two SO(3) coherent states.

The SO(3) coherent states with the fiducial vector $|j,-j\rangle$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, are given by

$$|j,\zeta\rangle = \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m)!(j-m)!} \right]^{1/2} \frac{\zeta^{j+m}}{(1+|\zeta|^2)^j} |j,m\rangle.$$

(3.1)

The resolution of the identity for these states may be written

$$\hat{1}_j = \frac{2j+1}{\pi} \int \frac{d^2\zeta}{(1+|\zeta|^2)^2} |j,\zeta\rangle\langle j,\zeta|,$$

(3.2)

where $d^2\zeta = d\operatorname{Re}\zeta\, d\operatorname{Im}\zeta$, and integration is over the entire complex plane projected from the unit sphere with the transformation $\zeta = -\tan(\theta/2)e^{-i\phi}$. In the hydrogenic realization, the representations of each copy of SO(3) are of equal di-

mension ($n=2j+1$) so the dimensions of the relevant representations of SO(4) are n^2, $n=1,2,3,\ldots$.

It is now straightforward to construct the coherent states for the full system. The coherent states for the hydrogen atom problem by this construction are given by

$$|s,\gamma,\zeta_1,\zeta_2\rangle = N(s^2) \sum_{n=0}^{\infty} \frac{s^n \exp(-i\gamma e_{n+1})(n+1)}{\sqrt{\rho_n}}$$
$$\times |n+1,\zeta_1,\zeta_2\rangle$$

(3.3)

with

$$|n,\zeta_1,\zeta_2\rangle = \sum_{m_1,m_2=-j}^{j} \frac{(2j)!\,\zeta_1^{j+m_1}\zeta_2^{j+m_2}|j,m_1\rangle|j,m_2\rangle}{[(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!]^{1/2}(1+|\zeta_1|^2)^j(1+|\zeta_2|^2)^j}.$$

(3.4)

The states $|j,m_1\rangle|j,m_2\rangle$ may be related to the standard hydrogen Hamiltonian eigenstates $|n,l,m\rangle$ via Clebsch-Gordon coefficients. The states Eq. (3.4) satisfy the resolution of the identity

$$\hat{1}_B = \frac{1}{\pi^2} \int d\mu(s,\gamma) \int \frac{d^2\zeta_1 d^2\zeta_2}{(1+|\zeta_1|^2)^2(1+|\zeta_2|^2)^2} |s,\gamma,\zeta_1,\zeta_2\rangle\langle s,\gamma,\zeta_1,\zeta_2|,$$

(3.5)

where the subscripted B is included to emphasize that this is more appropriately regarded as a projection operator into the bound portion of the Hilbert space. In the specific example of $\rho(u)=e^{-u}$, with moments $\rho_n=n!$, explicit form may be given to $N(s^2)$ and $k(u)$ by

$$N(s^2) = e^{-s^2/2}(1+3s^2+s^4)^{-1/2}$$

(3.6)

and

$$k(u) = 1+3u+u^2.$$

(3.7)

IV. SOME CLARIFICATION

At this point, a few observations are in order. Primarily, the term ''temporal stability'' in no way refers to the time evolution of the structure in configuration space. Only through a rather generous interpretation does this construction ''positively'' solve the long standing problem of forming nondispersing wave packets for the hydrogen atom. Temporal stability refers strictly to the mathematical property that the states evolve in time among themselves. With this property in mind, some authors [9] have grossly overstated the the nature of the configuration space time evolution, while other authors [23,19] have studied in detail the long-time evolution of individual states, even though there is no underlying physical basis either to provide for spatial coherence, or to presume states of this description are found in the laboratory at all. The question of how to prepare these states in the laboratory remains very much open.

Much of the study of generalized coherent states rests more in the mathematical than the physical nature of math-

ematical physics. Glauber's motivation in his study of coherent states [4] was not so much that coherent states are found in the laboratory, but that they provide a representation in which otherwise difficult calculations become feasible. Glauber noted that certain electric field operators have representations as sums over the modal annihilation operators. In diagonalizing these operators, one arrives at eigenstates of the modal annihilation operators. Restricted to a single mode, this corresponds to the annihilation operator definition of harmonic oscillator coherent states. Hence as annihilation operator coherent states, they arise from a representation, a point of mathematical convenience, not as a conclusion from the physics of the problem. In any case, generalizations of annihilation operator coherent states have appeared widely, though the physical motivation to study such definitions in any context besides as representations is unclear.

Glauber also showed how these states may be constructed through the action of a displacement operator on the ground state. This definition was generalized by Perelomov [24], a generalization which has been widely successful. This success is founded upon the properties of the dynamical group coming through into the set of coherent states, not from an assertion (which few researchers make) that an individual state by such a construction matches a state by some preparation in the laboratory. This success is of a mathematical, not physical, nature again resting upon the use of these states as a representation.

Of Glauber's original three definitions, the approach that appears to invest the most physics is the minimum uncertainty construction. Indeed, squeezed states, a generalization of this construction, are used as descriptions of physical as-

pects of certain quantum optical experiments in the laboratory. Nieto *et al.* [25] have also developed a generalization that minimizes the uncertainty product of a pair of ''natural'' operators. Ehrenfest's relations then lead to the initial evolution of the quantum expectation values approximating classical evolution. Though this approach is strongest in terms of an underlying physical motivation, these states still lack (in general) any physical hypothesis, which selects for states of this description in the laboratory. As an aside, the Nieto construction, though seen from time to time, is not as widely used as the Perelomov construction for perhaps two reasons. First, though it is generally applicable in principle, many systems are intractable to carry through to completion (when the Hamiltonian enters into the ''natural'' operators). Second, a certain tradeoff appears to be at work: This approach is somewhat less mathematically endowed than Perelomov's approach.

Now consider Klauder's construction. All of the attractions are mathematical in nature. As initially presented, no reservations are made for coherence in configuration space (i.e., semiclassical behavior) and there is no general physical mechanism that would result in finding these states in the laboratory. However, a certain degree of freedom remains in the construction, and two suggestions have separately appeared that a fourth requirement will simultaneously eliminate the degree of freedom and ensure for the behavior in configuration space. It is likely that a fourth requirement, if it exists, will be physical in nature. The requirement postulated by Majumdar and Sharatchandra [9] is that the measure found in the resolution of the identity corresponds to the ''canonical'' measure on classical phase space. They further assert that the measure uniquely identifies the set of coherent states. This assertion is false, as shown by Sixdeniers *et al.* [26] who demonstrate multiple measures corresponding to the same set of coherent states. Also, though it is convenient from a mathematical point of view, it is unclear why the measures should correspond at all from a physical point of view, or even if a meaningful identification (one to one) can always be made between individual coherent states and points in classical phase space.

A fourth requirement is also postulated by Gazeau and Klauder [10], which is motivated by an attempt to formalize the connection between the quantum parameters to the coherent state and the classical action-angle variables. Unfortunately, their requirement results in an angle variable whose rate of change with time is independent of the action, a rather special circumstance restricted to the harmonic oscillator and a small number of other systems. This is a severe limitation in terms of semiclassical behavior, since this is clearly at odds with how the angle variable evolves in, say, the hydrogen atom problem.

A degree of freedom also remains in Fox's construction [18] of Gaussian generalized coherent states, namely, the width of the Gaussian in question. Note that in this context, the distribution in energy level, not configuration space, is Gaussian. Fox does not give any criteria that are intended to specify a suitable width. As with the Klauder's construction, this degree of freedom may be optimized according to the aim in mind.

Hence in the absence of an acceptable fourth criterion (none is herein proposed), we carry on. This limits the construction to a mathematical tool, though an interesting mathematical tool it is. Note that the time-dependent Schrödinger equation for a time-independent Hamiltonian in this coherent state representation becomes (in atomic units)

$$\frac{\partial}{\partial \gamma} \langle s, \gamma | \psi \rangle = - \frac{\partial}{\partial t} \langle s, \gamma | \psi \rangle \tag{4.1}$$

whose solution is trivial. With this expression representing the unperturbed solution, this would make for an interesting point of departure towards a study of time-dependent perturbation theory. That is, coherent states are useful when considered as an ensemble of states (a representation), not as individual states.

Speaking now in the specific case, some authors [23,19] have suggested that the temporally stable construction of coherent states does not support the possibility of exhibiting full or fractional revivals as described by Averbukh and Perelman [27] or Nauenberg [17]. First, in order to decide whether a state is to be found in the laboratory, a physical mechanism for the preparation of these states must be postulated. Until this has been done, the presence or absence of a phenomenon, which is, after all, universal, is not relevant. Second, these authors did not exploit the degree of freedom that remains in the construction. Without supplying a physical motivation that would lead to finding these states in the laboratory, we shall see that by exploiting this degree of freedom wave functions may be formed by the present construction, which exhibit the full panoply of revivals.

V. DYNAMICS

Having thus constructed the states emphasizing, among other things, time evolution, it is now interesting to consider the behavior of the states as evolved in time. Other authors have defined hydrogen atom coherent states with a variety of constructions and with various reports of evolution in ''fictitious'' time [13], or evolution along circular [28,29] or Keplerian elliptical orbits with possible, eventual state revivals [16,17]. Coherent states also may be constructed by the present recipe, which travel along elliptical orbits and exhibit fractional revivals.

According to Averbukh and Perelman [27], fractional revivals are a universal phenomenon exhibited by wave functions provided third-order corrections and higher do not contribute significantly to a polynomial approximation to the energy eigenvalues over contributing energy eigenstates. Expanding about $n = \bar{n}$, the hydrogen atom energy levels are

$$e_n = - \frac{1}{2n^2} = - \frac{1}{2\bar{n}^2} + \frac{1}{\bar{n}^3}(n - \bar{n}) - \frac{3}{2\bar{n}^4}(n - \bar{n})^2$$

$$+ \frac{2}{\bar{n}^5}(n - \bar{n})^3 + \cdots. \tag{5.1}$$

The first two terms lead to phase angles equal to multiples of 2π for times in the vicinity of

$$t = T_r = \frac{2\pi}{3}\langle n\rangle^4. \qquad (5.2)$$

This signals a wave function revival at $t = T_r$ provided the cubic term is small at the edges of the distribution, i.e., $4\pi(\Delta n)^3 \ll 3\langle n\rangle$. Note that $T_r = T_{\mathrm{rev}}/2$ in Averbukh and Perelman's notation.

At the heart of these coherent states lies the function $\rho(u)$. With $\rho(u) = e^{-u}$, one obtains a distribution in eigenlevels characterized by

$$\langle n\rangle = s^2 \frac{s^4 + 5s^2 + 4}{s^4 + 3s^2 + 1} \qquad (5.3)$$

and

$$(\Delta n)^2 = s^2 \frac{s^8 + 6s^6 + 14s^4 + 10s^2 + 4}{s^8 + 6s^6 + 11s^4 + 6s^2 + 1}, \qquad (5.4)$$

so that, taking leading order behavior, $\Delta n \sim \sqrt{\langle n\rangle}$. Substituting into the above necessary condition for a revival at $t = T_r$ gives $4\pi\sqrt{\langle n\rangle}/3 \ll 1$, which is only satisfied for states in the immediate vicinity of the trivial coherent state, the ground state, and certainly violated by states with high principal quantum numbers.

However, the function $\rho(u)$ is a ''degree of freedom'' in the construction, and may be chosen according to the application in mind. Accordingly, consider instead $\rho(u) = \exp(-u^\alpha)$ for some constant $\alpha > 0$ with a view to constructing wave packets that exhibit strong revivals. The moments of this function are

$$\rho_n = \frac{1}{\alpha}\Gamma\left(\frac{n+1}{\alpha}\right). \qquad (5.5)$$

Many of the expressions involved in subsequent calculations may be handled using properties of the functions of Mittag-Leffler [30], though they will be treated instead by comparisons to expressions following from $\rho(u) = e^{-u}$. In fact, using $\rho(u) = \exp(-u^\alpha)$ results in a set of coherent states closely related to those described by Sixdeniers *et al.* [26].

Expressions for $\langle n\rangle$ and $(\Delta n)^2$ may be approximated by recognizing the scalings necessary to map expressions with $\alpha = 1$ onto those with general α:

$$n + 1 \rightarrow (n+1)/\alpha, \quad s \rightarrow s^\alpha. \qquad (5.6)$$

Hence, one obtains to leading order (large values of s will eventually be involved)

$$\langle n\rangle \sim \alpha s^{2\alpha}, \quad \Delta n \sim \alpha s^\alpha, \qquad (5.7)$$

so that $\Delta n \sim \sqrt{\alpha\langle n\rangle}$. Substituting this into the minimal condition for the first full revival gives $4\pi\alpha^{3/2}\sqrt{\langle n\rangle}/3 \ll 1$, which may be satisfied if α is chosen sufficiently small. Without discussing the effect of changing α any further, there will be a tradeoff between large and small α: Large α will introduce many significantly contributing energy levels for a given $\langle n\rangle$ yielding good spatial localization but weak or nonexistent revivals, whereas small α yields strong revivals of poorly
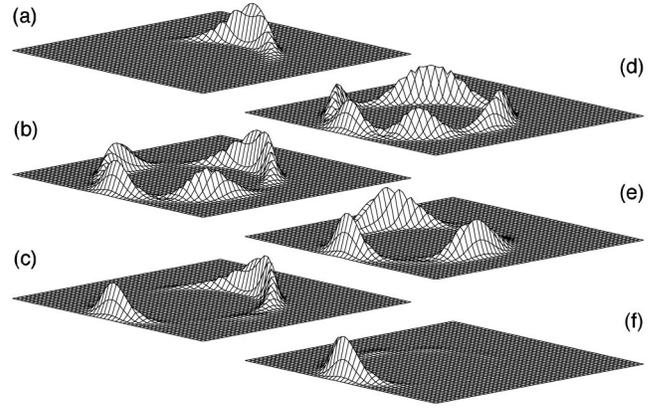
FIG. 1. Amplitude of coherent state on the x-y plane for times (a) $t = 0$, (b) $t = T_r/5$, (c) $t = T_r/4$, (d) $t = T_r/3$, (e) $t = T_r/2$, and (f) $t = T_r$.

localized states. Note that in this construction a small width in n follows from an appropriate choice for $\rho(u)$, whereas the same may be accomplished by Fox's construction [18] by simply specifying the width to be narrow.

As a typical example, consider the state depicted in Fig. 1. For this state, $\alpha = \frac{1}{32}$ and $s = 2.209 \times 10^{59}$. This results in a state centered at $\langle n\rangle = 160$ with a width of $\Delta n = \sqrt{5}$ for which one expects a full revival at $t = T_r = 1.373 \times 10^9$. The parameters $\zeta_{1,2}$ to the SO(4) coherent state were chosen to provide an elliptical orbit with eccentricity $\epsilon = 0.385$, the major axis parallel to the x axis, and the angular momentum parallel to the z axis. The vertical axes are amplitudes of the wave functions on the x-y plane, calculated at the times indicated on a square grid 80 000 units in width centered at the origin.

The evolution of this state is as expected. Initially localized, the state evolves semi-classically. The wave function then spreads out but remains close to the ellipse. As the expected times for the various fractional revivals arrive, the state exhibits the expected revival including the full revival

FIG. 2. Autocorrelation function of the coherent state at time t. For this state, the revival time is $T_r = 1.373 \times 10^9$ in atomic units.

FIG. 3. Distributions in energy level for $\alpha = 1(\times)$ and $\alpha = 1/32(+)$.

VI. CONCLUSION

This approach to defining temporally stable coherent states for the hydrogen atom problem is similar in spirit to those of Majumdar and Sharatchandra [9], Klauder [8], and Fox [18]. Majumdar and Sharatchandra make explicit use of the SO(4) degeneracy group, but treat the factor d_n differently. Although Klauder's construction is specific to the hydrogen atom problem rather than the present general approach, a similar term appears in that construction that disrupts normalization. Fox uses a somewhat different construction, which avoids the use of the moments of some function $\rho(u)$. Even so, one may construct coherent states in degenerate systems related to Fox's construction as the present construction is related to Klauder's original work.

The general construction of Eq. (2.7) provides states with many useful properties. They form a complete set of states (in the bound portion of the spectrum) and evolve in time among themselves. This makes them a clear candidate for use in representations of time evolved, bound states. Further, there is a freedom in their definition that stems from the choice of $\rho(u)$. If $\rho(u)$ is appropriately chosen, hydrogen atom coherent states may be defined that exhibit the full range of phenomena exhibited by other approaches: initial semiclassical behavior, interference between the head and tail of the state as it disperses about the Keplerian ellipse, localization on the Keplerian ellipse, and wave function revivals at predictable times.

The salient difference between these coherent states and other constructions is the natural and explicit manner in which the energy degeneracies are treated herein, via Perelomov's group theoretical construct of generalized coherent states.

at $t = T_r$, even though an examination of the minimal condition for the first revival gives the debatable result $0.29 \ll 1$.

Figure 2 depicts the autocorrelation function for the same state as above, exhibiting the typical pattern characterizing revivals (compare with Fig. 2 of Parker and Stroud [31]). Figure 3 demonstrates how a small α shrinks the width of the state in n.

Commenting again on the assertion by some authors that ''temporally stable'' states for the hydrogen atom cannot exhibit this phenomenon [23,19]: Their calculations involved, in terms of the present paper, $\alpha = 1$, hence wide distributions in n that exhibited no appreciable revivals over the time frames calculated. Wave function revivals are a universal phenomenon depending on the nature of the energy eigenlevel spacings, and in the case of the hydrogen atom, the width Δn. By exploiting the fact that one may choose $\rho(u)$ to one's liking, Δn may be reduced such that the resultant states do revive. Further, one study [23] used values of s leading to $\langle n \rangle$ equal to 25 and 400. With $\langle n \rangle = 400$, from Eq. (5.2), $T_r \sim 5.36 \times 10^{10}$ though their calculation only extends to $t = 5 \times 10^9$. For the other study [19], values of $\langle n \rangle$ exhibited were 10 and 200. A full revival should be found using $\langle n \rangle = 200$ at about $t = 3.35 \times 10^9$ though their calculation only extends to $t = 5 \times 10^8$.

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