Propagation of squeezed radiation through amplifying or absorbing random media

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We analyze how nonclassical features of squeezed radiation (in particular, the sub-Poissonian noise) are degraded when it is transmitted through an amplifying or absorbing medium with randomly located scattering centra. Both the cases of direct photodetection and of homodyne detection are considered. Explicit results are obtained for the dependence of the Fano factor (the ratio of the noise power and the mean current) on the degree of squeezing of the incident state, on the length and the mean free path of the medium, the temperature, and on the absorption or amplification rate.

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I. INTRODUCTION

Squeezed radiation is in a state in which one of the quadratures of the electric field fluctuates less than the other [1,2]. Such a nonclassical state is useful, because the fluctuations in the photon flux can be reduced below that of a Poisson process — at the expense of enhanced fluctuations in the phase. Sub-Poissonian noise is a delicate feature of the radiation; it is easily destroyed by the interaction with an absorbing or amplifying medium [3]. The noise from spontaneous-emission events is responsible for the degradation of the squeezing.

Because of the fundamental and practical importance, there exists a considerable literature on the propagation of squeezed and other nonclassical states of light through absorbing or amplifying media. We cite some of the most recent papers on this topic [4–10]. The main simplification of these investigations is the restriction to systems in which the scattering is one-dimensional, such as parallel dielectric layers. Each propagating mode can then be treated separately from any other mode. It is the purpose of the present paper to remove this restriction by presenting a general theory for three-dimensional scattering, and to apply it to a medium with randomly located scattering centra.

Our work builds on a previous paper [11], in which we considered the propagation of a coherent state through such a random medium. Physically, the problem considered here is different because a coherent state has Poisson noise, so that the specific nonclassical features of squeezed radiation do not arise in Ref. [11]. Technically, the difference is that a squeezed state, as most other nonclassical states, lacks a diagonal representation in terms of coherent states [1,2]. We cannot therefore directly extend the theory of Ref. [11] to the propagation of squeezed states. The basic idea of our approach remains the same: The photodetection statistics of the transmitted radiation is related to that of the incident radiation by means of the scattering matrix of the medium. The method of random-matrix theory [12] is then used to evaluate the noise properties of the transmitted radiation, averaged over an ensemble of random media with different positions of the scatterers.

The outline of this paper is as follows. In Sec. II we first summarize the scattering formalism and then show how the characteristic function of the state of the transmitted radiation can be obtained from that of the incident state. This allows us to compute the photocount statistics as measured in direct detection (Sec. III) and in homodyne photodetection measurements (Sec. IV). The expressions in Secs. II–IV are generally valid for any incident state. In Sec. V we specialize to the case that the incident radiation is in an ideal squeezed state (also known as a squeezed state of minimal uncertainty, or as a two-photon coherent state [1,2]). The statistics of direct and homodyne measurements are expressed in terms of the degree of squeezing of the incident state. The Fano factor, introduced in Sec. VI, quantifies the degree to which the squeezing has been destroyed by the propagation through an amplifying or absorbing medium. The ensemble average of the Fano factor is then computed using random-matrix theory in Sec. VII. We conclude in Sec. VIII.

II. SCATTERING FORMULATION

We consider an amplifying or absorbing disordered medium embedded in a waveguide that supports $N(\omega)$ propagating modes at frequency ω. The conceptual advantage of embedding the medium in a waveguide is that we can give a scattering formulation in terms of a finite-dimensional matrix. The outgoing radiation in mode n is described by an annihilation operator $a_n^{\rm out}(\omega)$, using the convention that modes $1,2,\ldots,N$ are on the left-hand side of the medium and modes $N+1,\ldots,2N$ are on the right-hand side. The vector $a^{\rm out}$ consists of the operators $a_1^{\rm out},a_2^{\rm out},\ldots,a_{2N}^{\rm out}$. Similarly, we define a vector $a^{\rm in}$ for incoming radiation.

These two sets of operators each satisfy the bosonic commutation relations

$$[a_n(\omega),a_m^\dagger(\omega')]=\delta_{nm}\delta(\omega-\omega'),\quad [a_n(\omega),a_m(\omega')]=0. \tag{2.1}$$

They are related by the input-output relations [13–15]

$$a^{\rm out}(\omega)=S(\omega)a^{\rm in}(\omega)+Q(\omega)b(\omega), \tag{2.2a}$$

$$a^{\rm out}(\omega)=S(\omega)a^{\rm in}(\omega)+V(\omega)c^\dagger(\omega), \tag{2.2b}$$

where the first equation is for an absorbing medium and the second for an amplifying medium. We have introduced the $2N\times 2N$ scattering matrix S, the $2N\times 2N$ matrices Q and V, and the vectors b and c of $2N$ bosonic operators. The scattering matrix can be decomposed into four $N\times N$ reflection and transmission matrices,

$$S=\begin{pmatrix} r' & t' \\ t & r \end{pmatrix}. \tag{2.3}$$

Reciprocity imposes the conditions $t'=t^T$, $r=r^T$, and $r'=r'^T$.

The operators b and c account for spontaneous emission in the medium. They satisfy the bosonic commutation relations (2.1), hence

$$QQ^\dagger=\mathbb{1}-SS^\dagger, \quad VV^\dagger=SS^\dagger-\mathbb{1}. \tag{2.4}$$

Their expectation values are

$$\langle b_n^\dagger(\omega)b_m(\omega')\rangle=\delta_{nm}\delta(\omega-\omega')f(\omega,T), \tag{2.5a}$$

$$\langle c_n(\omega)c_m^\dagger(\omega')\rangle=-\delta_{nm}\delta(\omega-\omega')f(\omega,T). \tag{2.5b}$$

The Bose-Einstein function

$$f(\omega,T)=[\exp(\hbar\omega/kT)-1]^{-1} \tag{2.6}$$

is evaluated at positive temperature T for an absorbing medium and at negative temperature for an amplifying medium.

It is convenient to discretize the frequency in infinitesimally small steps of Δ, so that $\omega_p=p\Delta$, and treat the frequency index p as a separate vector index (in addition to the mode index n). For example, $a_{np}^{\rm out}=a_n^{\rm out}(\omega_p)$ and $S_{np,n'p'}=S_{nn'}(\omega_p)\delta_{pp'}$.

The state of the outgoing radiation is described by the characteristic function

$$\chi_{\rm out}(\eta)=\left\langle :\exp\left(\Delta^{1/2}\sum_{n,p}[a_n^{\rm out\dagger}(\omega_p)\eta_n(\omega_p) - \eta_n^*(\omega_p)a_n^{\rm out}(\omega_p)]\right):\right\rangle$$
$$=\langle:\exp[\Delta^{1/2}(a^{\rm out\dagger}\eta-\eta^\dagger a^{\rm out})]:\rangle, \tag{2.7}$$

where $\langle:\cdots:\rangle$ indicates the expectation value of a normally ordered product of operators $a^{\rm out}$ and $a^{\rm out\dagger}$ (creation operators to the left of the annihilation operators). The vector η has elements $\eta_{np}=\eta_n(\omega_p)$. The density operator of the outgoing radiation is uniquely defined by the characteristic function $\chi_{\rm out}$ [1]. Similarly, the incoming state has a characteristic function,

$$\chi_{\rm in}(\eta)=\langle:\exp[\Delta^{1/2}(a^{\rm in\dagger}\eta-\eta^\dagger a^{\rm in})]:\rangle. \tag{2.8}$$

The characteristic function of the thermal radiation inside an absorbing medium is given by

$$\chi_{\rm abs}(\eta)=\langle:\exp[\Delta^{1/2}(b^\dagger\eta-\eta^\dagger b)]:\rangle$$
$$=\exp\left(-\sum_{n,p}\eta_{np}^*f(\omega_p,T)\eta_{np}\right)$$
$$\equiv\exp(-\eta^\dagger f\eta). \tag{2.9}$$

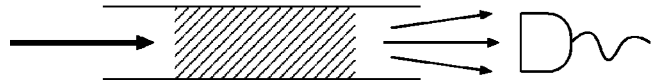

FIG. 1. Schematic illustration of direct detection: Radiation is incident on a random medium (shaded). The transmitted radiation is absorbed by a photodetector.

In the final equality f denotes the matrix with elements $f_{np,n'p'}=\delta_{nn'}\delta_{pp'}f(\omega_p,T)$. For an amplifying medium, replacing b by $c^\dagger$ and normal ordering by antinormal ordering, one finds instead

$$\chi_{\rm amp}(\eta)=\exp(\eta^\dagger f\eta). \tag{2.10}$$

A combination of Eqs. (2.2) and (2.4) with Eqs. (2.7)–(2.10) yields a relationship between the characteristic functions of the incoming and outgoing states,

$$\chi_{\rm out}(\eta)=\exp(-\eta^\dagger(\mathbb{1}-SS^\dagger)f\eta)\chi_{\rm in}(S^\dagger\eta). \tag{2.11}$$

This relation holds both for absorbing and amplifying media, because the difference in sign in the exponent of Eqs. (2.9) and (2.10) is canceled by the difference in sign between $QQ^\dagger=\mathbb{1}-SS^\dagger$ and $VV^\dagger=-(\mathbb{1}-SS^\dagger)$.

III. PHOTOCOUNT DISTRIBUTION

The photocount distribution is the probability $P(n)$ that n photons are absorbed by a photodetector within a certain time τ (see Fig. 1). The factorial cumulants κ_j of $P(n)$ [the first two being $\kappa_1=\overline{n}$ and $\kappa_2=\overline{n(n-1)}-\overline{n}^2$] are most easily obtained from the generating function [2]

$$F(\xi)=\sum_{j=1}^{\infty}\frac{\kappa_j\xi^j}{j!}=\ln\left(\sum_{n=0}^{\infty}(1+\xi)^nP(n)\right). \tag{3.1}$$

The generating function is determined by a normally ordered expectation value [16,17],

$$e^{F(\xi)}=\langle:e^{\xi W}:\rangle, \quad W=\int_0^\tau dt\sum_{n=1}^{2N}d_na_n^{\rm out\dagger}(t)a_n^{\rm out}(t). \tag{3.2}$$

Here $d_n\in[0,1]$ is the detection efficiency of the nth mode and the time-dependent operators are defined as

$$a_n^{\rm out}(t)=(2\pi)^{-1/2}\int_0^\infty d\omega\, e^{-i\omega t}a_n^{\rm out}(\omega). \tag{3.3}$$

Discretizing the frequencies as described in Sec. II, one can write

$$W=\frac{\Delta^2}{2\pi}\int_0^\tau dt\sum_n d_n\sum_{p,p'}e^{i\Delta(p-p')t}a_n^{\rm out\dagger}(\omega_p)a_n^{\rm out}(\omega_{p'}). \tag{3.4}$$

This expression can be simplified in the limit $\tau\to\infty$ of long counting times, when one can set $\Delta=2\pi/\tau$ and use

$$\int_0^\tau e^{i\Delta(p-p')t}dt=\tau\delta_{pp'}. \tag{3.5}$$

Hence, in the long-time limit the generating function is given by

$$e^{F(\xi)}=\left\langle :\exp\left(\xi\Delta\sum_{n,p} d_n a_n^{\rm out\dagger}(\omega_p)a_n^{\rm out}(\omega_p)\right):\right\rangle$$
$$\equiv\langle:\exp(\xi\Delta a^{\rm out\dagger}\mathcal{D}a^{\rm out}):\rangle, \tag{3.6}$$

where we have defined the matrix of detector efficiencies $\mathcal{D}_{np,n'p'}=d_n\delta_{nn'}\delta_{pp'}$.

Comparing Eqs. (2.7) and (3.6), we see that the generating function $F(\xi)$ can be obtained from the characteristic function $\chi_{\rm out}$ by convolution with a Gaussian,

$$e^{F(\xi)}=\frac{1}{\det(-\xi\pi\mathcal{D})}\int d\eta\,\chi_{\rm out}(\eta)\exp\left(\frac{1}{\xi}\eta^\dagger\mathcal{D}^{-1}\eta\right), \tag{3.7}$$

where $\int d\eta$ is an integration over the real and imaginary parts of η. We now substitute the relation (2.11) between $\chi_{\rm out}$ and $\chi_{\rm in}$, to arrive at a relation between $F(\xi)$ and $\chi_{\rm in}$:

$$e^{F(\xi)}=\frac{1}{\det(-\xi\pi\mathcal{D})}\int d\eta\,\chi_{\rm in}(S^\dagger\eta)$$
$$\times\exp\left(\frac{1}{\xi}\eta^\dagger\mathcal{D}^{-1}\eta-\eta^\dagger(\mathbb{1}-SS^\dagger)f\eta\right). \tag{3.8}$$

The fluctuations in the photocount are partly due to thermal fluctuations, which would exist even without any incident radiation. If we denote by $F_{\rm th}(\xi)$ the generating function of these thermal fluctuations, then Eq. (3.8) can be written in the form

$$F(\xi)=F_{\rm th}(\xi)+\ln\left[\frac{1}{\det(\pi M)}\int d\eta\,\chi_{\rm in}(\eta)\exp(-\eta^\dagger M^{-1}\eta)\right], \tag{3.9}$$

$$F_{\rm th}(\xi)=-\ln\det[\mathbb{1}-\xi\mathcal{D}(\mathbb{1}-SS^\dagger)f]. \tag{3.10}$$

We have defined the Hermitian matrix

$$M=-\xi S^\dagger[\mathbb{1}-\xi\mathcal{D}(\mathbb{1}-SS^\dagger)f]^{-1}\mathcal{D}S, \tag{3.11}$$

and we have performed a change of integration variables from η to $S^\dagger\eta$ [with Jacobian $\det(SS^\dagger)$].

The expression (3.10) generalizes the result of Ref. [15] to arbitrary detection-efficiency matrix $\mathcal{D}$. Returning to a continuous frequency, it can be written as (recall that $\Delta=2\pi/\tau$)

$$F_{\rm th}(\xi)=-\frac{\tau}{2\pi}\int_0^\infty d\omega\ln\det(\mathbb{1}-\xi D[\mathbb{1}-S(\omega)S^\dagger(\omega)]$$
$$\times f(\omega,T)), \tag{3.12}$$

where D is a $2N\times2N$ diagonal matrix containing the detection efficiencies d_n on the diagonal ($D_{nm}=d_n\delta_{nm}$). The first two factorial cumulants are

$$\kappa_1^{\rm th}=\tau\int_0^\infty\frac{d\omega}{2\pi}f(\omega,T)\,{\rm tr}\,D[\mathbb{1}-S(\omega)S^\dagger(\omega)], \tag{3.13}$$

$$\kappa_2^{\rm th}=\tau\int_0^\infty\frac{d\omega}{2\pi}f^2(\omega,T){\rm tr}(D[\mathbb{1}-S(\omega)S^\dagger(\omega)])^2. \tag{3.14}$$

Note that all factorial cumulants depend linearly on the detection time τ in the long-time limit.

If only the N modes at one side of the waveguide are detected (with equal efficiency d), then $d_n=0$ for $1\leqslant n\leqslant N$ and $d_n=d$ for $N+1\leqslant n\leqslant 2N$, hence

$$F_{\rm th}(\xi)=-\frac{\tau}{2\pi}\int_0^\infty d\omega\ln\det(\mathbb{1}-\xi d[\mathbb{1}-r(\omega)r^\dagger(\omega)$$
$$-t(\omega)t^\dagger(\omega)]f(\omega,T)), \tag{3.15}$$

in agreement with Ref. [18].

The difference $F(\xi)-F_{\rm th}(\xi)$ contains the noise from the incident radiation by itself as well as the excess noise due to beating of the incident radiation with the vacuum fluctuations. If the incident radiation is in a coherent state, then $\chi_{\rm in}(\eta)=\exp(\alpha^\dagger\eta-\eta^\dagger\alpha)$ for some vector α (called the displacement vector) with elements $\alpha_{np}=\alpha_n(\omega_p)$. Substitution into Eq. (3.9) gives the generating function

$$F(\xi)=F_{\rm th}(\xi)-\alpha^\dagger M\alpha$$
$$=F_{\rm th}(\xi)+\frac{\tau\xi}{2\pi}\int_0^\infty d\omega\,\alpha^\dagger(\omega)S^\dagger(\omega)$$
$$\times\{\mathbb{1}-\xi D[\mathbb{1}-S(\omega)S^\dagger(\omega)]f(\omega,T)\}^{-1}$$
$$\times DS(\omega)\alpha(\omega). \tag{3.16}$$

The first two factorial cumulants are

$$\kappa_1=\tau\int_0^\infty\frac{d\omega}{2\pi}\alpha^\dagger(\omega)S^\dagger(\omega)DS(\omega)\alpha(\omega)+\kappa_1^{\rm th}, \tag{3.17}$$

$$\kappa_2=2\tau\int_0^\infty\frac{d\omega}{2\pi}f(\omega,T)\alpha^\dagger(\omega)S^\dagger(\omega)D[\mathbb{1}-S(\omega)S^\dagger(\omega)]$$
$$\times DS(\omega)\alpha(\omega)+\kappa_2^{\rm th}. \tag{3.18}$$

If the incident coherent radiation is in a single mode m_0 and monochromatic with frequency ω_0, then Eqs. (3.16)–(3.18) simplify for detection in transmission to

$$F(\xi)=F_{\rm th}(\xi)+\tau\xi dI_0(t^\dagger\{\mathbb{1}-\xi d[\mathbb{1}-rr^\dagger-tt^\dagger]$$
$$\times f(\omega_0,T)\}^{-1}t)_{m_0m_0}, \tag{3.19}$$

$$\kappa_1=I_0\tau d[t^\dagger t]_{m_0m_0}+\kappa_1^{\rm th}, \tag{3.20}$$

$$\kappa_1^{\rm th}=\tau d\int_0^\infty\frac{d\omega}{2\pi}f(\omega,T){\rm tr}[\mathbb{1}-r(\omega)r^\dagger(\omega)-t(\omega)t^\dagger(\omega)], \tag{3.21}$$

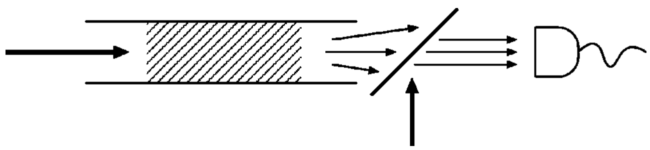

FIG. 2. Schematic illustration of homodyne detection: At the left, radiation is incident on a random medium (shaded). At the right, a strong coherent beam is superimposed onto the transmitted radiation, and the combined radiation is absorbed by a photodetector.

$$\kappa_2 = 2I_0\tau d^2 f(\omega_0,T)[t^\dagger(\mathbb{1}-rr^\dagger-tt^\dagger)t]_{m_0m_0}+\kappa_2^{\rm th}, \quad (3.22)$$

$$\kappa_2^{\rm th} = \tau d^2\int_0^\infty \frac{d\omega}{2\pi} f^2(\omega,T)\,{\rm tr}[\mathbb{1}-r(\omega)r^\dagger(\omega)-t(\omega)t^\dagger(\omega)]^2. \quad (3.23)$$

Here $I_0=(2\pi)^{-1}\int_0^\infty d\omega|\alpha|^2$ is the incident photon flux and the matrices r and t without frequency argument are to be evaluated at frequency ω_0. These are the results of Ref. [11].

IV. HOMODYNE DETECTION

The photocount measurement described in Sec. III (known as direct detection) cannot distinguish between the two quadratures of the electric field. Such phase-dependent information can be retrieved by homodyne detection, i.e., by superimposing a strong probe beam (described by operators $a^{\rm probe}$) onto the signal beam (see Fig. 2). The total radiation incident on the detector is described by the operator

$$a^{\rm total} = \kappa^{1/2}a^{\rm out}+(1-\kappa)^{1/2}a^{\rm probe}, \quad (4.1)$$

where the factor $\sqrt{\kappa}$ accounts for the attenuation of the signal beam by the beam splitter that superimposes it onto the probe beam. (For simplicity we assume a real scalar κ; more generally κ would be a complex coupling matrix.)

The characteristic function of $a^{\rm total}$ is the product of the characteristic functions of $a^{\rm out}$ and $a^{\rm probe}$. We assume that the probe beam is in the coherent state with displacement vector β, having elements $\beta_{np}=\beta_n(\omega_p)$. From Eq. (2.11) one gets

$$\chi_{\rm total}(\eta) = \exp[-\kappa\eta^\dagger(\mathbb{1}-SS^\dagger)f\eta+(1-\kappa)^{1/2}(\beta^\dagger\eta-\eta^\dagger\beta)] \times\chi_{\rm in}(S^\dagger\kappa^{1/2}\eta). \quad (4.2)$$

The generating function $F_{\rm homo}(\xi)$ of the photocount distribution in homodyne detection is given by [cf. Eq. (3.6)]

$$\exp[F_{\rm homo}(\xi)] = \langle{:}\exp(\xi\Delta a^{\rm total\dagger}\mathcal{D}a^{\rm total}){:}\rangle \quad (4.3)$$

$$\approx\exp[\xi(1-\kappa)\beta^\dagger\mathcal{D}\beta]\langle{:}\exp(\Delta^{1/2}\sqrt{\kappa(1-\kappa)}\xi \times[a^{\rm out\dagger}\mathcal{D}\beta+\beta^\dagger\mathcal{D}a^{\rm out}]){:}\rangle. \quad (4.4)$$

In the second approximate equality we have linearized the exponent with respect to $a^{\rm out}$, which is justified if the probe beam is much stronger than the signal beam. The remaining expectation value has the form of a characteristic function if we take ξ purely imaginary, so that $\xi^*=-\xi$. The result is

$$\begin{aligned}F_{\rm homo}(\xi) &= \xi(1-\kappa)\beta^\dagger\mathcal{D}\beta+\ln\chi_{\rm out}(\sqrt{\kappa(1-\kappa)}\xi\mathcal{D}\beta)\\ &=\xi(1-\kappa)\beta^\dagger\mathcal{D}\beta+\kappa(1-\kappa)\xi^2\beta^\dagger\mathcal{D}(\mathbb{1}-SS^\dagger)f\mathcal{D}\beta\\ &\quad+\ln\chi_{\rm in}(\sqrt{\kappa(1-\kappa)}\xi S^\dagger\mathcal{D}\beta).\end{aligned} \quad (4.5)$$

In the second equation we have substituted the relation (2.11) between $\chi_{\rm out}$ and $\chi_{\rm in}$.

V. SQUEEZED RADIATION

We consider the case that the incident radiation is in the ideal squeezed state $|\epsilon,\alpha\rangle=\mathcal{CS}|0\rangle$ [1,2], obtained from the vacuum state $|0\rangle$ by subsequent action of the squeezing operator

$$\mathcal{S} = \exp[\tfrac{1}{2}\Delta(a^{\rm in}\epsilon^* a^{\rm in}-a^{\rm in\dagger}\epsilon a^{\rm in\dagger})] \quad (5.1)$$

and the displacement operator

$$\mathcal{C} = \exp[\Delta^{1/2}(a^{\rm in\dagger}\alpha-\alpha^\dagger a^{\rm in})]. \quad (5.2)$$

As in the preceding sections, we have discretized the frequency, $\omega_p=p\Delta$, and used the vector of operators $a_{np}^{\rm in}=a_n^{\rm in}(\omega_p)$. The complex squeezing parameters $\epsilon_n(\omega)=\rho_n(\omega)e^{i\phi_n(\omega)}$ are contained in the diagonal matrix ϵ with elements $\epsilon_{np,n'p'}=\epsilon_n(\omega_p)\delta_{nn'}\delta_{pp'}$. Similarly, the vector α with elements $\alpha_{np}=\alpha_n(\omega_p)$ contains the displacement parameters.

The characteristic function of the incident radiation is given by [1,19]

$$\chi_{\rm in}(\eta) = \exp[\alpha^\dagger\eta-\eta^\dagger\alpha-\tfrac{1}{4}\eta^T(e^{-i\phi}\sinh 2\rho)\eta -\tfrac{1}{4}\eta^\dagger(e^{i\phi}\sinh 2\rho)\eta^*-\eta^\dagger(\sinh^2\rho)\eta]. \quad (5.3)$$

According to Eq. (2.11), we thus find for the characteristic function of the outgoing radiation

$$\chi_{\rm out}(\eta) = \exp(\alpha^\dagger S^\dagger\eta-\eta^\dagger S\alpha-\tfrac{1}{4}\eta^T S^*(e^{-i\phi}\sinh 2\rho)S^\dagger\eta -\tfrac{1}{4}\eta^\dagger S(e^{i\phi}\sinh 2\rho)S^T\eta^* -\eta^\dagger[f-S(f-\sinh^2\rho)S^\dagger]\eta). \quad (5.4)$$

The generating function $F(\xi)$ of the photocount distribution is obtained from $\chi_{\rm in}$ by convolution with a Gaussian, cf. Eq. (3.7). We find

$$F(\xi) = F_{\rm th}(\xi)-\tfrac{1}{2}\ln\det X-\tfrac{1}{2}\begin{pmatrix}\alpha^*\\ \alpha\end{pmatrix}^T X^{-1}\begin{pmatrix}M\alpha\\ M^*\alpha^*\end{pmatrix}, \quad (5.5)$$

where the matrix X is defined in terms of the matrix M by

$$X = \mathbb{1}+\begin{pmatrix}M\sinh\rho & -Me^{i\phi}\cosh\rho\\ -M^*e^{-i\phi}\cosh\rho & M^*\sinh\rho\end{pmatrix} \times\begin{pmatrix}\sinh\rho & 0\\ 0 & \sinh\rho\end{pmatrix}. \quad (5.6)$$

If squeezing is absent, $\rho=0$, hence $X=\mathbb{1}$ and Eq. (5.5) reduces to the result (3.16) for coherent radiation. For a squeezed vacuum $(\alpha=0)$ one has simply $F(\xi)=F_{\rm th}(\xi)-\frac{1}{2}\ln\det X$.

If the radiation is incident only in mode m_0, then we may compute the matrix inverse and the determinant in Eq. (5.5) explicitly. The matrix $M(\omega)$ defined in Eq. (3.11) may be replaced by its m_0, m_0 element,

$$M_{m_0m_0}(\omega) \equiv m = -\xi(S^\dagger(\mathbb{1}-\xi D[\mathbb{1}-SS^\dagger]f)^{-1}DS)_{m_0m_0}. \tag{5.7}$$

Note that m is real, since it is the diagonal element of a Hermitian matrix. The resulting generating function is

$$F(\xi) = F_{\rm th}(\xi) - \tfrac{1}{2}\tau\int_0^\infty \frac{d\omega}{2\pi}\ln(1+2m\sinh^2\rho - m^2\sinh^2\rho) - \tau\int_0^\infty\frac{d\omega}{2\pi}m|\alpha|^2 \times \frac{1+m\sinh\rho[\sinh\rho+\cosh\rho\cos(2\arg\alpha-\phi)]}{1+2m\sinh^2\rho - m^2\sinh^2\rho}. \tag{5.8}$$

The first two factorial cumulants, for detection in transmission, are

$$\kappa_1 = \kappa_1^{\rm th} + \tau d\int_0^\infty\frac{d\omega}{2\pi}(|\alpha|^2+\sinh^2\rho)[t^\dagger t]_{m_0m_0}, \tag{5.9}$$

$$\kappa_2 = \kappa_2^{\rm th} + 2\tau d^2\int_0^\infty\frac{d\omega}{2\pi}(|\alpha|^2+\sinh^2\rho) \times f[t^\dagger(\mathbb{1}-rr^\dagger-tt^\dagger)t]_{m_0m_0} + \tau d^2\int_0^\infty\frac{d\omega}{2\pi}[t^\dagger t]^2_{m_0m_0} \times[|\alpha\cosh\rho-\alpha^* e^{i\phi}\sinh\rho|^2 - |\alpha|^2 + \sinh^2\rho(\cosh^2\rho+\sinh^2\rho)], \tag{5.10}$$

where $\kappa_1^{\rm th}$ and $\kappa_2^{\rm th}$ are given by Eqs. (3.21) and (3.23).

The generating function for homodyne detection follows from Eqs. (4.5) and (5.4),

$$F_{\rm homo}(\xi) = \xi(1-\kappa)\beta^\dagger\mathcal{D}\beta + \xi\sqrt{\kappa(1-\kappa)} \times(\alpha^\dagger S^\dagger\mathcal{D}\beta + \beta^\dagger\mathcal{D}S\alpha) - \tfrac{1}{4}\xi^2\kappa(1-\kappa) \times[\beta\mathcal{D}S^*(e^{-i\phi}\sinh 2\rho)S^\dagger\mathcal{D}\beta + \beta^\dagger\mathcal{D}S(e^{i\phi}\sinh 2\rho)S^T\mathcal{D}\beta^*] + \xi^2\kappa(1-\kappa)\beta^\dagger\mathcal{D}[f-S(f-\sinh^2\rho)S^\dagger]\mathcal{D}\beta. \tag{5.11}$$

All factorial cumulants except for the first two vanish in the strong-probe approximation. We may simplify the generating function by assuming that the signal beam is incident in a single mode m_0 and that the probe beam is also in a single mode n_0. For detection in transmission one then has the factorial cumulants

$$\kappa_1 = \tau d\int_0^\infty\frac{d\omega}{2\pi}\{(1-\kappa)|\beta|^2 + 2\sqrt{\kappa(1-\kappa)}{\rm Re}[\alpha\beta^* t_{n_0m_0}]\}, \tag{5.12}$$

$$\kappa_2 = -\tau\kappa(1-\kappa)d^2\int_0^\infty\frac{d\omega}{2\pi}{\rm Re}[\beta^{*2}e^{i\phi}t^2_{n_0m_0}]\sinh 2\rho + 2\tau\kappa(1-\kappa)d^2\int_0^\infty\frac{d\omega}{2\pi}|\beta|^2[|t_{n_0m_0}|^2\sinh^2\rho + f(\mathbb{1}-rr^\dagger-tt^\dagger)_{n_0n_0}]. \tag{5.13}$$

VI. FANO FACTOR

For the application of these general formulas we focus our attention on the Fano factor $\mathcal{F}$, defined as the ratio of the noise power $P=\tau^{-1}\,{\rm var}\,n$ and the mean current $\overline{I} = \tau^{-1}\overline{n}$:

$$\mathcal{F} = P/\overline{I} = 1 + \kappa_2/\kappa_1. \tag{6.1}$$

(We have assumed the limit $\tau\to\infty$.) For coherent radiation $\mathcal{F}=1$, corresponding to Poisson statistics. Thermal radiation has $\mathcal{F}>1$ (super-Poissonian). Nonclassical states, such as squeezed states, can have $\mathcal{F}<1$.

We assume that the radiation is incident in a single mode m_0 and is detected in transmission (equal efficiency d per transmitted mode). We consider a frequency-resolved measurement, covering a narrow frequency interval around the central frequency ω_0 of the incident radiation. The thermal contributions $\kappa_1^{\rm th}$ and $\kappa_2^{\rm th}$ may then be neglected, since they are spread out over a wide frequency range. The incident radiation has Fano factor $\mathcal{F}_{\rm in}$, measured in direct detection with unit efficiency. For squeezed radiation, one has

$$\mathcal{F}_{\rm in} = 1 + \frac{|\alpha\cosh\rho - \alpha^* e^{i\phi}\sinh\rho|^2 - |\alpha|^2 + \sinh^2\rho(\cosh^2\rho+\sinh^2\rho)}{|\alpha|^2+\sinh^2\rho}. \tag{6.2}$$

We seek the Fano factor of the transmitted radiation, both for direct detection ($\mathcal{F}_{\rm direct}$) and for homodyne detection ($\mathcal{F}_{\rm homo}$). Combining Eqs. (5.9) and (5.10), we find for direct detection

$$\mathcal{F}_{\rm direct}-1=d(t^\dagger t)_{m_0m_0}(\mathcal{F}_{\rm in}-1)+2d\,f(\omega_0,T)\frac{[t^\dagger(\mathbb{1}-rr^\dagger-tt^\dagger)t]_{m_0m_0}}{(t^\dagger t)_{m_0m_0}}. \tag{6.3}$$

The first term is due entirely to the incident radiation. It is absent for coherent radiation (because then $\mathcal{F}_{\rm in}=0$). The second term is due to the beating of the incident radiation with the vacuum fluctuations. It is independent of the incident radiation and was studied in detail in Ref. [11]. Sub-Poissonian counting statistics, i.e., $\mathcal{F}_{\rm direct}<1$, is in an amplifying medium ($f\leqslant-1$) only possible when

$$2(t^\dagger tt^\dagger t)_{m_0m_0}<(t^\dagger t)^2_{m_0m_0}+2(t^\dagger t)_{m_0m_0}-2(t^\dagger rr^\dagger t)_{m_0m_0}. \tag{6.4}$$

In the absence of reflection ($r=0$) and intermode scattering (t diagonal), this reduces to the well-known condition [20–22] $(t^\dagger t)_{m_0m_0}<2$. Since $(t^\dagger tt^\dagger t)_{m_0m_0}=\Sigma_k|(t^\dagger t)_{m_0k}|^2\geqslant(t^\dagger t)^2_{m_0m_0}$, the presence of intermode scattering decreases the maximally allowed amplification factor $(t^\dagger t)_{m_0m_0}$.

The Fano factor in the strong-probe approximation ($|\beta|\to\infty$) follows from Eqs. (5.12) and (5.13), with the result

$$\begin{aligned}\mathcal{F}_{\rm homo}-1&=2\,d\kappa|t_{n_0m_0}|^2\sinh^2\rho+2\,d\kappa\,f(\omega_0,T)\\&\quad\times(\mathbb{1}-rr^\dagger-tt^\dagger)_{n_0n_0}-d\kappa\\&\quad\times{\rm Re}[e^{i(\phi-2\arg\beta)}t^2_{n_0m_0}]\sinh 2\rho.\end{aligned} \tag{6.5}$$

In the strong-probe approximation, it is independent of α and $|\beta|$. Similarly to Eq. (6.3), the first term is entirely due to the incident radiation, vanishing for coherent radiation ($\rho=0$), and the second term is due the beating with vacuum fluctuations. The additional third term describes the effect of the phase of the probe beam on the measurement. Typically, in a measurement one would vary the phase of the probe beam until the Fano factor is minimized, which occurs when $\arg\beta=\frac{1}{2}\phi+\arg t_{n_0m_0}$. The resulting Fano factor $\mathcal{F}^{\rm min}_{\rm homo}$ is given by

$$\mathcal{F}^{\rm min}_{\rm homo}=1-2\,d\kappa|t_{n_0m_0}|^2e^{-\rho}\sinh\rho+2\,d\kappa\,f(\omega_0,T)(\mathbb{1}-rr^\dagger-tt^\dagger)_{n_0n_0}. \tag{6.6}$$

Squeezing in the outgoing radiation for an amplifying medium is only possible when $|t_{n_0m_0}|^2<2-2(rr^\dagger)_{n_0n_0}-\Sigma_{k\neq m_0}|t_{n_0k}|^2$. The single-mode limit $|t_{n_0m_0}|^2=2$ is thus decreased by both reflection and intermode scattering.

VII. ENSEMBLE AVERAGES

The expressions for the Fano factor given in the preceding section contain the reflection and transmission matrices of the waveguide. These are N-dimensional matrices that depend on the positions of the scatterers inside the waveguide. The distribution of these matrices in an ensemble of disordered waveguides is described by random-matrix theory [12]. Ensemble averages of moments of $rr^\dagger$ and $tt^\dagger$ for $N\gg1$ have been computed by Brouwer [23], as a function of the mean free path l and the amplification (absorption) length $\xi_a=\sqrt{D\tau_a}$, where $1/\tau_a$ is the amplification (absorption) rate and $D=cl/3$ is the diffusion constant. It is assumed that both ξ_a and L are small compared to the localization length Nl but large compared to the mean free path l. Obviously, this requires a large number N of propagating modes. The relative size of L and ξ_a is arbitrary.

As sample-to-sample fluctuations are small for $N\gg1$, we can take in Eq. (6.3) the averages of numerator and denominator separately. The dependence on the index m_0 of the incident mode drops out on averaging, $\langle\cdots\rangle_{m_0m_0}=N^{-1}\langle{\rm tr}\cdots\rangle$. For an absorbing disordered waveguide, we find

$$\mathcal{F}_{\rm direct}=1+\frac{4ld}{3\xi_a\sinh s}(\mathcal{F}_{\rm in}-1)+\frac{d}{2}f(\omega_0,T)\times\left[3-\frac{2s+{\rm cotanh}\,s}{\sinh s}-\frac{s\,{\rm cotanh}\,s-1}{\sinh^2 s}+\frac{s}{\sinh^3 s}\right]. \tag{7.1}$$

We have abbreviated $s=L/\xi_a$. In the limit of strong absorption, $s\to\infty$, the Fano factor approaches the universal limit [24] $\mathcal{F}_{\rm direct}=1+\frac{3}{2}df$. The Fano factor $\mathcal{F}_{\rm in}$ is given by Eq. (6.2) for an incident squeezed state, but Eq. (7.1) is more generally valid for any state of the incident radiation.

The result for an amplifying disordered waveguide follows by the replacement $\tau_a\to-\tau_a$, hence $\xi_a\to i\xi_a$:

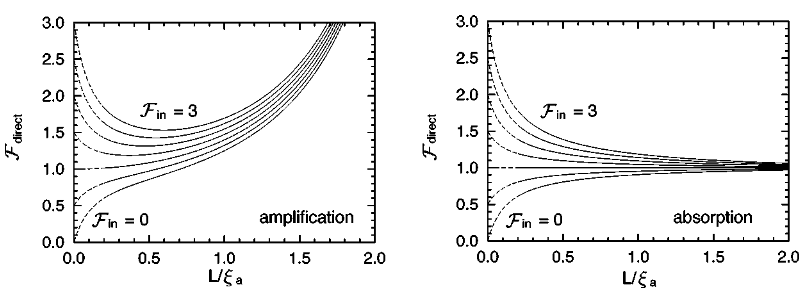

FIG. 3. Average Fano factor $\mathcal{F}_{\rm direct}$ for direct detection as a function of the length of the waveguide. The left panel is for an amplifying medium [Eq. (7.2), $f=-1$], the right panel for an absorbing medium [Eq. (7.1), $f=0$]. In both cases we took $l/\xi_a=0.1$, $d=1$, and values of $\mathcal{F}_{\rm in}$ increasing from 0 to 3 in steps of 0.5. The dotted parts of the curves are extrapolations in the range $L\leqslant l$ that is not covered by Eqs. (7.1) and (7.2).

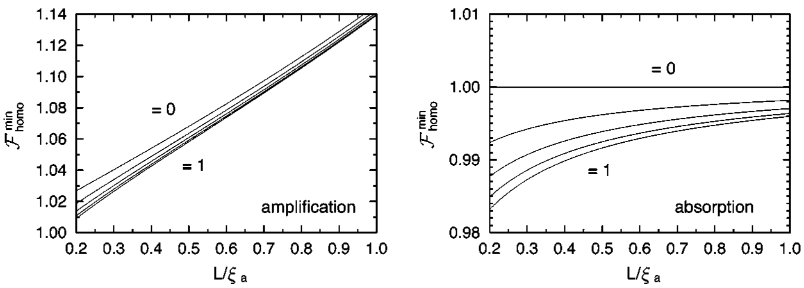

FIG. 4. Average minimal Fano factor for homodyne detection, from Eqs. (7.3) and (7.4). Same parameter values as in Fig. 3, with $N=10$, $\kappa=\frac{1}{2}$, and ρ increasing from 0 to 1 in steps of 0.25. For $L\lesssim l$ the curves extrapolate either to 1 (if $n_0\neq m_0$) or to $1-e^{-\rho}\sinh\rho$ (if $n_0=m_0$). (This extrapolation is not shown in the figure.)

$$\mathcal{F}_{\rm direct}=1+\frac{4ld}{3\xi_a\sin s}(\mathcal{F}_{\rm in}-1)+\frac{d}{2}f(\omega_0,T)\times\left[3-\frac{2s-\operatorname{cotan}s}{\sin s}+\frac{s\operatorname{cotan}s-1}{\sin^2 s}-\frac{s}{\sin^3 s}\right]. \tag{7.2}$$

The Fano factor diverges at the laser threshold $s=\pi$. The function $f(\omega_0,T)$ now has to be evaluated at a negative temperature. For a complete population inversion of the atomic states $f\to-1$.

The minimal Fano factor in homodyne detection is given by Eq. (6.6). The average $\langle|t_{n_0m_0}|^2\rangle$ is again independent of the mode indices, hence it can be replaced by $N^{-2}\langle\operatorname{tr}tt^\dagger\rangle$. For an absorbing waveguide we find

$$\mathcal{F}_{\rm homo}^{\rm min}=1-\frac{8ld\kappa}{3N\xi_a\sinh s}e^{-\rho}\sinh\rho+\frac{8ld\kappa}{3\xi_a}f(\omega_0,T)\times\left[\operatorname{cotanh}s+\frac{1}{\sinh s}\right], \tag{7.3}$$

and for an amplifying waveguide

$$\mathcal{F}_{\rm homo}^{\rm min}=1-\frac{8ld\kappa}{3N\xi_a\sin s}e^{-\rho}\sinh\rho+\frac{8ld\kappa}{3\xi_a}f(\omega_0,T)\left[\operatorname{cotan}s-\frac{1}{\sin s}\right]. \tag{7.4}$$

Measurement of the ensemble average $\mathcal{F}_{\rm homo}^{\rm min}$ requires that for every sample the phase of the probe beam is readjusted so as to minimize the Fano factor. This is common practice in a homodyne measurement. If the phase of the probe beam is fixed, the random phase of $t_{n_0m_0}$ will average to zero the third term in Eq. (6.5). In Eqs. (7.3) and (7.4) this amounts to the substitution $e^{-\rho}\to-\sinh\rho$.

A graphical presentation of the results (7.1)–(7.4) is given in Figs. 3–5. For the absorbing case we have taken $f=0$ (appropriate for optical frequencies at room temperature). For the amplifying case we have taken $f=-1$ (complete population inversion). The formulas above cannot be used for $L\lesssim l$. The values of $\mathcal{F}_{\rm direct}$, $\mathcal{F}_{\rm homo}$, and $\mathcal{F}_{\rm homo}^{\rm min}$ for $L=0$ can be read off from Eqs. (6.3)–(6.6), $\mathcal{F}_{\rm direct}\to1+d(\mathcal{F}_{\rm in}-1)$, $\mathcal{F}_{\rm homo}=1+2\delta_{n_0m_0}d\kappa\sinh^2\rho$, and $\mathcal{F}_{\rm homo}^{\rm min}=1-2\delta_{n_0m_0}d\kappa e^{-\rho}\sinh\rho$. An extrapolation to $L=0$ is shown by dashes in Fig. 3.

The common feature of the Fano factors plotted in Figs. 3–5 is a convergence as the length of the waveguide becomes longer and longer. For an absorbing medium the $L\to\infty$ limit is independent of the state of the incident radiation. For an amplifying medium, complete convergence is preempted by the laser threshold at $L=\pi\xi_a$.

VIII. CONCLUSIONS

In conclusion, we have derived general expressions for the photodetection statistics in terms of the scattering matrix of the medium through which the radiation has propagated. These expressions are particularly well suited for evaluation by means of random-matrix theory, as we have shown by an explicit example, namely the propagation of squeezed radiation through an amplifying or absorbing waveguide. The sub-Poissonian noise that can occur in a squeezed state (characterized by a Fano factor smaller than unity) is destroyed by thermal fluctuations in an absorbing medium or by spontaneous emission in an amplifying medium. The theory presented here describes this interaction of nonclassical radiation with matter in a quantitative way, without the restriction to one-dimensional scattering of earlier investigations.

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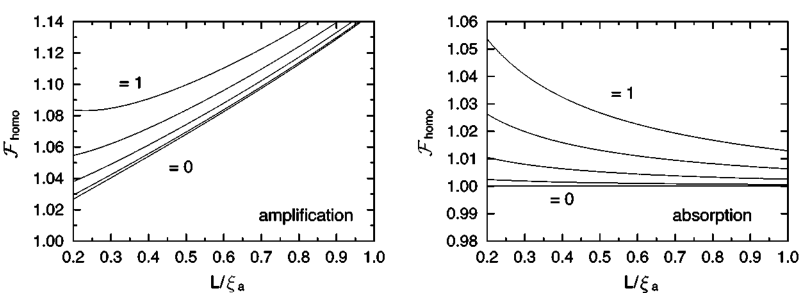

FIG. 5. Average Fano factor for homodyne detection, from Eqs. (7.3) and (7.4) after the substitution $e^{-\rho}\to-\sinh\rho$, otherwise identical to Fig. 4. For $L\lesssim l$ the curves extrapolate either to 1 (if $n_0\neq m_0$) or to $1+\sinh^2\rho$ (if $n_0=m_0$). (This extrapolation is not shown in the figure.)

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