Matter-wave phase measurement: A noninterferometric approach

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(Received 25 October 1999; published 17 May 2000)

We show that noninterferometric techniques can produce quantitative phase measurements of quantum-mechanical wave fields with coherence requirements that are considerably reduced over those for interferometry.

PACS number(s): 03.75.Be, 03.75.Dg, 03.65.Bz

I. INTRODUCTION

The measurement of phase has been a key part of the development of physics. Optical phase measurement has opened up precision measurement in a number of fields. X-ray phase measurement techniques are being rapidly developed as third-generation synchrotrons become more widely utilized [1,2]. Moreover, measurement of the phase of matter-wave fields, including both neutrons and atoms, is opening up further areas of precision measurement [3,4]. Historically, phase measurement has relied on interferometry, which typically requires a high degree of coherence. In this paper we explore the measurement of phase using noninterferometric propagation-based approaches which have considerably reduced coherence requirements over those of interferometry [5]. Although previously published work has concentrated on the measurement of the phase of optical [6–11], x-ray [1,2], and electron [12–14] wave fields, in this paper we express these ideas in the language of quantum-mechanical waves. We also explore whether these phase measurement ideas can be usefully adapted to quantum-mechanical matter-wave fields.

We introduce a generalized notion of phase for a wide class of quantum-mechanical and classical fields and then go on to show that this generalized phase interacts with a potential distribution in a manner identical to that of the conventional phase (when it exists). Phase-sensitive measurements are therefore possible with far less stringent requirements than those for interferometry. In particular, the approach to phase measurement considered here allows phase measurements to be performed without the need for a high degree of coherence.

We begin in Sec. II with a review of the definitions of phase based on the probability flow vector. This definition leads to the generalized concept of a scalar phase and a vector phase, which are somewhat analogous to the scalar and vector potentials of classical electrodynamics. In Sec. III we consider the uniqueness of the solutions for phase based on the observation of the probability distribution. In Sec. IV we briefly consider higher-order contributions to the flow of the wave. In Sec. V we consider the interaction of the generalized phase with a potential and show that it behaves in precisely the same manner as does the phase of a coherent wave. We then go on to make some experimental considerations in Sec. VI. In Sec. VII we give three examples of our concept of phase and we conclude the paper in Sec. VIII.

II. DEFINITIONS OF PHASE

Consider a wave field with an associated probability flow vector. In a region of space that is free of sources and sinks, the principle of energy conservation implies that the ensemble-averaged flow vector must obey the continuity equation

$$\vec{\nabla}\cdot\langle\vec{j}(\vec{r})\rangle=0, \tag{1}$$

where $\vec{j}(\vec{r})$ is the flow vector of the wave (i.e., the probability current, or its appropriate analog), and we use angular brackets to denote an ensemble average so as to permit the inclusion of partially coherent fields in the discussion. If the wave field is coherent (in the optical sense) or, equivalently, the wave field is in a stationary state (in the quantum-mechanical sense) then its spatial part may be written as $\psi(\vec{r})=\sqrt{\rho(\vec{r})}\exp[iS(\vec{r})/\hbar]$, where $\rho(\vec{r})$ is the probability density, $S(\vec{r})$ is the phase, and $\hbar$ is the reduced Planck constant $\hbar\equiv h/2\pi$. In this case the probability current is time invariant and assumes the form $\vec{j}(\vec{r})=\rho(\vec{r})\vec{\nabla}S(\vec{r})/m$, where m is the mass of the object [15]. Evidently, the phase and probability density determine the probability current. Since both the current and probability distribution are observables [16], we conclude that the phase may be defined in terms of observables, without any reference to interferometry. In this paper we explore the observation that a meaningful and very general definition of phase may be based on the current vector. While the current vector is considered to be an observable [16], our approach differs from other optical frameworks based on observables in that other approaches use correlation functions as their starting point [17,18].

In general, the probability current associated with a given radiation field will be a function of time. If we assume the field to be statistically stationary [19], then we may meaningfully introduce the ensemble average of the probability current for a partially coherent field, $\langle\vec{j}(\vec{r})\rangle$. This is a well-defined vector field and will be used as the basis for our formulation. This notion remains well defined for partially coherent fields and reduces to the conventional definition of phase in the coherent limit of vortex-free waves (for the case of a scalar field). It will prove useful in the noninterferometric measurement of phase using matter waves with low coherence.

We define the normalized probability current in terms of the ensemble-average current using

$$\langle \hat{j}(\vec{r}) \rangle \equiv m \lim_{\varepsilon \to 0^+} \frac{\langle \vec{j}(\vec{r}) \rangle}{\rho(\vec{r}) + \varepsilon}. \tag{2}$$

Over regions of nonzero time-averaged probability density, Eq. (2) describes a well-defined vector field, which may therefore be Helmholtz decomposed into a potential and a rotational component in the usual way [20]. Performing this decomposition, we are able to rewrite the probability current in the following form, which is analogous to the expression for the current vector in the presence of both scalar and vector electromagnetic potentials [21]:

$$\langle \vec{j}(\vec{r}) \rangle = \frac{1}{m} \rho(\vec{r}) [\vec{\nabla} \phi_S(\vec{r}) + \vec{\nabla} \times \vec{\phi}_V(\vec{r})]. \tag{3}$$

We regard Eq. (3) as defining the scalar phase ϕ_S, which is single valued, and the vector phase $\vec{\phi}_V$, which is divergence-free, in terms of the ensemble-averaged probability current $\langle \vec{j}(\vec{r}) \rangle$ and the ensemble-averaged probability density $\rho(\vec{r})$. Equation (3) may be inverted to express the phase components in terms of the probability current $\langle \vec{j}(\vec{r}) \rangle$ [5]. This decomposition is unique up to a vectorial constant that may float between the two components; we place this vectorial constant in the gradient term. The phase so defined obeys the following Poisson-type differential equations:

$$\nabla^2 \phi_S(\vec{r}) = \vec{\nabla} \cdot \langle \hat{j}(\vec{r}) \rangle, \tag{4a}$$

$$\nabla^2 \vec{\phi}_V(\vec{r}) = -\vec{\nabla} \times \langle \hat{j}(\vec{r}) \rangle. \tag{4b}$$

Thus these two functions are related to the current vector and thereby to the phase of the wave via the following integrals [20]:

$$\phi_S(\vec{r}) = -\frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \langle \hat{j}(\vec{r}\,') \rangle}{|\vec{r} - \vec{r}\,'|} d^3 r', \tag{5a}$$

$$\vec{\phi}_V(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{\nabla} \times \langle \hat{j}(\vec{r}\,') \rangle}{|\vec{r} - \vec{r}\,'|} d^3 r'. \tag{5b}$$

These are the fundamental quantities discussed in this paper.

We conclude this section by showing that the definition of phase given in Eqs. (5) reduces to the usual definition of phase in the coherent limit, provided of course that one is dealing with a scalar wave field. In the coherent limit, where $\vec{j}(\vec{r}) = \rho(\vec{r}) \vec{\nabla} S(\vec{r})/m$ [15], Eqs. (5) reduce to

$$\phi_S(\vec{r}) = -\frac{1}{4\pi} \int \frac{\nabla^2 S(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3 r', \tag{5c}$$

$$\phi_S(\vec{r}) = -\frac{1}{4\pi} \int \frac{\vec{\nabla} \times \vec{\nabla} S(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3 r'. \tag{5d}$$

Note that the vector phase $\vec{\phi}_V(\vec{r})$ will vanish if $\vec{\nabla} \times \vec{\nabla}(\vec{r}) = \vec{0}$. Thus, in the coherent limit, the vector phase will vanish if the conventional phase of the wave is single valued and continuous. In the case of a coherent field, then, the vector phase is nonzero only if the phase of the wave field is discontinuous, or multiply valued, and so corresponds to a topological phase [5,22]. It is also apparent from Eq. (5c) that the scalar phase reduces to the conventional phase when the field is coherent and the phase is continuous [5].

III. PHASE RECOVERY AND ITS UNIQUENESS

We begin by considering the case of a coherent wave, examples of which might include a monoenergetic beam of electrons or an atom laser [23]. Energy conservation implies the following well-known equation from the hydrodynamic formulation of quantum mechanics [15,24]:

$$\vec{\nabla} \cdot [\rho(\vec{r}) \vec{\nabla} S(\vec{r})] = 0. \tag{6}$$

This equation can be shown to have a unique solution for the phase S provided that the probability distribution is known and is always greater than zero [25]. Thus, given these conditions, the phase of a wave is uniquely determined by its distribution of probability density in three dimensions.

The measurement procedure involves a three-dimensional measurement of the probability distribution. The phase is deduced from the probability measurements. Thus at no time do we attempt to measure both the phase and position of a single electron. That this approach is possible implies that we must require the statistical properties of the electron wave field to be time invariant. It also means that this work is quite distinct from the efforts to find a quantum-mechanical operator for phase [26].

In our previous work we have used the paraxial version of Eq. (6) to achieve quantitative noninterferometric phase measurements of paraxial fields using visible light, x rays, and electrons [1,6–11,14]. Paraxiality is a well-defined concept in physical optics and requires that the wave field travel predominantly along the z axis and so obeys the approximation $\psi(\vec{r}_\perp, z) \approx \psi_z(\vec{r}_\perp) e^{i(p/\hbar)z}$, where $\psi_z(\vec{r}_\perp)$ is slowly varying compared with $e^{i(p/\hbar)z}$, and $\vec{r}_\perp$ is the position vector in the plane perpendicular to the z axis. Under the paraxial approximation, Eq. (6) becomes the following transport equation, which for the case of optical fields has been termed the ''transport-of-intensity equation'' [26,27]:

$$\frac{\partial \rho(\vec{r}_\perp)}{\partial z} = -\frac{\hbar}{p} \vec{\nabla}_\perp \cdot [\rho(\vec{r}_\perp) \vec{\nabla}_\perp S(\vec{r}_\perp)]. \tag{7}$$

The derivative of the probability along the z axis and the probability distribution in that plane are both observable quantities, and so Eq. (7) offers a direct approach to the quantitative measurement of phase from noninterferometric measurements of probability density. This approach will require two consecutive measurements of the probability distribution and so we require that the wave field be statistically stationary. We also point out that we do not measure the amplitude and phase of a single particle and so the approach does not violate the uncertainty principle.

We now explore the conditions on the uniqueness of the solution of Eq. (7). We do this by considering four separate subcases in turn.

A. No intensity zeros present

In this case, Eq. (7) has elsewhere been proven [7] to have a unique solution for the phase, up to a physically meaningless additive constant. This additive constant is meaningless as the wave equation is invariant under a shift in the origin of time. Thus, for this case, we have a well-defined noninterferometric phase measurement scheme based on Eq. (7).

B. Intensity zeros in the field

1. Intensity zeros present but phase known to be continuous

We assert that if we know *a priori* that the phase is continuous we may uniquely recover the phase even in the presence of intensity zeros, provided the region of positive ensemble-averaged probability density is connected (but not necessarily simple connected). The proof of this claim is as follows.

We assume that we measure the ensemble-average probability density $\rho(\vec{r}_\perp)$ and its longitudinal derivative $\partial\rho(\vec{r}_\perp)/\partial z$, with $\rho(\vec{r}_\perp)$ being greater than zero over a connected two-dimensional region, and zero outside this region. The probability density and phase satisfy the transport equation as written in Eq. (7).

Over the two-dimensional connected surface over which the phase is being measured, we now consider adding an infinitesimal perturbation ε to the probability density so that the perturbed quantum-mechanical field may be written

$$\psi_\varepsilon(\vec{r}_\perp)=\lim_{\varepsilon\to 0}\{\sqrt{\rho(\vec{r}_\perp)+\varepsilon}\exp[iS(\vec{r}_\perp)/\hbar]\}. \tag{8}$$

Note that if the phase is continuous $\lim_{\varepsilon\to 0}\psi_\varepsilon(\vec{r}_\perp)=\psi(\vec{r}_\perp)$. The corresponding transport-of-intensity equation becomes

$$\frac{p}{\hbar}\frac{\partial\rho(\vec{r}_\perp)}{\partial z}=-\vec{\nabla}_\perp\cdot[\rho(\vec{r}_\perp)\vec{\nabla}_\perp S(\vec{r}_\perp)]-\varepsilon\nabla_\perp^2 S(\vec{r}_\perp). \tag{9}$$

This equation has a unique solution [7], and the phase may be recovered by taking the limit as $\varepsilon\to 0$. Thus the phase may be uniquely determined for the case where intensity zeros are present but the phase is known *a priori* to be continuous.

2. Hidden screw dislocations (phase vortices) in the field

In this case the paraxial transport equation may be written

$$\frac{p}{\hbar}\frac{\partial\rho(\vec{r}_\perp)}{\partial z}=-\vec{\nabla}_\perp\cdot[\rho(\vec{r}_\perp)\vec{\nabla}_\perp\phi_S(\vec{r}_\perp)] -\vec{\nabla}_\perp\rho(\vec{r}_\perp)\cdot\vec{\nabla}\times\vec{\phi}_V(\vec{r}_\perp). \tag{10}$$

If the vector phase $\vec{\phi}_V(\vec{r}_\perp)$ is such that $\vec{\nabla}_\perp\rho(\vec{r})$ and $\vec{\nabla}\times\vec{\phi}_V(\vec{r})$ are everywhere perpendicular, this component of the phase will be invisible. In other words, topological phase components may be hidden by symmetries in the probability density distribution. In this case, the hidden phases are not recoverable but, since the scalar phase is continuous, it will be uniquely recoverable. Thus, in the case of phases hidden by symmetry, the scalar phase may be recovered but the vector phases remain completely undetermined.

3. Edge dislocations

Edge dislocations in a wave are characterized by a discontinuous jump in the phase of the wave as one crosses a given line that extends across the direction of propagation [28]. In free space, the edge dislocation forms either an infinitely long line that extends to infinity in both directions, or a closed loop. In either case, two or more regions of the probability distribution are completely disconnected and it follows that their relative phase cannot be determined. In general, the relative phase of two physically separate regions of probability cannot be determined by noninterferometric means. To prove this, we need only note that the ensemble-averaged probability density is unchanged upon the introduction of an absolute phase shift in one of the physically separated beams.

4. Arbitrary case

We now consider the case of a field containing a discontinuous phase distribution. It has been established that phase dislocations may, in general, be considered as a sum of screw and edge dislocations [28]. The edge dislocation has been discussed above and we henceforth disregard these.

It can be further shown that the multivalued phase associated with multiple screw dislocations may be added directly [29]. It follows that the transport equation (7) may be written in the form

$$\vec{\nabla}_\perp\cdot[\rho(\vec{r}_\perp)\vec{\nabla}_\perp\phi_S(\vec{r}_\perp)]+\sum_j\left(\frac{m_j}{r_j}\frac{\partial}{\partial\theta_j}\right)\rho(\vec{r}_\perp)=-\frac{p}{\hbar}\frac{\partial\rho(\vec{r}_\perp)}{\partial z}. \tag{11}$$

Here, m_j is the topological charge of the jth dislocation, r_j is the distance between the jth dislocation and $\vec{r}_\perp$, and θ_j is the polar angle measured about the jth dislocation. The physical picture implied by this equation is that the effect of the scalar phase is a lateral *translation* as we move along the axis z. The effect of the screw dislocation is a *rotation* as we move along z. It follows, therefore, that the presence of screw dislocations has a characteristic signature in the propagation of the probability distribution.

An algorithm therefore suggests itself for the complete recovery of phase. For ergodic fields, the measurement of the time-averaged probability density over multiple measurement planes will in principle allow translation and rotation effects to be identified and decoupled. As they must be in the form of screw dislocations or edges, then the charge of the screw dislocations may be found and the vortex cores of the screw dislocations located. In this case, the appropriate terms describing the effect of the dislocations may be written down to permit the rearrangement of Eq. (11) into the form

$$\vec{\nabla}_\perp\cdot[\rho(\vec{r}_\perp)\vec{\nabla}_\perp\phi_S(\vec{r}_\perp)]=-\sum_j\left(\frac{m_j}{r_j}\frac{\partial}{\partial\theta_j}\right)\rho(\vec{r}_\perp)-k\frac{\partial\rho(\vec{r}_\perp)}{\partial z}. \tag{12}$$

As the scalar phase appearing on the left-hand side of this equation is continuous by construction, and the right-hand side is known, Eq. (12) may be uniquely solved for the scalar phase using existing methods. For example, the phase-retrieval algorithm developed in [5] may be applied to solve Eq. (12) for the scalar phase, given the right-hand side. Thus, as all of the vector phase components have been identified via the screw dislocations, we can see that the entire observable phase structure will have been determined.

In the case that some phase components are invisible as a result of a symmetry in the probability distribution, it will be possible to break the symmetry and thereby reveal the presence of the phase component. The observation that all the hidden phases must be able to be expressed as a sum of screw and edge dislocations places severe restrictions on the class of symmetries that are of significance. In particular, we note that the screw dislocation is represented as an eigenfunction of angular momentum where the direction of the rotation (the sign of the topological charge) is not visible (see Secs. III B 2 and VII C). However, the topological charge may be rendered visible by, for example, passing the vortex through a cylindrical lens [30], which destroys the azimuthal symmetry of the distribution and thereby renders rotation visible.

IV. HIGHER-ORDER CONTRIBUTIONS AND PARTIAL COHERENCE

Consider the probability current for a random statistically stationary ergodic quantum-mechanical wave field given by [31]

$$\langle \vec{j}(\vec{r}) \rangle = \frac{\hbar}{im} \lim_{\vec{x}\to 0} \vec{\nabla}_{\vec{x}} \rho(\vec{r}+\vec{x}/2, \vec{r}-\vec{x}/2), \tag{13}$$

where

$$\rho(\vec{r}_1, \vec{r}_2) \equiv \langle \psi(\vec{r}_1) \psi^*(\vec{r}_2) \rangle. \tag{14}$$

Here, $\rho(\vec{r}) \equiv \rho(\vec{r},\vec{r})$ is the ensemble-averaged probability density and $\rho(\vec{r}_1,\vec{r}_2)$ is the time-averaged density matrix of the wave field [31]; the arguments of ρ will always be used so there should be no danger of ambiguity. We examine this probability current in terms of the Wigner distribution function defined by

$$\begin{aligned} W(\vec{r},\vec{p}) &= \frac{1}{(2\pi\hbar)^3} \int \langle \psi(\vec{r}+\vec{x}/2) \psi^*(\vec{r}-\vec{x}/2) \rangle e^{-i\vec{p}\cdot\vec{x}/\hbar} d\vec{x} \\ &= \frac{1}{(2\pi\hbar)^3} \int \rho(\vec{r}+\vec{x}/2, \vec{r}-\vec{x}/2) e^{-i\vec{p}\cdot\vec{x}/\hbar} d\vec{x}. \end{aligned} \tag{15}$$

Inverting Eq. (15), we obtain

$$\rho(\vec{r}+\vec{x}/2, \vec{r}-\vec{x}/2) = \int W(\vec{r},\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} d\vec{p}, \tag{16}$$

which may then be substituted into Eq. (13) to obtain

$$\langle \vec{j}(\vec{r}) \rangle = \frac{1}{m} \int \vec{p} W(\vec{r},\vec{p}) d\vec{p}. \tag{17}$$

The time-averaged probability current is therefore an appropriately normalized average of the momentum vector over the Wigner distribution [32]. In the particular case of a stationary-state state scalar wave function described by $\psi(\vec{r}) = \sqrt{\rho(\vec{r})} e^{iS(\vec{r})/\hbar}$, the probability current is easily shown to reduce to the familiar expression $\vec{j}(\vec{r}) = (1/m) \rho(\vec{r}) \nabla S(\vec{r})$ [15].

In the paraxial approximation, the Wigner function obeys the propagation expression

$$W(\vec{r}_\perp, \Delta z, \vec{p}_\perp) = W(\vec{r}_\perp - (\vec{p}_\perp / p) \Delta z, 0, \vec{p}_\perp), \tag{18}$$

where $\vec{p}_\perp$ is the transverse momentum component [33]. The resulting probability density distribution can therefore be Taylor expanded into the form

$$\begin{aligned} \rho(\vec{r}_\perp, \Delta z) &= \rho(\vec{r}_\perp, 0) - \frac{1}{k} \Delta z \vec{\nabla}_\perp \cdot \langle \vec{j}(\vec{r}_\perp, 0) \rangle \\ &\quad + \tfrac{1}{2} \Delta z^2 \int (\vec{p}_\perp \cdot \vec{\nabla}_\perp)^2 W(\vec{r}_\perp, 0, \vec{p}_\perp) d\vec{p}_\perp + \cdots. \end{aligned} \tag{19}$$

In the paraxial approximation it can be seen that the first-order term gives the flow vector for the field, which yields a quantity identical in properties to the phase of a coherent field. The second-order term relates to the diffusion of the probability density on propagation and yields information on the coherence properties of the field. The precise nature of this information will be the subject of further work. However, Eq. (19) will be used in the experimental considerations presented in Sec. VI.

With our definitions we are able to meaningfully discuss phase for noncoherent fields for which phase has not hitherto been defined. In the next section we will demonstrate that the phase as defined by us behaves, in many circumstances, in precisely the same manner as it does for coherent fields. Thus noncoherent fields may be used to make phase-sensitive measurements. However, fields for which the previous concept of phase breaks down may be described through correlations using the theory of partial coherence. If knowledge of the correlation properties is desired, then the more complex noninterferometric technique of phase-space tomography may be used. In this case very large (four-dimensional) data sets are required [30,34,35].

V. MEASURING THE PHASE OF A MATTER-WAVE FIELD

In a previous publication, it was shown that noninterferometric measurement of the generalized phase associated with a partially coherent scalar electromagnetic wave could be used as a means of quantitatively probing the refractive-index distribution of an object through which the radiation field had passed [10]. In this section, we show how these results can be extended to the case of partially coherent quantum-mechanical wave fields. Specifically, we shall use the paraxial transport equation associated with our generalized notion of phase to explore a technique that permits us to determine the projected potential of a region through which partially coherent quantum-mechanical wave fields have passed.

Consider a general partially coherent scalar wave function of the form

$$\Psi(\vec{r},t)=\sum_{\nu} a_{\nu}\psi_{\nu}(\vec{r})e^{2\pi i\nu t}, \tag{20}$$

where a_{ν} denote the amplitudes of the component wave functions and ν denotes the corresponding frequencies. The time-averaged density matrix for this wave function is

$$\rho(\vec{r}_1,\vec{r}_2)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{\nu}\sum_{\nu'} a_{\nu}a^{*}_{\nu'}\psi_{\nu}(\vec{r}_1)\psi^{*}_{\nu'}(\vec{r}_2)\times e^{2\pi i(\nu-\nu')t}. \tag{21}$$

If we take this limit, the density matrix reduces to

$$\rho(\vec{r}_1,\vec{r}_2)=\sum_{\nu} |a_{\nu}|^2\psi_{\nu}(\vec{r}_1)\psi^{*}_{\nu}(\vec{r}_2), \tag{22}$$

so that the Wigner function of this wave function is

$$W(\vec{r},\vec{p})=\sum_{\nu} |a_{\nu}|^2 W_{\nu}(\vec{r},\vec{p}), \tag{23}$$

where

$$W_{\nu}(\vec{r},\vec{p})\equiv\frac{1}{(2\pi\hbar)^3}\int \psi_{\nu}\left(\vec{r}+\frac{\vec{x}}{2}\right)\psi^{*}_{\nu}\left(\vec{r}-\frac{\vec{x}}{2}\right)e^{-i\vec{p}\cdot\vec{x}/\hbar}d\vec{x}. \tag{24}$$

Thus, making use of Eq. (17), we arrive at

$$\langle\vec{j}(\vec{r})\rangle=\frac{1}{m}\sum_{\nu} |a_{\nu}|^2\int \vec{p}W_{\nu}(\vec{r},\vec{p})d\vec{p}. \tag{25}$$

We now use this result to determine the effect of a potential on the probability current.

Atom interferometry probes the effect of a potential on the phase of a wave function. We consider the analogous situation where a wave function encounters a slice of potential perpendicular to the z axis that induces a phase shift given by $S_V(\vec{r},\nu)$, as shown in Fig. 1. We assume that the potential is located in the plane at $z=0$ and that the potential does not affect the probability density in that plane. We write the component wave functions in Eq. (23) as $\psi_{\nu}(\vec{r})=\sqrt{\rho_{\nu}(\vec{r})}e^{iS_{\nu}(\vec{r})/\hbar}$. The phase change produced by the potential will influence the probability density distribution elsewhere in space and we write the modified distribution in a plane z as $\rho^{(z)}_{\nu}(\vec{r})$, where $\rho^{(0)}_{\nu}(\vec{r})=\rho_{\nu}(\vec{r})$. We may also write the phase everywhere in space in the form $S^{(z)}_{\nu}(\vec{r})=S_{\nu}(\vec{r})+S^{(z)}_{V}(\vec{r},\nu)$, where $S^{(0)}_{V}(\vec{r},\nu)=S_V(\vec{r},\nu)$, which is the phase change induced by the potential in the plane of the potential. The resulting Wigner function is therefore given by

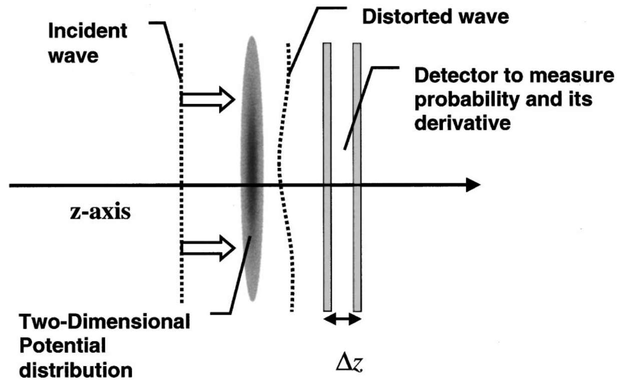

FIG. 1. A schematic of a phase measurement experiment. The longitudinal derivative in the probability density may be measured by acquiring the density over two closely spaced planes, as indicated here. The effects of the incident wave field may be removed by obtaining data with the object removed.

$$W^{(z)}_{\nu}(\vec{r},\vec{p})=\frac{1}{(2\pi\hbar)^3}\int \sqrt{\rho^{(z)}_{\nu}(\vec{r}+\vec{x}/2)}\times e^{i[S_{\nu}(\vec{r}+\vec{x}/2)+S^{(z)}_{V}(\vec{r}+\vec{x}/2,\nu)]/\hbar}\sqrt{\rho^{(z)}_{\nu}(\vec{r}-\vec{x}/2)}\times e^{-i[S_{\nu}(\vec{r}-\vec{x}/2)+S^{(z)}_{V}(\vec{r}-\vec{x}/2,\nu)]/\hbar}e^{i\vec{p}\cdot\vec{x}/\hbar}d\vec{x}. \tag{26}$$

We now wish to find the probability current in the plane of the potential and we do this by substituting Eq. (26) into Eq. (25) and using the values of the probability density and the phase in this plane. We find

$$\langle\vec{j}^{(0)}(\vec{r})\rangle=\frac{1}{m}\sum_{\nu} |a_{\nu}|^2\rho_{\nu}(\vec{r})\vec{\nabla}[S_{\nu}(\vec{r})+S_V(\vec{r},\nu)]. \tag{27}$$

By examining Eq. (27), we can see that this may be rewritten as

$$\langle\vec{j}^{(0)}(\vec{r})\rangle=\langle\vec{j}_0(\vec{r})\rangle+\frac{1}{m}\sum_{\nu} |a_{\nu}|^2\rho_{\nu}(\vec{r})\vec{\nabla}S_V(\vec{r},\nu), \tag{28}$$

where $\langle\vec{j}_0(\vec{r})\rangle$ is equal to the incident probability current at $z=0$. The second term in Eq. (28) is a sum over frequencies. Assume the phase term may be factorized into the form $S_V(\vec{r},\nu)=\Phi(\vec{r})f(\nu)$, where we define $f(\nu)$ to be such that $f(\overline{\nu})=1$, $\overline{\nu}$ being the average frequency of the incident wave function. In this case, the sum in Eq. (28) can be written as $\nabla\Phi(\vec{r})\Sigma_{\nu}|a_{\nu}|^2 f_{\nu}\rho_{\nu}(\vec{r})$, where f_{ν} are the values of $f(\nu)$ evaluated at the frequencies in the sum. Now assume that $f_{\nu}\approx 1$ over the spread of frequencies in the wave function; that is, we assume dispersion is negligible over the frequency width of the wave function. In this case, using the properties of the Wigner function, we obtain

$$\sum_{\nu} |a_{\nu}|^2 f_{\nu}\rho^{(0)}_{\nu}(\vec{r})\simeq\sum_{\nu} |a_{\nu}|^2\rho^{(0)}_{\nu}(\vec{r})=\rho(\vec{r}), \tag{29}$$

so that Eq. (28) may be written

$$\langle \vec{j}^{(0)}(\vec{r}) \rangle \simeq \langle \vec{j}_0(\vec{r}) \rangle + \frac{1}{m} \rho(\vec{r}) \vec{\nabla} \Phi(\vec{r}). \tag{30}$$

This allows us to write

$$\langle \vec{j}^{(0)}(\vec{r}) \rangle \simeq \frac{1}{m} \rho(\vec{r}) \vec{\nabla} [S(\vec{r}) + \Phi(\vec{r})]. \tag{31}$$

The probability current leaving the potential has a form identical to that of the coherent probability current where the generalized phase, defined in Eqs. (5a) and (5b), acts precisely as would the conventionally defined phase. Thus, propagation-based phase determination techniques can be applied even though the incident wave does not have a conventionally defined phase.

In an interferometric experiment, the phase and amplitude properties of the illuminating wave may be measured and removed from the data in order to recover the effects of the object on the wave field. We assume that the properties of the incident probability current $\langle \vec{j}_0(\vec{r}) \rangle$ can be similarly removed. Given this, Eq. (30) obeys $\vec{\nabla} \cdot [\rho(\vec{r}) \vec{\nabla} \Phi(\vec{r})] = 0$, which is precisely the equation that we have established may be uniquely solved for the phase, given a knowledge of the probability density. A determination of the probability current will therefore allow the accurate probing of the phase modification of the wave function by the medium.

The formalism just described permits an experiment of the form sketched in Fig. 1 to be used to measure the effect of a potential on a quantum-mechanical wave function. We now explore the practical imitations on such a measurement.

VI. EXPERIMENTAL CONSIDERATIONS

Consider the experiment in Fig. 1, where a phase change is imprinted on an incident wave field and we make a measurement over a surface. The paraxial form of Eq. (1) may then be written

$$\frac{\partial \rho(\vec{r}_\perp)}{\partial z} = -\frac{1}{2\pi\hbar} \vec{\nabla}_\perp \cdot [\rho(\vec{r}) \langle \lambda \vec{\nabla}_\perp \Phi \rangle_\lambda], \tag{32}$$

where here $\langle \, \rangle_\lambda$ denotes an average over wavelength. In practice, a measurement of the spatial derivative of the probability density will entail a measurement via the approximation

$$\frac{\partial \rho(\vec{r}_\perp)}{\partial z} \simeq \frac{\rho(\vec{r}_\perp, +\Delta z/2) - \rho(\vec{r}_\perp, -\Delta z/2)}{\Delta z}. \tag{33}$$

This requires a measurement of the probability over two closely spaced planes separated by Δz. However, the momentum distribution in the probability current will blur the measurement of these distributions even though the current at a point defines the phase precisely [see Eq. (19)]. This blurring will limit the precision of the measurement. We therefore wish to estimate the coherence requirements on the wave field for this effect to be experimentally negligible.

The blurring will have two components: (i) that due to the distribution in transverse momentum in the incident beam, and (ii) that due to the additional transverse momentum distribution produced by dispersion (frequency-dependent phase shift) in the potential. We consider each of these.

First assume, for simplicity, that the transverse momentum distribution follows the appropriately normalized Gaussian form $\exp(-p_\perp^2/2\Delta p_\perp^2)$, where $\vec{p}_\perp$ is the transverse momentum. In the case of a displacement of measurement planes of $\pm \Delta z/2$ the probability density will be spread by a distance

$$\Delta x_p \approx \Delta z \Delta p_\perp / 2p. \tag{34}$$

We use Eq. (16) to relate the momentum distribution to the lateral coherence length l_{lat}. Ignoring any spatial dependence in the Wigner function of the wave function, and taking the coherence function as being described by $\exp(-x^2/2l_{\text{lat}}^2)$, we find that

$$l_{\text{lat}} = \hbar \sqrt{2} / \Delta p_\perp . \tag{35}$$

This is simply a form of the uncertainty principle. The condition that there be no significant blurring of the probability density measurement is $\Delta x_p < \Delta q$, where Δq is the spatial resolution of the detector. We thus require that the coherence length satisfy the condition

$$l_{\text{lat}} > \frac{1}{2\pi\sqrt{2}} \frac{\lambda}{\gamma_{\min}}, \tag{36}$$

where $\gamma_{\min} \equiv \Delta q / \Delta z$ is the minimum angle to which the experiment is sensitive. For the sake of a specific case, if we assume that $\gamma_{\min} \approx 1$ mrad (e.g., $\Delta q = 20\ \mu$m, $\Delta z = 20$ mm) then we require $l_{\text{lat}} > 100\lambda$.

The second source of degradation arises through the frequency dependence in the phase shift. The resulting blurring is approximated by

$$\Delta x_\nu \approx \frac{1}{p} \frac{\partial \nabla S(\vec{r}, \nu)}{\partial \nu} \Delta \nu \, \Delta z. \tag{37}$$

For simplicity, again assume $S(\vec{r}, \nu) = \Phi(\vec{r}) f(\nu)$. In this case, we obtain

$$\Delta x_\nu \approx \frac{1}{p} \nabla \Phi(\vec{r}) \frac{\partial f(\nu)}{\partial \nu} \Delta z. \tag{38}$$

If we introduce a dimensionless variable to describe the dispersion,

$$\beta \equiv \bar{\nu} \frac{\partial f(\nu)}{\partial \nu} \bigg|_{\bar{\nu}}, \tag{39}$$

then

$$\Delta x_{\bar{\nu}} \approx \frac{1}{p} \nabla \Phi(\vec{r}) \beta \frac{\Delta \nu}{\bar{\nu}} \Delta z. \tag{40}$$

As an example, consider a frequency-independent potential. The resulting phase shift is frequency dependent and may be written $S(r,\nu)\simeq\Phi(\vec{r})(\overline{\nu}/\nu)^{1/2}$, so that $\beta=-\frac{1}{2}$. The sign of β is irrelevant to the argument, and so we assume that it is characteristically a number of order unity. The typical phase gradient measured will presumably need to be much greater than the system angular resolution; that is, $\nabla\Phi(\vec{r})/p\gg\Delta q/\Delta z$. We require $\Delta x_\nu\approx\Delta q$ which therefore implies that $\Delta\nu/\overline{\nu}\ll 1$. In terms of coherence length, using $l_{\text{long}}\approx(\nu/\Delta\nu)\lambda$, this translates to

$$l_{\text{long}}\gg\lambda. \tag{41}$$

It can be seen, therefore, that the lateral and longitudinal coherence requirements for this approach to phase determination are substantially less demanding than those for comparable interference experiments.

VII. EXAMPLES OF PHASE

We now present three examples of phase from the viewpoint developed in this paper.

A. A scalar phase: The gravitational phase shift

In this first example, we show how noninterferometric quantum phase imaging may be used to detect gravitationally induced phase shifts. The gravitational potential at height y about the surface of the earth is given by

$$V(y)=mgy, \tag{42}$$

where g is the acceleration due to gravity at the surface of the earth and m is the mass of the particle. Suppose that nonrelativistic collimated monoenergetic matter waves of energy E are incident from a point source located at height γ_0 above the surface of the earth. This point source lies somewhere upstream of a certain plane, which is perpendicular to the surface of the earth. It is easy to show that the de Broglie wavelength of these particles as a function of height over the reference plane is given by

$$\lambda=\frac{h}{\sqrt{2m[E-mg(y-y_0)]}}. \tag{43}$$

Since the de Broglie wavelength λ_0 in a field-free space would be given by setting $g=0$,

$$\lambda_0=\frac{h}{\sqrt{2mE}}, \tag{44}$$

we conclude that the ''effective'' refractive index $n_{\text{eff}}(y)$ associated with the gravitational field is given by the ratio of the quantities in Eqs. (43) and (44):

$$n_{\text{eff}}(y)\equiv\frac{\lambda_0}{\lambda}=\sqrt{1-[mg(y-y_0)]/E}\approx 1-[mg(y-y_0)]/2E, \tag{45}$$

where we assume that the source is sufficiently removed that the binomial approximation is valid over the range of heights of interest. A thin slice of a gravitational potential with thickness Δz therefore produces a phase shift

$$\Delta\phi(z)=-\frac{2\pi}{\lambda_0}\frac{mg(y-y_0)}{2E}\Delta z \tag{46}$$

and the continuity equation gives

$$\vec{\nabla}\cdot\{\rho(\vec{r})\vec{\nabla}[S(\vec{r})+\Delta\phi(\vec{r})]\}=0. \tag{47}$$

Using the earlier results of this paper, it is clearly possible to recover the gravitational phase shift using this noninterferometric approach. It is easy to show that the resulting phase is entirely consistent with earlier investigations of this gravitational phase shift of quantum-mechanical particles [36].

B. A vector phase: Angular momentum

As an example, let us calculate the vector phase associated with Gauss-Laguerre beams. These are defined via [37]

$$\psi_{pm}(r,\theta,z)=\frac{Cr^m 2^{m/2}}{w^m(z)\sqrt{1+(z/z_R)^2}}L_p^m\left(\frac{2r^2}{w^2(z)}\right)\times\exp\left(-\frac{r^2}{w^2(z)}\right)\exp\left[\frac{ikr^2z}{2(z^2+z_R^2)}-im\theta+i(2p+m+1)\tan^{-1}\left(\frac{z}{z_R}\right)\right], \tag{48}$$

where (r,θ,z) are cylindrical polar coordinates, p and m are integers, z_R is the Rayleigh range, $w(z)$ is the radius of the beam, L_p^m is the associated Laguerre polynomial, and C is a constant. Note the presence of the nonintegrable vortex phase term $\exp(-im\theta)$, of topological charge $-m$. Making use of Eq. (4b), we obtain the following differential equation for the vector phase associated with a given Gauss-Laguerre beam $\psi_{pm}(r,\theta,z)$:

$$\nabla^2\vec{\phi}_V(\vec{r})=-\frac{4\pi}{\omega}\vec{\nabla}\times\hat{j}(\vec{r})=m\vec{\nabla}\times\vec{\nabla}\theta=2\pi m\,\delta(r)\hat{z}, \tag{49}$$

where δ is the Dirac delta and $\hat{z}$ is the unit vector aligned with the z axis. Hence we obtain an expression for the vector phase, which depends solely on the presence of the vortex phase term and contains no contribution from the continuous portion of the phase [38]:

$$\vec{\phi}_V(\vec{r})=-m\log_e(r)\hat{z}. \tag{50}$$

Since the probability density associated with a given Gauss-Laguerre beam is rotationally symmetric, $\vec{\nabla}\rho(\vec{r}_\perp)$ will have a nonzero component only in the $\hat{r}$ direction, and so Eq. (10) implies that $\vec{\nabla}_\perp\rho(\vec{r}_\perp)\cdot\vec{\nabla}\times\vec{\phi}_V(\vec{r}_\perp)=0$. For this case, therefore, the topological phase component is hidden by the (rotational) symmetry in the probability density distribution.

C. A more complex case: The Aharonov-Bohm phase shift

We now consider a more sophisticated example of the structure of phase in this formalism. We consider the case of an electron passing through a region of space containing a vector potential. In this case, the probability flow vector has the form

$$\vec{j}(\vec{r}) = \frac{1}{m_e}\rho(\vec{r})\left(\vec{\nabla}S(\vec{r}) - \frac{e}{c}\vec{A}(\vec{r})\right). \tag{51}$$

If the electron is passing around an infinite perfect solenoid then the region of space will not contain a field and so we demand that the probability flow vector not be changed by the presence of the vector potential. In order for this to be the case, a phase must be induced that precisely acts to cancel out the effect of the vector potential:

$$\vec{j}(\vec{r}) = \frac{1}{m_e}\rho(\vec{r})\left(\vec{\nabla}S(\vec{r}) + \vec{\nabla}\times\vec{\phi}_V(\vec{r}) - \frac{e}{c}\vec{A}(\vec{r})\right), \tag{52}$$

where we demand

$$\vec{\nabla}\times\vec{\phi}_V(\vec{r}) = \frac{e}{c}\vec{A}(\vec{r}). \tag{53}$$

Thus we find

$$\begin{aligned}\vec{\phi}_V(\vec{r}) &= -\frac{e}{4\pi c}\int \frac{\vec{\nabla}\times\vec{A}(\vec{r})}{|\vec{r}-\vec{r}\,'|}d^3r' \\ &= -\frac{e}{4\pi c}\int \frac{B(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'.\end{aligned} \tag{54}$$

The vector Aharonov-Bohm phase depends *nonlocally* on the magnetic field. Note that this argument is precisely analogous to that used in the discussion of gauge transformations with the exception that the induced phase is discontinuous.

The Aharonov-Bohm phase is typically measured using an interferometric method in which the wave passes around both sides of a region of magnetic field without passing through the region. The phase difference between the two paths is obtained using the expression

$$\Delta\phi_{\text{int}} = \oint [\vec{\nabla}S(\vec{r}) + \vec{\nabla}\times\vec{\phi}_V(\vec{r})]\cdot d\vec{l}. \tag{55}$$

Therefore

$$\Delta\phi_{\text{int}}(\vec{r}) = -\frac{e}{c}\oint A(\vec{r})\cdot d\vec{l}. \tag{56}$$

Note that the observability of this phase using interferometry depends critically on the phase being discontinuous. Equation (56) is identical to earlier results [3,39].

VIII. CONCLUSIONS

In this paper we have established that, in the absence of hidden phases, the probability distribution of a quantum-mechanical wave field fully determines the probability current for the wave function. Phase measurement may therefore be performed simply via a measurement of the probability distribution. We have analyzed the form of the probability current when it encounters a sheet of potential (or material) that induces a phase change of the wave function. We established that, with very few limitations, the probability current leaving the sheet of potential responds in precisely the manner that it would with coherent illumination. We then examined some of the practical requirements on such a phase measurement and concluded that noninterferometric probing of potential distributions may be performed using wave fields with very limited coherence requirements. We therefore believe that this noninterferometric approach to phase measurement represents a practical, forgiving, and flexible approach to the probing of wave fields and their interaction with potentials.

ACKNOWLEDGMENTS

The authors wish to acknowledge the support of the Australian Research Council in this work. The authors also acknowledge useful discussions with Robert Scholten, Brendan Allman, Anton Barty, and Lincoln Turner.

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