Method of small rotations and effective Hamiltonians in nonlinear quantum optics

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We propose a general method for diagonalizing a wide class of nonlinear Hamiltonians describing different quantum optical models. This method makes use of a nonlinear deformation of the usual su(2) algebra and when some physical parameter, dictated by the particular model under consideration, becomes small, it gives a diagonal effective Hamiltonian that describes correctly the dynamics for arbitrary states and long times. We apply as well the technique to three-level systems interacting with quantum fields, showing that it is possible to engineer resonant interactions through nonresonant processes.

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I. INTRODUCTION

The amount of quantum optical models where the angular-momentum theory [1,2], which is generated by the well-known algebra su(2), plays a crucial role is immense. Just to quote a few relevant problems we recall the time evolution of two coupled harmonic oscillators [3], a two-level atom interacting with a classical external field [4], or the nonrelativistic analysis of the free electron laser [5]. Due to this symmetry, all these models reduce to linear problems that can be efficiently solved in an analytical form.

On the other hand, the quantum dynamics of nonlinear optical processes can be described by simple Hamiltonians with cubic or higher terms in creation and annihilation operators. Among others, typical examples are kth harmonic generation, k-wave mixing, and generalized Dicke models [6]. In all these cases, a nonlinear or deformed su(2) algebra naturally arises [7].

The importance of this deformed structure is that it allows us to handle the problem in close analogy with the usual treatment for an angular momentum. In particular, we get a decomposition of the Hilbert space into direct sums of invariant subspaces, and the dynamical problem generated by the corresponding Hamiltonian can be reduced to the diagonalization of a finite-dimensional matrix.

In spite of this considerable achievement, convenient analytic expressions for the eigenvalues and eigenstates are unknown. Exact solutions can be written in the framework of the algebraic Bethe *ansatz* [8], but the resulting formulas are too unwieldy for practical calculations, and several approximations have been devised relying on specific features of each particular problem under study [9]. Apart from numerical approaches, these approximations assume quasiclassical limits for one or more modes, make use of perturbative techniques, or employ short-time expansions, and therefore their validity is limited to some special regimes.

The aim of this work is to propose a new approach to the problem based on obtaining approximate effective Hamiltonians that can be diagonalized in an exact form. In Sec. II we explain the motivation of the method, whose origins lie in the physics of a particle of spin j in a magnetic field [i.e., a su(2) Hamiltonian], which can be exactly solved by performing an appropriate rotation.

We generalize this idea to the polynomial deformed su(2) algebra. In such a case, the action of this rotation is, in general, rather involved, but when a physical parameter (dictated by the particular model under consideration) becomes small, it generates an approximate effective Hamiltonian that is diagonal and describes correctly the evolution for arbitrary states and even for long times. We apply the method to some relevant nonlinear problems in quantum optics; namely, three-wave mixing, kth harmonic generation, and Dicke model, discussing some interesting dynamical features arising from the description in terms of the corresponding effective Hamiltonians.

In this context, it is worth noting that the su(3) algebra is the natural extension of su(2) to study the dynamical evolution of three-level atoms [10]. Far from being a mathematical curiosity, this evolution is central to the discussion of many physically fascinating problems, such as two-photon coherence [11], resonant Raman scattering [12], superradiance [13], and three-level echoes [14]. As one could expect, when these three-level systems interact with quantum fields, a nonlinear or deformed structure of su(3) naturally emerges. In this case, the advantages of our method are remarkable, as demonstrated in Sec. III: the obtaining of dynamical effective Hamiltonians appears a natural and systematic task, in contrast with the standard approach of adiabatic elimination of variables, which is cumbersome and not one-to-one (in fact, depending on the term eliminated, the final Hamiltonian could be different, as shown in Ref. [15]). Moreover, and perhaps more important, we show how one can manage to *engineer* resonant interactions in this framework. The conclusions of this work are summarized in Sec. IV, while the Appendix discusses at length the range of validity of this approximation.

II. NONLINEAR SU(2) DYNAMICS AND EFFECTIVE HAMILTONIANS

A. Motivation for the method

In order to introduce the physical ideas underlying the method, let us start with the very simple example of a par-

ticle of spin j in a magnetic field (note that a collection of $A=2j$ identical two-level atoms pumped by a classical field is described by the same model). The Hamiltonian for this system has the following form (in units $\hbar=1$, which will be used throughout all this paper)

$$H=\omega S_3+g(S_+ +S_-), \tag{2.1}$$

where g is the coupling constant and the operators S_3, S_+, and S_- constitute a $(2j+1)$-dimensional representation of the su(2) algebra, obeying the usual commutation relations

$$[S_3,S_\pm]=\pm S_\pm,$$

$$[S_+,S_-]=2S_3. \tag{2.2}$$

In the traditional angular momentum basis $|j,m\rangle (m=-j, -j+1,\ldots,j-1,j)$ the operator S_3 is diagonal

$$S_3|j,m\rangle=m|j,m\rangle, \tag{2.3}$$

while the action of the ladder operators $S_\pm$ is nondiagonal and is given by

$$S_\pm|j,m\rangle=[(j\mp m)(j\pm m+1)]^{1/2}|j,m\pm 1\rangle. \tag{2.4}$$

The Hamiltonian (2.1) belongs to the class of the so-called linear Hamiltonians and admits an exact solution. For our purposes, a very convenient way of finding this solution is to apply the unitary transformation

$$U=\exp[\alpha(S_+ -S_-)], \tag{2.5}$$

and by recalling that

$$e^A Be^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots, \tag{2.6}$$

the transformed Hamiltonian, which is unitarily equivalent to the original one, becomes

$$\tilde{H}=UHU^\dagger=[\omega\cos(2\alpha)+2g\sin(2\alpha)]S_3 + \frac{1}{2}[2g\cos(2\alpha)-\omega\sin(2\alpha)](S_+ +S_-). \tag{2.7}$$

Now, the idea is to choose the parameter α so as to cancel the nondiagonal terms appearing in (2.7). This can be accomplished by taking

$$\tan(2\alpha)=\frac{2g}{\omega}, \tag{2.8}$$

and, in consequence, the transformed Hamiltonian reduces to

$$H_{\rm eff}=\omega\sqrt{1+\frac{4g^2}{\omega^2}}S_3. \tag{2.9}$$

Since this effective Hamiltonian is diagonal in the angular-momentum basis, the dynamical problem is completely solved. The key observation for our purpose is that when $\omega \gg g$ we can approximate Eq. (2.8) by $\alpha\simeq g/\omega$, and (2.5) can be substituted by

$$U_{\rm app}=\exp\left[\frac{g}{\omega}(S_+ -S_-)\right]. \tag{2.10}$$

This small rotation *approximately* (i.e., up to second-order terms in g/ω) diagonalizes the original Hamiltonian (2.1), originating the effective Hamiltonian

$$H_{\rm eff}=U_{\rm app}HU_{\rm app}^\dagger=\left(\omega+2\frac{g^2}{\omega}\right)S_3 \tag{2.11}$$

which obviously coincides with the exact solution after expanding (2.9) in a series of g^2/ω^2.

A direct application of the standard time-independent perturbation theory [16] to the original Hamiltonian (2.1) leads immediately to the same results that the effective Hamiltonian (2.11) for the eigenvalues and eigenstates in the same order of approximation. However, we stress that our method is settled in a fully operatorial form, which avoids the tedious work of computing the successive corrections as sums over all the accessible states.

B. Small rotations and effective Hamiltonians

Having in mind the previous example, we shall proceed by considering more involved nonlinear Hamiltonians. Let us start with the general case in which the system admits some integrals of motion N_j and the interaction part of the Hamiltonian can be written in the form

$$H_{\rm int}=\Delta X_3+g(X_+ +X_-), \tag{2.12}$$

where g is a coupling constant, Δ is a parameter usually representing the detuning between frequencies of different subsystems (although it is not necessary), and the operators $X_\pm$ and X_3 maintain the first commutation relation of su(2) in (2.2)

$$[X_3,X_\pm]=\pm X_\pm, \tag{2.13}$$

but the second one is modified in the following way:

$$[X_+,X_-]=P(X_3,N_j), \tag{2.14}$$

where $P(X_3,N_j)$ refers to an arbitrary polynomial function of the diagonal operator X_3 with coefficients perhaps depending on the integrals of motion N_j. This is the origin of the name of polynomial deformations of the su(2) algebra.

Let us suppose that for some physical reasons (depending on the particular model under consideration) the condition

$$\Delta\gg g \tag{2.15}$$

is fulfilled. Then, it is clear that (2.12) is *almost* diagonal in the basis that diagonalizes X_3. In fact, a standard perturbation analysis immediately shows that the first-order corrections introduced by the nondiagonal part $g(X_+ +X_-)$ to the eigenvalues of X_3 vanish and those of second order are pro-

portional to $g/\Delta \ll 1$. According to our discussion for the linear su(2) model, we apply the following unitary transformation to (2.12) (which, in fact, is a *small* nonlinear rotation)

$$U=\exp\left[\frac{g}{\Delta}(X_+ - X_-)\right], \tag{2.16}$$

so that

$$H_{\rm eff}=UH_{\rm int}U^\dagger. \tag{2.17}$$

Keeping terms up to order $(g/\Delta)^2$ we get

$$H_{\rm eff}=\Delta X_3+\frac{g^2}{\Delta}P(X_3,N_j), \tag{2.18}$$

and we have an effective Hamiltonian that is diagonal in the basis of eigenstates of the operator X_3. With this approach, the evolution (as well as the spectral) problem is completely solved. Besides the advantage of having the effective Hamiltonian expressed in an operatorial form, the method has the virtue of generality, since it is valid for any model whose Hamiltonian could be written down in terms of the generators of an arbitrary polynomial deformation of su(2). In order to gain more physical insight into the method, we shall apply it to some relevant models in nonlinear quantum optics.

C. Three-wave mixing

The nonlinear dynamics of the process of three-wave mixing can be described by the Hamiltonian

$$\begin{aligned} H&=H_0+H_{\rm int}\\ &=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+\omega_3 a_3^\dagger a_3+g(a_1 a_2 a_3^\dagger + a_1^\dagger a_2^\dagger a_3), \end{aligned} \tag{2.19}$$

where g is a coupling constant proportional to the second-order susceptibility and a_j ($a_j^\dagger$) are the annihilation (creation) bosonic operators for the jth mode. It can be directly checked that this Hamiltonian admits two integrals of motion

$$\begin{aligned} N_1&=a_1^\dagger a_1+a_2^\dagger a_2+2a_3^\dagger a_3,\\ N_2&=a_1^\dagger a_1-a_2^\dagger a_2. \end{aligned} \tag{2.20}$$

The interaction Hamiltonian can be represented in the general form (2.12) if, for example, we choose the operators $X_\pm$, and X_3 as

$$\begin{aligned} X_+&=a_1^\dagger a_2^\dagger a_3, \quad X_-=a_1 a_2 a_3^\dagger,\\ X_3&=a_1^\dagger a_1, \end{aligned} \tag{2.21}$$

and now Δ is the detuning

$$\Delta=\omega_1+\omega_2-\omega_3. \tag{2.22}$$

After some simple calculations, the corresponding function in Eq. (2.14) is the second-order polynomial

$$\begin{aligned} P(X_3,N_1,N_2)&=(X_3+1)(X_3+1-N_2)-(2X_3-N_2+1)\\ &\times[(N_1+N_2)/2-X_3+1]. \end{aligned} \tag{2.23}$$

Let us consider the limit in which the fields interact in a dispersive cavity, which corresponds to the case [17]

$$|\Delta| \gg g\sqrt{(\overline{n}_1+1)(\overline{n}_2+1)(\overline{n}_3+1)}, \tag{2.24}$$

where $\overline{n}_i (i=1,2,3)$ are the average number of photons in the corresponding modes. We emphasize that this is a limit physically *realizable* in practice. Then, the effective Hamiltonian associated with (2.19) can be conveniently represented as

$$H_{\rm eff}=\Delta a_1^\dagger a_1-\frac{g^2}{\Delta}[a_1^\dagger a_1(a_3^\dagger a_3-a_2^\dagger a_2)+a_2 a_2^\dagger a_3^\dagger a_3], \tag{2.25}$$

whose spectrum is directly known.

As a special case of (2.19), let us examine the situation when $\omega_2=-\omega_3=\omega_0/2$. By using the Schwinger representation, we can introduce the following operators satisfying the su(2) commutation relations

$$\begin{aligned} S_+&=a_2 a_3^\dagger, \quad S_-=a_2^\dagger a_3,\\ S_3&=\frac{1}{2}(a_3^\dagger a_3-a_2^\dagger a_2). \end{aligned} \tag{2.26}$$

Then, (2.19) reduces to

$$H=\omega_1 a_1^\dagger a_1+\omega_0 S_3+g(aS_+ + a^\dagger S_-), \tag{2.27}$$

and the integrals of motion (2.20) imposes that $a_2^\dagger a_2+a_3^\dagger a_3=A$ is a constant. In other words, the trilinear Hamiltonian (2.19) is equivalent to the Dicke model, describing the interaction of a single-mode field of frequency ω_1 with a collection of A identical two-level atoms with transition frequency ω_0.

Therefore, as a consequence of the general result (2.25), we can conclude that the Dicke model in the dispersive limit $|\omega_1-\omega_0| \gg gA\sqrt{(\overline{n}_1+1)}$ can be represented by the effective Hamiltonian

$$H_{\rm eff}=\Delta a_1^\dagger a_1+\frac{g^2}{\Delta}[S_3^2-(2a_1^\dagger a_1+1)S_3-A/2(A/2+1)], \tag{2.28}$$

which coincides with the result obtained previously in Ref. [18] by a quite different method.

D. kth harmonic generation

Let us consider now the following Hamiltonian

$$H=H_0+H_{\rm int}=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+g(a_1^k a_2^{\dagger n}+a_1^{\dagger k}a_2^n), \tag{2.29}$$

which describes the conversion of k photons of the fundamental mode 1 into n photons of the signal mode 2, and admits the integral of motion

$$N=\frac{ka_1^\dagger a_1+na_2^\dagger a_2}{n+k}. \tag{2.30}$$

The interaction Hamiltonian can be represented in the form (2.12) if we introduce

$$X_+=a_1^k a_2^{\dagger n}, \quad X_-=a_1^{\dagger k}a_2^n,$$

$$X_3=\frac{1}{n+k}(a_2^\dagger a_2-a_1^\dagger a_1), \tag{2.31}$$

and now $\Delta=n\omega_2-k\omega_1$. This generates also a nonlinear deformation of su(2) where the polynomial function $P(X_3,N)$ takes the form

$$P(X_3,N)=\phi_k(N-kX_3)\psi_{n-1}(N+nX_3) - \phi_n(N+nX_3)\psi_{k-1}(N-kX_3), \tag{2.32}$$

where $\phi_k(m)=m^{(k)}=m(m-1)\cdots(m-k+1)$ and $\psi_{k-1}(m)=\phi_k(m+k)-\phi_k(m)$ are polynomials of order k and $k-1$, respectively.

According to our general result (2.18), the effective Hamiltonian for this model in the dispersive limit, when $\Delta \gg g(\bar{n}_1+1)^{k/2}(\bar{n}_2+1)^{n/2}$ ($\bar{n}_1$ and $\bar{n}_2$ denote the average photon numbers in modes 1 and 2, respectively) takes the form

$$H_{\rm eff}=\frac{n\omega_2-k\omega_1}{n+k}(a_2^\dagger a_2-a_1^\dagger a_1)+\frac{g^2}{\Delta} \times[\phi_n(a_2^\dagger a_2)\psi_{k-1}(a_1^\dagger a_1)-\phi_k(a_1^\dagger a_1)\psi_{n-1}(a_2^\dagger a_2)]. \tag{2.33}$$

Except for the case $n=k=1$, this effective Hamiltonian is still nonlinear on operators $a_1^\dagger a_1$ and $a_2^\dagger a_2$. Perhaps, the most important example of this kind of processes would be the second-harmonic generation, that corresponds to $k=2$ and $n=1$. In such a case, the nonlinear part of the Hamiltonian (2.33) takes the form

$$H_{\rm eff}^{(2)}=\frac{g^2}{\Delta}[4a_2^\dagger a_2 a_1^\dagger a_1-(a_1^\dagger a_1)^2], \tag{2.34}$$

and we think this is an important result, since it shows that in the dispersive limit, the second-harmonic generation behaves just as a Kerr media [19] in the presence of a dynamical Stark shift [20], explaining some interesting physical effects previously discovered for this model, such as the generation of Schrödinger's cat states [21].

E. Strong-field Dicke model

Until now we have analyzed only Hamiltonians describing dispersive interactions. Nevertheless, the above approach can be applied to the resonant case too. As a relevant example, we consider the Dicke model [described by (2.27)] for the case of exact resonance between the field and the atoms and in the strong-field limit. The interaction Hamiltonian can be written as

$$H_{\rm int}=g(aS_+ + a^\dagger S_-), \tag{2.35}$$

and admits the integral of motion

$$N=a^\dagger a+S_3. \tag{2.36}$$

It was shown in Ref. [22] that when $\sqrt{\bar{n}}\gg A$, (2.35) can be diagonalized in the field space and expanded in a series of the *small* operator

$$\varepsilon=\frac{1}{2\sqrt{n-A/2+1/2}}, \tag{2.37}$$

where $n=a^\dagger a$, in the form

$$H=\frac{1}{\varepsilon}S_x-\varepsilon\{S_3,S_x\}-\frac{\varepsilon^3}{2}\{S_3,\{S_3,S_x\}\}+\mathcal{O}(\varepsilon^5), \tag{2.38}$$

where $\{,\}$ stands for the anticommutator and $S_\pm=S_x\pm iS_y$. It is obvious that the second term in (2.38) does not contribute, in first order, to the eigenvalues of the above Hamiltonian and, therefore, can be eliminated by some small rotation. To make the direct reduction of the Hamiltonian (2.38) to the form (2.12) we first apply the rotation

$$U_y=\exp\left(i\frac{\pi}{2}S_y\right), \tag{2.39}$$

and then the transformed Hamiltonian $\tilde{H}=U_yHU_y^\dagger$ takes the form

$$\tilde{H}=\frac{1}{\varepsilon}X_3+\varepsilon(X_+ + X_-)-\frac{\varepsilon^3}{2}\{S_x,X_+ + X_-\}, \tag{2.40}$$

where we have introduced the operators

$$X_+=\frac{1}{2}\{S_3,S_+\}, \quad X_-=\frac{1}{2}\{S_3,S_-\},$$

$$X_3=S_3, \tag{2.41}$$

and the associated polynomial is

$$P(X_3,\mathbf{S}^2)=4X_3^3-\frac{1}{2}X_3(4\mathbf{S}^2-1), \tag{2.42}$$

with

$$\mathbf{S}^2=\frac{A}{2}\left(\frac{A}{2}+1\right). \tag{2.43}$$

Now, we can apply to this form the general result (2.18) and obtain the effective Hamiltonian (we write it down directly in terms of atomic operators)

$$\tilde{H}_{\rm eff}=\frac{1}{\varepsilon}S_3+\varepsilon^3P(X_3=S_3,\mathbf{S}^2)-\frac{\varepsilon^3}{2}\,\mathrm{diag}\,\{S_x,\{S_x,S_3\}\}, \tag{2.44}$$

where we have taken the diagonal part of the last term in (2.40) [which does not transform under (2.16), but contributes to corrections of order ε^3 to the eigenvalues of the initial Hamiltonian]. Finally, we perform the inverse transformation of (2.39) and we get

$$H_{\rm eff}=\frac{1}{\varepsilon}S_x+\varepsilon^3[5S_x^3-(3\mathbf{S}^2-1)S_x]. \tag{2.45}$$

This effective Hamiltonian describes the dynamics for times $gt\leqslant\overline{n}^{5/2}$ and is nonlinear on the atomic operators. It follows that the atomic collective effects in the strong-field limit play a relevant role for times of order $gt\sim\overline{n}^{3/2}$ (we recall that up to times $gt\leqslant\overline{n}$, which include various revivals of atomic population, the collective effects are not very important in such a limit, since they induce only a rescaling of the average values [23]). This leads, for example, to the modulations of the collapse-revival structure appearing in the initial strong coherent field and also to the modulation of the Rabi oscillations for a field initially in a number state.

III. NONLINEAR SU(3) DYNAMICS AND EFFECTIVE HAMILTONIANS

A. The description of three-level atoms in terms of su(3) dynamics

The method of approximate diagonalization embodied in Eqs. (2.12), (2.16), and (2.18) can be applied not only to Hamiltonians having a nonlinear su(2) structure, but also to quantum systems with a more complicated algebraic structure.

In what follows we consider Hamiltonians that can be represented in terms of the su(3) algebra. This algebraic structure naturally arises when describing atomic systems with three relevant energy levels. It is well known that in this case three possible configurations (commonly called Ξ, V, and Λ) are admissible [10]. The Hamiltonian governing the evolution of a collection of A identical three-level atoms (for definiteness, we consider the case of a cascade or Ξ configuration) pumped by a classical field has the form

$$H=H_0+H_{\rm int},=E_1S^{11}+E_2S^{22}+E_3S^{33}+g_{12}(S_+^{12}+S_-^{12}) +g_{23}(S_+^{23}+S_-^{23}), \tag{3.1}$$

where $S^{ii}(i=1,2,3)$ are the population operators of the ith energy level, and $S_\pm^{ij}(i<j)$ describe transitions between levels i and j. The operators $(S^{11},S^{22},S_\pm^{12})$ and $(S^{22},S^{33},S_\pm^{23})$ form two u(2) subalgebras (each one of them describes transitions $1\leftrightarrow2$ and $2\leftrightarrow3$, independently) and accordingly they satisfy the commutation relations

$$[S^{11},S_\pm^{12}]=\mp S_\pm^{12},\quad [S^{22},S_\pm^{12}]=\pm S_\pm^{12},$$
$$[S_+^{12},S_-^{12}]=S^{22}-S^{11},$$
$$[S^{22},S_\pm^{23}]=\mp S_\pm^{23},\quad [S^{33},S_\pm^{23}]=\pm S_\pm^{23},$$
$$[S_+^{23},S_-^{23}]=S^{33}-S^{22}. \tag{3.2}$$

Nevertheless, because the transitions $1\leftrightarrow2$ and $2\leftrightarrow3$ are not physically independent, one needs to add to the above relations the following ones:

$$[S_+^{12},S_+^{23}]=-S_+^{13},\quad [S_-^{12},S_-^{23}]=S_-^{13},\quad [S_+^{12},S_-^{23}]=0, \tag{3.3}$$

where the operators $S_\pm^{13}$ have the meaning of transition operators between levels 1 and 3 [note that these transitions cannot appear in the Hamiltonian (3.1) simultaneously with $1\leftrightarrow2$ and $2\leftrightarrow3$, due to the existence of dipole selection rules]. The operators $S^{kk},S_\pm^{ij},(i,j=1,2,3)$ form a $(A+1)(A+2)/2$-dimensional representation of the u(3) algebra.

The sum $S^{11}+S^{22}+S^{33}=A$ is an integral of motion and determines the total number of atoms, which allows us to rewrite the free Hamiltonian H_0 as

$$H_0=-\Delta_{12}S^{11}+\Delta_{23}S^{33}+E_2A, \tag{3.4}$$

with

$$\Delta_{12}=E_2-E_1,\quad \Delta_{23}=E_3-E_2. \tag{3.5}$$

We note here that the Hamiltonian (3.1) can be rewritten in terms of the usual su(3) algebra by introducing traceless operators $S_3^{12}=(S^{22}-S^{11})/2$ and $S_3^{23}=(S^{33}-S^{22})/2$.

It is clear that if $|g_{12}/\Delta_{12}|\ll1$, the nondiagonal term $g_{12}(S_+^{12}+S_-^{12})$ in (3.1) can be eliminated [up to terms of order $(g_{12}/\Delta_{12})^2$] by a transformation analogous to (2.16) devised for su(2), with

$$U_{12}=\exp\left[\frac{g_{12}}{\Delta_{12}}(S_+^{12}-S_-^{12})\right]. \tag{3.6}$$

Nevertheless, in contrast to the su(2) case, this transformation necessarily generates [through the commutator with $(S_+^{23}+S_-^{23})$] a term proportional to $(S_+^{13}+S_-^{13})$ (i.e., transitions between levels 1 and 3, which were absent in the initial Hamiltonian).

If the physical conditions enables also the condition $|g_{23}/\Delta_{23}|\ll1$ to be satisfied, the term $g_{23}(S_+^{23}+S_-^{23})$ can be removed by the transformation

$$U_{23}=\exp\left[\frac{g_{23}}{\Delta_{23}}(S_+^{23}-S_-^{23})\right]. \tag{3.7}$$

In what follows, we extend these ideas to the case of polynomial deformed su(3) algebra.

B. Effective Hamiltonians for three-level systems

Having in mind the previous analysis, and in analogy with the developments for su(2) in Sec. II, we shall proceed by considering more involved nonlinear models in su(3). Let us start with the typical Hamiltonian

$$H=E_1X^{11}+E_2X^{22}+E_3X^{33}+g_{12}(X_+^{12}+X_-^{12}) +g_{23}(X_+^{23}+X_-^{23}), \tag{3.8}$$

where the operators X^{ij} satisfy the commutation relations

$$[X^{ii},X^{kk}]=0,\quad [X_+^{ij},X_+^{ik}]=0,\quad [X_-^{ij},X_-^{ik}]=0,$$
$$[X_+^{ij},X_+^{jk}]=-Y_+^{ik},\quad [X_-^{ij},X_-^{jk}]=Y_-^{ik},j>k, \tag{3.9}$$
$$[X^{kk},X_\pm^{ij}]=\pm X_\pm^{ij}(\delta_{kj}-\delta_{ik}),$$
$$[X^{kk},Y_\pm^{ij}]=\pm Y_\pm^{ij}(\delta_{kj}-\delta_{ik}),$$

which are the same as those of the usual u(3) algebra in (3.2), but the rest of them are modified in the following way:

$$[X_+^{ij},X_-^{ij}]=P(X^{ii},X^{kk}),$$
$$[Y_+^{ij},Y_-^{ij}]=Q(X^{ii},X^{kk}), \tag{3.10}$$

where $P(X^{ii},X^{kk})$ and $Q(X^{ii},X^{kk})$ are arbitrary polynomials function of the diagonal operators X^{ii} $(i=1,2,3)$ and define a polynomial deformation of u(3). The Hamiltonian (3.8) admits the following integral of motion:

$$N=X^{11}+X^{22}+X^{33}. \tag{3.11}$$

It proves convenient to rewrite (3.8) as

$$H=E_2N+H_{\rm int}, \tag{3.12}$$

with

$$H_{\rm int}=-\Delta_{12}X^{11}+\Delta_{23}X^{33}+g_{12}(X_+^{12}+X_-^{12})+g_{23}(X_+^{23}+X_-^{23}) \tag{3.13}$$

and the detunings Δ_{12} and Δ_{23} are defined in Eq. (3.5).

Now, let us suppose that $|\Delta_{12}|\gg g_{12}$. It is clear from our previous analysis that by applying the small nonlinear rotation

$$U_{12}=\exp\left[\frac{g_{12}}{\Delta_{12}}(X_+^{12}-X_-^{12})\right] \tag{3.14}$$

we can eliminate, up to order $(g_{12}/\Delta_{12})^2$, the interaction term $g_{12}(X_+^{12}+X_-^{12})$, representing transitions $1\leftrightarrow 2$, and obtain the effective Hamiltonian

$$H_{\rm eff}^{(1)}=-\Delta_{12}\ X^{11}+\Delta_{23}X^{33}+g_{23}(X_+^{23}+X_-^{23})$$
$$-\frac{g_{12}g_{23}}{\Delta_{12}}(Y_+^{13}+Y_-^{13})+\frac{g_{12}^2}{\Delta_{12}}P(X^{11},X^{22},N)$$
$$+\frac{g_{12}g_{23}}{\Delta_{12}}([X_+^{12},X_-^{23}]+[X_+^{23},X_-^{12}]). \tag{3.15}$$

It is worth noting that by eliminating the transitions $1\leftrightarrow 2$, we have generated an effective transition $1\leftrightarrow 3$ (represented by the operators $Y_\pm^{13}$), which was absent in the starting Hamiltonian.

It is obvious that if, in addition $\Delta_{23}\gg g_{23}$, the nondiagonal term $g_{23}(X_+^{23}+X_-^{23})$ can be also eliminated in (3.15) by applying a second small rotation

$$U_{23}=\exp\left[\frac{g_{23}}{\Delta_{23}}(X_+^{23}-X_-^{23})\right]. \tag{3.16}$$

The transformed Hamiltonian becomes then

$$H_{\rm eff}^{(2)}=-\Delta_{12}X^{11}+\Delta_{23}X^{33}+\frac{g_{12}g_{23}}{\Delta_{12}}(Y_+^{13}+Y_-^{13})$$
$$+\frac{g_{12}g_{23}}{\Delta_{12}}([X_+^{12},X_-^{23}]+[X_+^{23},X_-^{12}])$$
$$+\frac{g_{12}^2}{\Delta_{12}}P(X^{11},X^{22},N)+\frac{g_{23}^2}{\Delta_{23}}P(X^{22},X^{33},N). \tag{3.17}$$

We could further proceed by eliminating the third and/or fourth terms in this equation in an analogous way. But, as we shall see in a moment, this possibility strongly depends on the resonance conditions satisfied by Δ_{12} and Δ_{23}. As we did in Sec. II, we shall apply these general considerations to some relevant models of broad interest in quantum optics.

C. Three-level atoms interacting with quantum fields

Let us consider the interaction of a collection of A identical three-level atoms in a cascade configuration with a single-mode quantum field of frequency ω_f. The Hamiltonian of this model is

$$H=\omega_f a^\dagger a+E_1S^{11}+E_2S^{22}+E_3S^{33}+g_{12}(aS_+^{12}+a^\dagger S_-^{12})$$
$$+g_{23}(aS_+^{23}+a^\dagger S_-^{23}). \tag{3.18}$$

Apart from $S^{11}+S^{22}+S^{33}=A$, this Hamiltonian admits now the integral of motion

$$N=a^\dagger a+S^{33}-S^{11}, \tag{3.19}$$

and thus, it can be recast as

$$H=\omega_f N+E_2A+H_{\rm int}, \tag{3.20}$$

where $H_{\rm int}$ has exactly the same form of (3.13) with

$$X^{11}=S^{11},\quad X^{33}=S^{33},$$
$$X_+^{12}=aS_+^{12},\quad X_+^{23}=aS_+^{23},$$
$$\Delta_{12}=E_2-E_1-\omega_f,\quad \Delta_{23}=E_3-E_2-\omega_f. \tag{3.21}$$

If we assume the dispersive regime in which $|\Delta_{12}|\gg g_{12}A\sqrt{\bar{n}}$, where $\bar{n}$ is the average number of photons in the field, we can apply the general transformation (3.14), and obtain the effective Hamiltonian just as in Eq. (3.15), provided we make the identifications

$$Y_+^{13}=a^2S_+^{13},\quad Y_-^{13}=a^{\dagger 2}S_-^{13},$$
$$P(X^{11},X^{22})=S_+^{12}S_-^{12}+a^\dagger a(S^{22}-S^{11}). \tag{3.22}$$

In this particular case we can still try to eliminate the transitions $1\leftrightarrow 3$ [represented by the term $(Y_+^{13}+Y_-^{13})$] in (3.15) by applying a further rotation

$$U_{13}=\exp[\varphi(Y_+^{13}-Y_-^{13})], \tag{3.23}$$

where the parameter φ in the above equation must be chosen as

$$\varphi=\frac{g_{12}g_{23}}{\Delta_{12}(\Delta_{12}+\Delta_{23})}. \tag{3.24}$$

Taking into account that $\Delta_{12}+\Delta_{23}=E_3-E_1-2\omega$, we have that $\varphi\ll 1$ only if the transition $1\leftrightarrow 3$ is far from the two-photon resonance condition. If this is the case, the term $(Y_+^{13}+Y_-^{13})$ is eliminated by this transformation and no additional contributions of order g/Δ will appear in (3.15). Then the effective Hamiltonian takes the form

$$H_{\rm eff}^{(1)}=-\Delta_{12}X^{11}+\Delta_{23}X^{33}+g_{23}(X_+^{23}+X_-^{23})+\frac{g_{12}^2}{\Delta_{12}}P(X^{11},X^{22}) + \frac{g_{12}g_{23}}{\Delta_{12}}([X_+^{12},X_-^{23}]+[X_+^{23},X_-^{12}]). \tag{3.25}$$

Note that the last term in (3.25) can now by eliminated by means of the small rotation

$$U=\exp\left[\frac{g_{12}g_{23}}{\Delta_{12}(\Delta_{12}-\Delta_{23})}([X_+^{12},X_-^{23}]+[X_+^{23},X_-^{12}])\right], \tag{3.26}$$

due to the fact that $|\Delta_{12}-\Delta_{23}|=|E_1+E_3-2E_2|=|E_3-E_1-\omega|\gg g_{12}$ and no additional terms will appear in (3.25).

Let us impose the resonance condition $\Delta_{23}=0$ (i.e., resonant interaction between levels 2–3 and the field). Putting $S^{11}=0$ [absence of initial population in level 1, which will be conserved due to the fact that S^{11} is an integral of motion for (3.25)], we obtain

$$P(X^{11},X^{22})=S^{22}(a^\dagger a+1). \tag{3.27}$$

Thus, we conclude that the effective Hamiltonian that describes the resonant interaction of a collection of two-level atoms with a single-mode quantum field, taking into account the existence of an off-resonant level, has the form

$$H_{\rm eff}^{(1)}=g_{23}(aS_+^{23}+a^\dagger S_-^{23})+\frac{g_{12}^2}{\Delta_{12}}S^{22}(a^\dagger a+1). \tag{3.28}$$

This means that the far-lying level produces a mark in the system in the form of a dynamical Stark-shift term.

Let us now envisage the very different situation in which the two-photon resonance condition between levels 1–3 is fulfilled: $E_3-E_1=2\omega$. This means that the transition generated by the operators $Y_\pm^{13}$ cannot be removed. Now, we have $\Delta_{12}=-\Delta_{23}$, and the term $g_{23}(X_+^{23}+X_-^{23})$ in (3.15), that generates transition between levels 2–3, can be eliminated by the transformation (3.16) with the rotation parameter $|g_{23}/\Delta_{23}|\ll 1$. Moreover, the last term in (3.15) can be eliminated once again without any additional contribution. Therefore, we obtain as an effective Hamiltonian for this case

$$H_{\rm eff}^{(2)}=-\Delta_{12}(X^{11}+X^{33})-\frac{g_{12}g_{23}}{\Delta_{12}}(Y_+^{13}+Y_-^{13}) + \frac{g_{12}^2}{\Delta_{13}}P(X^{11},X^{22})-\frac{g_{23}^2}{\Delta_{12}}P(X^{22},X^{33}), \tag{3.29}$$

where

$$P(X^{22},X^{33})=S_+^{23}S_-^{23}+a^\dagger a(S^{33}-S^{22}). \tag{3.30}$$

Finally, by imposing the condition of the absence of an initial population in level 2, we obtain the effective two-photon Dicke Hamiltonian including the dynamical Stark shift [15]

$$H_{\rm eff}^{(2)}=\frac{g_{12}g_{23}}{\Delta_{12}}(a^2S_+^{13}+a^{\dagger 2}S_-^{13})+\left(S_3^{13}+\frac{A}{2}\right) \times\left[a^\dagger a\left(\frac{g_{23}^2}{\Delta_{12}}-\frac{g_{12}^2}{\Delta_{12}}\right)+\frac{g_{23}^2}{\Delta_{12}}\right]+A\frac{g_{12}^2}{\Delta_{12}}a^\dagger a, \tag{3.31}$$

where we have inverted the common sign of the Hamiltonian.

IV. ENGINEERING RESONANT INTERACTIONS THROUGH NONRESONANT PROCESSES

The dynamics generated by su(3) is, obviously, richer than that of su(2), given the existence of a greater number of physical degrees of freedom. In this section, we wish to show how the method of small rotations can be used to tailor resonant interactions from nonresonant process.

Instead of discussing an abstract formalism, we shall illustrate the main idea by resorting to relevant examples. To this end, let us start from the process of the nonresonant three-wave mixing in the presence of nonresonant linear mode conversion. The Hamiltonian describing this system has the form

$$H=\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2+\omega_3a_3^\dagger a_3+g_1(a_1a_2a_3^\dagger+a_1^\dagger a_2^\dagger a_3) + g_2(a_1a_2^\dagger+a_1^\dagger a_2), \tag{4.1}$$

and does not admit any integral of motion. When the process of three-photon mixing is far-off resonant, $|\Delta_1|\gg g_1\sqrt{(\bar{n}_1+1)(\bar{n}_2+1)(\bar{n}_3+1)}$, the detuning being $\Delta_1=\omega_3-\omega_2-\omega_1$, and we can eliminate the interaction term $(a_1a_2a_3^\dagger+a_1^\dagger a_2^\dagger a_3)$ by applying the small rotation

$$U_1=\exp\left[\frac{g_1}{\Delta_1}(a_1a_2a_3^\dagger-a_1^\dagger a_2^\dagger a_3)\right]. \tag{4.2}$$

The transformed Hamiltonian $H_{\rm eff}^{(1)}=U_1HU_1^\dagger$ takes the form

$$H_{\rm eff}^{(1)}=\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2+\omega_3a_3^\dagger a_3+g_2(a_1a_2^\dagger+a_1^\dagger a_2) + \frac{g_1g_2}{\Delta_1}(a_1^2a_3^\dagger+a_1^{\dagger 2}a_3)+\frac{g_1g_2}{\Delta_1}(a_2^2a_3^\dagger+a_2^{\dagger 2}a_3) + \frac{g_1^2}{\Delta_1}P(a_1^\dagger a_1,a_2^\dagger a_2,a_3^\dagger a_3), \tag{4.3}$$

where

$$P(a_1^\dagger a_1, a_2^\dagger a_2, a_3^\dagger a_3) = a_2 a_2^\dagger (a_3^\dagger a_3 - a_1^\dagger a_1) + a_3^\dagger a_3 a_1 a_1^\dagger . \tag{4.4}$$

If we impose the condition that the process of linear mode conversion is also out of resonance, $|\Delta_2| \gg g_2\sqrt{(\bar{n}_1+1)(\bar{n}_2+1)}$ with $\Delta_2=\omega_2-\omega_1$, then the term $a_1 a_2^\dagger + a_1^\dagger a_2$ can be also eliminated by the transformation

$$U_2 = \exp\left[\frac{g_2}{\Delta_2}(a_1 a_2^\dagger - a_1^\dagger a_2)\right]. \tag{4.5}$$

Therefore we obtain for the Hamiltonian $H_{\rm eff}^{(2)} = U_2 H_{\rm eff}^{(1)} U_2^\dagger$

$$\begin{aligned} H_{\rm eff}^{(2)} = {} & \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \omega_3 a_3^\dagger a_3 + \frac{g_1^2}{\Delta_1} P(a_1^\dagger a_1, a_2^\dagger a_2, a_3^\dagger a_3) \\ & + \frac{g_2^2}{\Delta_2} Q(a_1^\dagger a_1, a_2^\dagger a_2) + \frac{g_1 g_2}{\Delta_1}(a_1^2 a_3^\dagger + a_1^{\dagger 2} a_3) \\ & + \frac{2 g_1 g_2}{\Delta_1}(a_2^2 a_3^\dagger + a_2^{\dagger 2} a_3), \end{aligned} \tag{4.6}$$

where

$$Q(a_1^\dagger a_1, a_2^\dagger a_2) = a_2 a_2^\dagger - a_1^\dagger a_1 . \tag{4.7}$$

Now it is easy to observe that if we can satisfy the resonant condition $2\omega_1=\omega_3$, then (due to the previous nonresonant condition) the field 2 is far-off resonant with the fields 1 and 3. Thus we can remove the last term in (4.6) by a similar transformation [which does not add any additional term to (4.6) of the order g/Δ], meanwhile the resonant interaction described by $a_1^2 a_3^\dagger + a_1^{\dagger 2} a_3$ cannot be eliminated. Imposing the condition that field 2 is initially unexcited, we get $P(a_1^\dagger a_1, a_2^\dagger a_2, a_3^\dagger a_3) = a_3^\dagger a_3 a_1 a_1^\dagger$, $Q(a_1^\dagger a_1, a_2^\dagger a_2) = a_1^\dagger a_1 (a_1^\dagger a_1 - 1)$ (because $a_2^\dagger a_2$ is now an integral of motion) and the effective Hamiltonian takes the form

$$\begin{aligned} H_{\rm eff}^{(2)} = {} & \omega_1 a_1^\dagger a_1 + 2\omega_1 a_3^\dagger a_3 + \frac{g_1 g_2}{\Delta_1}(a_1^2 a_3^\dagger + a_1^{\dagger 2} a_3) \\ & + \frac{g_1^2}{\Delta_1}(a_3^\dagger a_3 a_1 a_1^\dagger) - \frac{g_2^2}{\Delta_1} a_1^\dagger a_1 . \end{aligned} \tag{4.8}$$

The point is that this effective Hamiltonian [which is essentially different from the initial one (4.1)] describes the dynamics of the (quasi) resonant process of second-harmonic generation (this process can be easily made resonant by changing slightly the initial resonance conditions).

One can treat in a similar way more and more involved examples of the initial system. Let us mention the process of second-harmonic generation and simultaneous three-photon mixing. The Hamiltonian describing this system has the form

$$\begin{aligned} H = {} & \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \omega_3 a_3^\dagger a_3 + g_1(a_1 a_2 a_3^\dagger + a_1^\dagger a_2^\dagger a_3) \\ & + g_2(a_1^2 a_2^\dagger + a_1^{\dagger 2} a_2) \end{aligned} \tag{4.9}$$

and does not admit any integral of motion. By repeating the same steps as before in the dispersive limit for both processes and imposing that the frequencies of fields 1 and 3 satisfy the resonance condition $3\omega_1=\omega_3$, which means that the field 2 is far-off resonant with fields 1 and 3, we obtain the final form of the corresponding effective Hamiltonian as

$$\begin{aligned} H_{\rm eff}^{(2)} = {} & \omega_1 a_1^\dagger a_1 + 3\omega_1 a_3^\dagger a_3 + \frac{g_1 g_2}{\Delta_1}(a_1^3 a_3^\dagger + a_1^{\dagger 3} a_3) \\ & + \frac{g_1^2}{\Delta_1}(a_3^\dagger a_3 a_1 a_1^\dagger) + \frac{g_2^2}{\Delta_1} a_1^\dagger a_1 (a_1^\dagger a_1 - 1). \end{aligned} \tag{4.10}$$

The above effective Hamiltonian describes the (quasi) resonant process of third-harmonic generation. In summary, we see that by imposing some resonance conditions on the initial model, we obtain that the resulting dynamics is governed by an effective Hamiltonian that has a quite different form.

V. CONCLUSIONS

In this paper we have investigated in a systematic way a nonperturbative method of small rotations that has allowed us to obtain the description of a quantum model in terms of effective Hamiltonians that can be diagonalized in an exact form.

The approach is inspired by simple linear models having a su(2) symmetry. In such a case, the small-rotation method gives the exact solution in a straightforward way. However, when dealing with nonlinear optical processes, a nonlinear (or polynomial deformation) of the su(2) algebra naturally arises. Unfortunately, the action of the proposed rotation is now rather involved. But when an appropriate (controllable) parameter becomes small, the method provides an easy and elegant solution, free from the drawbacks of other previous approximations.

We have applied this technique to some relevant nonlinear problems in quantum optics, discussing some nontrivial dynamical features derived from the corresponding effective Hamiltonians.

We have extended also the method to the nonlinear deformation of su(3), which emerges when treating the interaction of three-level systems with quantum fields. In such a case, the description in terms of effective Hamiltonians has allowed us to show how one can manipulate resonance conditions to tailor resonant interactions from very different and nonresonant ones.

Finally, it is worth emphasizing that the validity of this technique is not restricted to the problems examined in this paper. It can be easily generalized to more involved situations, such as *N*-level systems interacting with quantum fields, whose physical interest seems more than a curiosity.

APPENDIX A

In this appendix we refine the method of small rotations by taking into account higher-order terms, which will allow us to explore its range of validity and to compare it with the results of the standard perturbation theory. To be concrete, we shall consider only the case of nonlinear deformations of

the su(2) algebra and thus Hamiltonians of the form (2.12) with a small parameter we shall rewrite as $\varepsilon = g/\Delta$.

To proceed further we shall need some algebraic relations from the polynomial deformation of this su(2) algebra. First, we introduce the structural polynomial function [7]

$$\phi(X_3) = X_+ X_- . \tag{A1}$$

Then the polynomial (2.14) in terms of $\phi(X_3)$ takes the form

$$P(X_3) = -\nabla\phi(X_3) = \phi(X_3) - \phi(X_3+1). \tag{A2}$$

Taking into account (2.14), we can easily obtain the following commutation relations:

$$[X_3, X_+ f(X_3)] = -X_+ \nabla f(X_3),$$

$$[X_+, f(X_3)X_-] = -\nabla[\phi(X_3) f(X_3-1)],$$

$$[X_+ f(X_3), f(X_3)X_-] = -\nabla[\phi(X_3) f^2(X_3-1)], \tag{A3}$$

where $f(X_3)$ is an arbitrary function of X_3.

Applying the transformation (2.16) to each term of the Hamiltonian (2.12) we get

$$UX_3U^\dagger = X_3 - \varepsilon V - \sum_{p=2}^{\infty} \frac{\varepsilon^p}{p!} \mathrm{ad}_{\mathrm{T}}^{\mathrm{p}}(\mathrm{V}),$$

$$UVU^\dagger = V + \sum_{p=1}^{\infty} \frac{\varepsilon^p}{p!} \mathrm{ad}_{\mathrm{T}}^{\mathrm{p}}(\mathrm{V}), \tag{A4}$$

where

$$V = X_+ + X_- ,$$

$$T = X_+ - X_- , \tag{A5}$$

and ad_{T} is the adjoint operator defined as

$$\mathrm{ad}_{\mathrm{T}}(\mathrm{V}) = [\mathrm{T}, \mathrm{V}]. \tag{A6}$$

Therefore we obtain

$$H_{\mathrm{eff}} = UH_{\mathrm{int}}U^\dagger = \Delta X_3 + g \sum_{p=1}^{\infty} \varepsilon^p \frac{p}{(p+1)!} \mathrm{ad}_{\mathrm{T}}^{\mathrm{p}}(\mathrm{V}). \tag{A7}$$

By using the commutation relations (A3) and after some calculations we get

$$\mathrm{ad}_{\mathrm{T}}(\mathrm{V}) = -2\nabla\phi(X_3),$$

$$\mathrm{ad}_{\mathrm{T}}^2(\mathrm{V}) = 2[X_+ \nabla^2\phi(X_3) + \nabla^2\phi(X_3)X_-],$$

$$\mathrm{ad}_{\mathrm{T}}^3(\mathrm{V}) = -2[X_+^2 \nabla^3\phi(X_3) + \nabla^3\phi(X_3)X_-^2] - 4\nabla[\phi(X_3)\nabla^2\phi(X_3-1)], \ldots . \tag{A8}$$

The effective Hamiltonian takes then the form

$$H_{\mathrm{eff}} = \Delta X_3 - \varepsilon g \nabla\phi(X_3) + \tfrac{2}{3}\varepsilon^2 g[X_+ \nabla^2\phi(X_3) + \nabla^2\phi(X_3)X_-] - \tfrac{1}{4}\varepsilon^3 g\{X_+^2 \nabla^3\phi(X_3) + \nabla^3\phi(X_3)X_-^2 + 2\nabla[\phi(X_3)\nabla^2\phi(X_3-1)]\} + \mathcal{O}(\varepsilon^4). \tag{A9}$$

Besides, one can compute $\mathrm{ad}_{\mathrm{T}}^4(\mathrm{V})$ to prove that the term proportional to ε^4 is completely nondiagonal.

We can now devise how to continue the diagonalization procedure. First, one eliminates the nondiagonal term of order ε^2 applying the transformation

$$U_1 = \exp\left\{\frac{2}{3}\varepsilon^2[X_+ \nabla^2\phi(X_3) - \nabla^2\phi(X_3)X_-]\right\}. \tag{A10}$$

This transformation will generate some additional contributions of order ε^4 and higher. In much the same way, one removes the terms of order ε^3 and so on.

It is worth emphasizing that the structure of these additional contributions [they always have the form $X_+^p f(X_3) + \mathrm{H.c.} + \Phi(X_3)$] makes the procedure of removing of nondiagonal terms trivial at each step, in the sense that it is always obvious which transformation should be applied. For example, to eliminate terms of the form

$$\varepsilon^q [X_+^p f(X_3) + f(X_3) X_-^p], \tag{A11}$$

we should apply the following transformation

$$\exp\left[\frac{1}{p}\varepsilon^{q+1} T_p\right], \tag{A12}$$

with

$$T_p = X_+^p f(X_3) - f(X_3) X_-^p , \tag{A13}$$

since the first commutator of ΔX_3 with T_p, cancels the corresponding term in the Hamiltonian. However, one should be careful when applying this method to more complex deformed algebras, due to the additional resonance conditions that could be imposed in that case, as we have seen in Sec. III.

Because the transformed Hamiltonian has the form of an expansion in the small parameter ε, it is clear that eigenvalues of the effective Hamiltonian will coincide with those obtained using the standard perturbation theory. The advantage of our method consists in obtaining *diagonal effective Hamiltonians* for *any system* that could be described in terms of the generators of a nonlinear deformation of su(2), which avoids the necessity of calculating cumbersome series of matrix elements.

Nevertheless, the convergence of our method can be substantially improved if at each step (i.e., before applying the correspondent small rotation) we separate the Hamiltonian into diagonal and nondiagonal parts, rather than into free and interaction parts. For example, it could be advantageous to apply the following rotation instead of the second transformation (A10)

$$U_1=\exp\left\{\frac{2}{3}\varepsilon^3\left[X_+\frac{\nabla^2\phi(X_3)}{1-\varepsilon\nabla^2\phi(X_3)}-\text{H.c.}\right]\right\}, \quad \text{(A14)}$$

which follows from the separation of (A9) into a diagonal part $\Delta X_3-\varepsilon g\nabla\phi(X_3)$ and the rest of the Hamiltonian. Now, no additional corrections [with respect to (A9)] to the terms of order ε^4 will appear in the Hamiltonian transformed via (A14), and all the diagonal terms of order ε^5 and higher will have a different form than in the perturbation theory. This corresponds to a partial resummation of the perturbation series and the final effective diagonal Hamiltonian up to $\mathcal{O}(\varepsilon^5)$ reads as

$$H_{\text{eff}}=\Delta X_3-\varepsilon g\nabla\phi(X_3)-\frac{1}{2}\varepsilon^3 g\nabla[\phi(X_3)\nabla^2\phi(X_3-1)]. \quad \text{(A15)}$$

This allows us to explore the limit of applicability of the effective Hamiltonian (2.18). For definiteness, let us consider the case of the second-harmonic generation as treated in Sec. II D. Then

$$X_3=\frac{1}{3}(a_2^\dagger a_2-a_1^\dagger a_1),$$

$$N=\frac{1}{3}(a_2^\dagger a_2+2a_1^\dagger a_1),$$

$$\phi(X_3)=(N+X_3+2)(N+X_3+1)(N-2X_3),$$

$$\nabla\phi(X_3)=6X_3^2+3X_3(N+1)+3(N+3),$$

$$\nabla^2\phi(X_3)=3(4X_3+N+3), \quad \text{(A16)}$$

where N is the excitation number operator. It is easy to see that the corrections to the effective Hamiltonian (2.33) are of order $\varepsilon^3\bar{n}^3$, which means that the approximation (2.33) describes correctly the dynamics up to times $gt\sim(\Delta/g)^3\bar{n}^{-3}$, where $\bar{n}$ is the average number of excitations in the system.

Finally, let us point out that our technique also provides a valuable method of obtaining corrections to the eigenstates of the Hamiltonian (2.12). Effectively, from (2.17) it is easy to realize that the eigenstates of the interaction Hamiltonian (2.12) can be approximated as

$$|\psi_m\rangle=U^\dagger|m\rangle, \quad \text{(A17)}$$

where $|m\rangle$ is an eigenstate of the diagonal operator X_3 and U is the corresponding small rotation (or the composition of several of them). Since the rotation operators and $|m\rangle$ do not depend on time, the operator U can be applied to $|m\rangle$ in the form of an expansion in series of ε. For example, the eigenstate $|\psi_m\rangle$ up to order ε^2 takes the form

$$|\psi_m\rangle=\left\{1-\frac{\varepsilon^2}{2}[\phi(X_3)+\phi(X_3+1)]\right\}|m\rangle-\varepsilon(X_+-X_-)|m\rangle+\frac{\varepsilon^2}{2}(X_+^2+X_-^2)|m\rangle. \quad \text{(A18)}$$

This representation is especially advantageous if we construct the space of states of the model as a representation space of the deformed su(2) algebra using the raising operator X_+ [7]:

$$|m\rangle\sim X_+^m|0\rangle, \quad \text{(A19)}$$

where $|0\rangle$ is a lowest weight vector fulfilling the standard condition $X_-|0\rangle=0$.

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