

Comment on “Conditionally exactly soluble class of quantum potentials”

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We detect an omission in the paper “Conditionally exactly soluble class of quantum potentials” by A. de Souza Dutra [Phys. Rev. A **47**, R2435 (1993)]. There, two strongly singular s -wave bound-state problems have been claimed completely solvable in closed form. Unfortunately, all the displayed wave functions are merely asymptotically correct solutions which do not satisfy the appropriate threshold boundary condition. We show that the incorporation of the threshold boundary conditions only leads to a very partial exact solvability at a single energy and for special couplings.

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For the two strongly singular s -wave potentials given, in the units $\hbar = 2\mu = 1$, by the formulas

$$V_1(r) = \frac{A}{r} + \frac{B}{r^{1/2}} + \frac{G}{r^2}, \quad G = G_0 = -\frac{3}{16} \quad (1)$$

and

$$V_2(r) = A r^{2/3} + \frac{B}{r^{2/3}} + \frac{G}{r^2}, \quad G = g_0 = -\frac{5}{36}, \quad (2)$$

de Souza Dutra [1] offered the explicit elementary wave functions as well as closed formulas for all their bound-state energies. One of the three couplings is not free: This entitled him to coin their “conditionally” exactly soluble (CES) status. In what follows we intend to demonstrate that in the sense of the Ushveridze’s monograph [2] both these forces $V_{1,2}(r)$ only remain *partially* solvable at specific values of the energies E and couplings B .

Our present main point is that all the solutions presented in Ref. [1] still have to satisfy an appropriate and, for reasons to be made understandable here, forgotten boundary condition in the origin. Indeed, it is well known that for a central potential, the Schrödinger equation $-\Delta\Psi(\vec{r}) + V(|\vec{r}|)\Psi(\vec{r}) = E\Psi(\vec{r})$ degenerates to an infinite set of the ordinary (often called radial) decoupled differential equations

$$-\frac{d^2}{dr^2}\psi(r) + \frac{l(l+1)}{r^2}\psi(r) + V(r)\psi(r) = E\psi(r),$$

$$l = 0, 1, \dots \quad (3)$$

for the separate angular-momentum components of the whole original wave function. The Newton’s excellent review [3] summarizes the details. *Under the assumption of the analyticity of $V(r)$ in the origin* it shows that and why the standard physical requirement of normalizability of bound states $||\Psi(\vec{r})|| < \infty$ is *strictly* equivalent to the integrability of their partial waves,

$$\psi(r) \in L_2(0, \infty). \quad (4)$$

For $l = 1, 2, \dots$ the unphysical component $\psi_{\text{irregular}}(r) \approx r^{-l}$ of the general threshold solution of eq. (3) is manifestly nonintegrable near $r \approx 0$.

In the s wave with $l = 0$ a more subtle argumentation is needed [3]. In practice, the subtlety is usually avoided by the replacement of Eq. (4) by the boundary condition

$$\lim_{r \rightarrow 0} \psi(r) = 0. \quad (5)$$

Even when we solve the ordinary harmonic oscillator the latter boundary condition in the origin offers a more straightforward recipe for numerical calculations. Let us repeat: *for analytic potentials*, Eqs. (4) and (5) are equivalent but the proof [3] of their equivalence immediately fails for the “very next” nonanalytic $V(r) \approx G r^{-2}$, $r \approx 0$, say, in the Kratzer’s solvable phenomenological model [4] with $G \neq 0$, etc. One must reanalyze the whole quantization procedure anew, even for the harmonic oscillator, at $G \rightarrow 0$ [5].

For all the similar singular forces with the finite limit in Eq. (3),

$$G = \lim_{r \rightarrow 0} r^2 V(r) \neq 0,$$

we have to redefine $l(l+1) + G = \mathcal{L}(\mathcal{L}+1)$. The new parameter $\mathcal{L} = \sqrt{(l + \frac{1}{2})^2 + G} - \frac{1}{2}$ enters then the modified threshold solutions $\psi_{\text{regular}}(r) \approx r^{\mathcal{L}+1}$ and $\psi_{\text{irregular}}(r) \approx r^{-\mathcal{L}}$. The irregular one is eliminated as manifestly violating the normalizability (4) at $\mathcal{L} \geq 1/2$.

The latter bound means $G \geq 3/4$ in s wave with $l = 0$. Below such a strength of repulsion the Hamiltonian *ceases to be self-adjoint*. The conclusion is strongly counterintuitive. Mathematically, the problem is serious. First spotted and analyzed by Case [6], it means that at $G < 3/4$, the textbook quantization of the Kratzer-like singular models is *not unique at all*. A more detailed discussion may be found in the literature (cf., e.g., [7] or [8]). In its light, the physics community currently accepts a unique way of quantization which is, mathematically speaking, a mere regularization. It is often supported by the various sufficiently robust *ad hoc* arguments (cf., e.g., [5] on pp. 157 and 167 or Ref. [9]).

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TABLE I. Nonvanishing zeros $X=X(n,k)$ of the first few Hermite polynomials $H_n(X)$.

$n \backslash k$	4	3	2	1
0				
1				
2		$-\sqrt{1/2}$	$\sqrt{1/2}$	
3		$-\sqrt{3/2}$	$\sqrt{3/2}$	
4	$-\sqrt{(3+\sqrt{6})/2}$	$-\sqrt{(3-\sqrt{6})/2}$	$\sqrt{(3-\sqrt{6})/2}$	$\sqrt{(3+\sqrt{6})/2}$
5	$-\sqrt{(5+\sqrt{10})/2}$	$-\sqrt{(5-\sqrt{10})/2}$	$\sqrt{(5-\sqrt{10})/2}$	$\sqrt{(5+\sqrt{10})/2}$

For our present purposes, in the physical textbook language [10], the correct recipe may be formulated as follows.

(i) In the domain of a weak repulsion we distinguish between the physical $\psi_{\text{regular}}(r) \approx r^{\mathcal{L}+1}$ and unphysical $\psi_{\text{irregular}}(r) \approx r^{-\mathcal{L}}$. As long as both of them remain normalizable, we impose an extra, *stronger* boundary condition in the origin,

$$\lim_{r \rightarrow 0} \psi(r) = 0, \quad G \in (0, 3/4). \quad (6)$$

It coincides with Eq. (5) but its mathematical meaning of a convenient choice of the most plausible self-adjoint extension is new.

(ii) In the domain of weak attraction, both solutions $\psi_{\text{regular}}(r) \approx r^{\mathcal{L}+1}$ and $\psi_{\text{irregular}}(r) \approx r^{-\mathcal{L}}$ are compatible with Eq. (6). In a sensible physical theory which distinguishes between the two, the replacement of Eq. (6) by an even stronger artificial constraint is needed,

$$\lim_{r \rightarrow 0} \psi(r)/\sqrt{r} = 0, \quad G \in (-1/4, 0). \quad (7)$$

(iii) Below the lower bound $G \leq -1/4$ one cannot prevent the spectrum from collapse by any means. Particles would definitely fall in the origin.

We may summarize: In practice, bound-state solutions of the Schrödinger differential Eq. (3) may be constructed in two ways, namely as follows.

(iv) As the regular solutions $\psi_{\text{regular}}(r)$ (RS) constrained by the asymptotic normalizability condition

$$\psi_{\text{regular}}(R) = 0, \quad R \rightarrow \infty.$$

(v) From the so-called Jost solutions $\psi_{\text{Jost}}(r)$ (JS), *always* exhibiting the square-integrable asymptotic decrease by definition.

The former regular-solution approach (RS) proves useful within the framework of the standard Taylor series method [11] and in nonnumerical context [12]. Schrödinger equation (3) becomes converted into the exactly solvable two-term recurrences for harmonic oscillator, into the three-term recurrences for sextic forces, etc. [13]. Rather unexpectedly, for several other potentials, a few bound states may still appear in an exact polynomial (i.e., terminating Taylor series) form. An explicit construction of these exceptional elementary

states is based on the solution of the Magyari's nonlinear algebraic equations [14]. They determine a few energy levels exactly and restrict also the free variability of the available couplings.

In the latter context, potentials $V_{1,2}(r)$ exhibit a certain incomplete dynamical symmetry and play an exceptional role as quasieactly solvable in a certain narrower sense (cf. Ref. [2] for more details). This would make the ambitious conclusions of Ref. [1], if they were all true, even more important.

Their analysis must be based on the alternative option (JS) which requires the threshold boundary conditions (5)–(7) [7]. This is the core of our present message. For the particular forces (1) and (2) such an approach has already thoroughly been tested numerically in Ref. [9]. The Liouvillean [15] change of variables $r \rightarrow x = r^{\text{const}}$ and $\psi(r) \rightarrow x^{\text{const}} \chi(x)$ has been employed there. As long as it leaves the form of the Schrödinger equation unchanged, it reduces all the bound-state problems with forces of types (1) and (2) to their ‘‘canonical’’ equivalents with polynomial potentials

$$V_T(x) = ar^{-2} + br^2 + cr^4 + \dots + yr^{4q} + zr^{4q+2},$$

$$a > -1/4. \quad (8)$$

On this basis, we may easily deduce the leading-order solutions (near the origin) also for our singular potentials $V_{1,2}(r)$ of Eqs. (1) and (2),

$$\psi_{1,\text{regular}}(r) \sim r^{3/4}, \quad \psi_{1,\text{irregular}}(r) \sim r^{1/4},$$

$$\psi_{2,\text{regular}}(r) \sim r^{5/6}, \quad \psi_{2,\text{irregular}}(r) \sim r^{1/6}.$$

This is to be compared with the Dutra's wave functions: Say, for potential $V_1(r)$ we may quote Eq. (9) from [1],

$$\begin{aligned} \psi_1^{(D)}(r) = & C r^{1/4} \exp \left[-\frac{1}{2} \beta^2 \left(r^{1/2} - \frac{B}{2E} \right)^2 \right] \\ & \times H_n \left[\beta \left(r^{1/2} - \frac{B}{2E} \right) \right]. \end{aligned} \quad (9)$$

Its energies $E = -\beta^4/4$ are parametrized by $\beta = \beta_n$ and numbered by an integer $n = 0, 1, \dots$. The formula also contains a certain normalization constant $C = C_n$ and Hermite polyno-

TABLE II. Parameters of the first few simplest quasixact states in $V_1(r)$. M counts nodes in $\psi_1(r)$: $M=0$ means ground state, etc.

M	n	k	Fixed coupling $B' = B/8$		Binding energy E	
0	2	1	$\sqrt{1/9^3}$	~ 0.0370	$-(4/9)^2$	~ -0.197
	3	1	$\sqrt{3/11^3}$	~ 0.0475	$-(4/11)^2$	~ -0.055
	4	1	$\sqrt{(3+\sqrt{6})/(15-\sqrt{6})^3}$	~ 0.0525	$-[4/(15+\sqrt{6})]^2$	~ -0.029
	5	1	$\sqrt{(5+\sqrt{10})/(17-\sqrt{10})^3}$	~ 0.0555	$-[4/(17+\sqrt{10})]^2$	~ -0.018
1	2	2	$-\sqrt{1/9^3}$	~ -0.037	$-(4/9)^2$	~ -0.132
	4	2	$\sqrt{(3-\sqrt{6})/(15+\sqrt{6})^3}$	~ 0.0102	$-[4/(15-\sqrt{6})]^2$	~ -0.047
	5	2	$\sqrt{(5-\sqrt{10})/(17+\sqrt{10})^3}$	~ 0.0150	$-[4/(17-\sqrt{10})]^2$	~ -0.028
2	3	2	$-\sqrt{3/11^3}$	~ -0.047	$-(4/11)^2$	~ -0.055
	4	3	$-\sqrt{(3-\sqrt{6})/(15+\sqrt{6})^3}$	~ -0.010	$-[4/(15-\sqrt{6})]^2$	~ -0.047
3	4	4	$-\sqrt{(3+\sqrt{6})/(15-\sqrt{6})^3}$	~ -0.053	$-[4/(15+\sqrt{6})]^2$	~ -0.029
	5	3	$-\sqrt{(5-\sqrt{10})/(17+\sqrt{10})^3}$	~ -0.015	$-[4/(17-\sqrt{10})]^2$	~ -0.028
4	5	4	$-\sqrt{(5+\sqrt{10})/(17-\sqrt{10})^3}$	~ -0.056	$-[4/(17+\sqrt{10})]^2$	~ -0.039

mials $H_n(x)$. We immediately detect an irregularity of the latter solution at the origin.

Similar observation is also made for $\psi_2(r)$ from Eq. (13) in [1]: None of the Dutra's wave functions satisfies the physical boundary condition (7). An explanation of this obvious misunderstanding is, in fact, not too difficult: The solutions were merely constrained by the too weak (though, in practice, much more frequently encountered) and, hence, inapplicable threshold condition (5). We may summarize that this inconsequent use of the boundary conditions would lead to a physically absurd spectrum covering the whole real line, $E \in (-\infty, \infty)$.

The Dutra's "nonanonymous" (i.e., Hermite-polynomial) solutions have already evoked a non-negligible response in the current literature. As an example one might quote Ref. [16]. Its authors relied on the physical correctness of the Dutra's argumentation and were misguided in their mathematical appreciation of the role of supersymmetry in the similar problems. Still, the majority of their argument remains valid. Hence, let us show in the conclusion how the correct physical bound states coincide with Dutra's wavefunctions at certain exceptional couplings and energies.

Obviously, one has to incorporate simply the necessary constraint (7). An inspection, say, of our sample equation (9) reveals that $\psi_1^{(D)}(r)$ satisfies condition (7) *if and only if* its Hermite-polynomial component acquires an exact nodal zero in the origin. In terms of the known numbers $X = X(n, k)$ [calculated as the k th nontrivial zeros of $H_n(X)$, cf. Table I] this requirement, unfortunately, fixes the non-Coulombic coupling as a function of the energy $E = -\beta^4/4$,

$$B = \frac{1}{2} X \beta^3 \neq 0. \quad (10)$$

This makes both these values coupled to the additional (in fact, Magyari's [14]) constraint. As a cubic equation for the

energy E it appears under Eq. (8) in Ref. [1]. This algebraic self-consistency condition must be combined with Eq. (10). The resulting polynomial equation in β (of the twelfth degree) is easily factorized in closed form. The real roots we need are

$$\beta = \beta(n, k) = 2 \sqrt{\frac{-A}{2n+1-X^2(n, k)}}.$$

They all exist for any $A < 0$. This is an important conclusion: let us note that Eq. (8) of Ref. [1] reappears as Eq. (16) in Ref. [16], etc.

For illustration, let us finally fix the scale $A = -1$ and display the first few non-numerical specifications of energies $E = -\beta^4$ and their couplings (10) in Table II. The same parameters are to be used also in the definition (9) of the correct bound-state wave function. *Mutatis mutandis*, the entirely parallel "return to validity" applies also to $\psi_2^{(D)}(r)$ in [1]. We omit the details here, reemphasizing only that both Dutra's expressions $\psi_{1,2}^{(D)}(r)$ are elementary and still satisfy the Schrödinger differential equation, exhibiting also the correct asymptotic behavior. Thus, we may return, say, to the paper by Dutt *et al.* [17], originally motivated by Ref. [1] as well. In the light of our present notes, the importance of the latter paper increases: Its authors have, involuntarily, found and constructed (to the best of our knowledge) *the first* CES example in one dimension.

Years long discussions of the subject with my colleagues in Theory Group of NPI in Rež and with authors of Refs. [16] and [17] contributed to this paper. The reference to the highly relevant paper [9] was kindly communicated to me by A. de Souza Dutra. He also informed me about his correspondence with F. H. Stillinger, the subsequent private communication with whom is also acknowledged.

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