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Macroscopic quantum fluctuations in the Josephson dynamics of two weakly linked Bose-Einstein condensates

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We study analytically the quantum corrections to the Gross-Pitaevskii equation for two weakly linked Bose-Einstein condensates. The goals are (1) to investigate dynamical regimes at the borderline between the classical and quantum behavior of the bosonic field, and (2) to search macroscopic quantum coherence phenomena not observable with other superfluid/superconducting systems. Quantum fluctuations renormalize the classical Josephson oscillation frequencies. Large amplitude phase oscillations are modulated, exhibiting collapses and revivals. We describe an interwell oscillation mode with a vanishing (ensemble averaged) mean value of the observables, but with oscillating mean square fluctuations. We finally discuss the limit in which the classical Gross-Pitaevskii (Josephson) dynamics is recovered.

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The experimental observation of the Bose-Einstein condensation of a trapped, dilute gas of alkali-metal atoms [1], and the high accuracy of the engineering [2,3], are opening a new avenue to investigate the interplay between macroscopics and quantum coherence. Foundational problems of quantum theory [4,5] and condensed matter [4,6] can be addressed through real (and not just ‘‘gedanken’’) experiments; dynamical regimes not accessible with other superconducting/superfluid systems might be testable.

The main goal of this work is to study analytically the quantum corrections to the classical Gross-Pitaevskii dynamics [7] of two weakly linked Bose-Einstein condensates (BEC's). The Gross-Pitaevskii equation (GPE) has been shown to describe quite accurately the dynamical regimes experimentally investigated so far [8]. On the other hand, BEC's can be experimentally created with a number of atoms ranging from a few thousand to several millions, and in a wide variety of confining geometries [1–3]. This is opening the possibility of studying dynamical regimes at the borderline between the quantum and classical nature of the bosonic field, and, more generally, to search new macroscopic quantum phenomena. Several predicted effects, like, for example, the collapse and the revival of the relative phase of two coupled condensates, have never been observed with other superfluid/superconducting systems.

Different Josephson dynamical regimes are characterized by the ratio of the ‘‘Josephson coupling energy’’ E_J and the ‘‘on-site energy’’ E_c [9–11]. In the limit $E_J \gg E_c$ (often referred to in the literature as ‘‘classical’’ [9,10]), both the phase difference and the relative number of condensate atoms are well defined. In this limit the classical Josephson equations can be ‘‘microscopically’’ derived by the GPE in the ‘‘two-mode’’ approximation [14–24]. On the other hand, we will see that quantum corrections can significantly modify the classical dynamics even for $E_J/E_c \sim 10^2$, a regime accessible in current BEC experiments. Quantum fluctuations renormalize the classical Josephson oscillation frequencies. Large amplitude phase oscillations are modulated by partial collapses and revivals. It is well known that the relative phase of two decoupled condensates diffuses ballistically [4,12–14] and subsequently revives [12] after a much longer time even though the Josephson coupling is absent. Numerical analysis has shown that even in two *coupled* condensates (in a symmetric double-well potential), the relative phase diffuses in the self-trapped, running-phase [16,17] regime. Then, partial or complete revivals could occur, due to the finite number of condensate atoms [16,17]. An asymmetric potential can also induce phase diffusion [18]. There have also been studies involving atom-number fluctuations and decoherence [14,19–21] and finite-temperature effects and damping [19,22,23].

For large condensates, the dynamical equations for the mean values of the physical observables decouple from the equations governing the respective quantum fluctuations. This describes the smooth crossover from the quantum to the classical GPE regime.

The classical boson Josephson junction (BJJ) equations, derived by the GPE in the ‘‘two-mode’’ approximation [14–24], can be cast in terms of two canonically conjugate variables: the relative population $N=\frac{1}{2}(N_1-N_2)$ and phase $\phi=\phi_1-\phi_2$ between the two traps. Quantizing BJJ, the c numbers N and ϕ are replaced by the corresponding operators, satisfying the commutation relation $[\hat{\phi},\hat{N}]=i$ [25]. Then the Hamiltonian of two weakly coupled condensates reads [11]

$$\hat{H}=\frac{E_c}{2}\hat{N}^2-E_J\cos\hat{\phi}+\Delta E_0\hat{N}, \tag{1}$$

where $E_J(\sim N_T^{\alpha};\alpha\sim 1)$ is the ‘‘Josephson coupling energy’’; $E_c(\sim N_T^{-\beta}$, with $\beta=3/5$ in $3d$ traps) is the ‘‘on-site energy,’’ the analog of the charging energy in SJJ; N_T is the total number of condensate atoms. ΔE_0 is the zero-point energy difference in two asymmetric traps [15], or an applied chemical potential difference (induced, for example, by the gravitational potential in vertical traps [3]). The coefficients E_J, E_c are determined by the BJJ geometry and the total number of atoms. They can be (consistently) calculated, for example, as overlap integrals of two orthogonal one-body Gross-Pitaevskii wave functions [15,26].

In the phase representation, the operators in Eq. (1) are expressed as $\hat{N}=-i(\partial/\partial\phi),\hat{\phi}=\phi$. Then the dynamical equation for the amplitude $\Psi(\phi,t)$ is ($\hbar=1$),

$$i\frac{\partial\Psi(\phi,t)}{\partial t}=-\frac{\partial^2\Psi(\phi,t)}{\partial\phi^2}-\Gamma\cos\phi\Psi(\phi,t)-iE_0\frac{\partial\Psi(\phi,t)}{\partial\phi}, \tag{2}$$

where $\Gamma=2E_J/E_c$, $E_0=2\Delta E_0/E_c$, and the time has been rescaled as $(E_c/2)t\rightarrow t$. Since we are considering an isolated, energy-conserving system, the ''potential'' is periodic and defined on a 2π ring, with the wave-function boundary conditions $\Psi(\phi)=\Psi(\phi+2\pi)$.

In the context of superconductors, Eq. (2) describes low-capacitance Josephson junctions [9,10]. The effect of dissipation on its quantum statistical properties (also displaying phase transitions from normal to superconducting phases) has been extensively studied [27]. In superconducting, no current-biased junctions, the phase is localized in the minimum of the potential. By driving the system with an external current, the phase runs (Bloch oscillations), leading to nonzero (high frequency $\sim$GHz) oscillating voltages across the junction. Such high-frequency oscillations are difficult to observe directly. Quantum effects are usually manifested as a renormalized Josephson critical current with respect to the classical prediction [28], although quantum fluctuations have been observed in the measurements of quantum noise in resistively shunted superconducting Josephson junctions (SJJs) [29]. We note here that a related but quite different and well-studied phenomenon is macroscopic quantum tunneling between different minima, occuring in mesoscopic SJJ [9,10,30].

In an (*isolated*) BJJ, on the other hand, density oscillations can be induced by shifting the position of the laser barrier or tailoring the traps [2,3]. (A similar argument holds when considering Raman transitions between two condensate in different hyperfine levels of a single traps). The small frequency, less than 1 kHz, oscillations of the population imbalance, and the corresponding mean square deviations, might be directly measurable by destructive or nondestructive techniques. It is worth stressing, therefore, that the set of experimentally accessible observables in BJJ is quite different from its SJJ counterpart. Furthermore, because BEC's of trapped gas atoms are dilute, weakly interacting, and electrically neutral, it may be easier to uncover new collective quantum phenomena than in the Josephson tunneling of superconducting/superfluid systems.

In this paper we study analytically the quantum corrections to the classical Gross-Pitaevskii-Josephson equations. We provide a quite simple, though accurate, framework to study quantitatively the phenomena that occur in a mesoscopic BJJ [31].

We consider a time-dependent variational approach. The time evolution of the variational parameters is characterized by the minimization of an action with the effective Lagrangian,

$$L(q_i,\dot{q}_i)=i\langle\Psi\dot{\Psi}\rangle-\langle\Psi\hat{H}\Psi\rangle \tag{3}$$

with $\hat{H}=-\partial^2/\partial\phi^2-\Gamma\cos\phi-iE_0(\partial/\partial\phi)$ and $\Psi(\phi,q_i(t))$, q_i being the time-dependent parameters. This provides the familiar Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}=\frac{\partial L}{\partial q_i}. \tag{4}$$

We choose the time-dependent variational wave function as

$$\Psi(\phi,t)=R\left[\frac{\phi-\phi_c(t)}{\lambda(t)}\right]e^{iS[\phi-\phi_c(t),p(t),\delta(t)]} \tag{5}$$

with the square root of the probability density R and the dynamical phase S being real, and

$$S=p(t)[\phi-\phi_c(t)]+\frac{\delta(t)}{2}[\phi-\phi_c(t)]^2. \tag{6}$$

The pairs of time-dependent parameters $\phi_c(t),p(t)$ and $\lambda(t),\delta(t)$ are canonically conjugate,

$$\dot{\phi}_c=\frac{\partial H}{\partial p}=2p+E_0, \tag{7a}$$

$$\dot{p}=-\frac{\partial H}{\partial\phi_c}=-\frac{\partial}{\partial\phi_c}\langle V(\phi)\rangle, \tag{7b}$$

$$\dot{\lambda}=\frac{\partial H}{\partial\delta}=4\lambda\delta, \tag{7c}$$

$$\dot{\delta}=-\frac{\partial H}{\partial\lambda}=-2\delta^2+\frac{\partial}{\partial\lambda}\left[\int_{\phi_c-\pi}^{\phi_c+\pi}RR''d\phi-\langle V(\phi)\rangle\right], \tag{7d}$$

with the effective Hamiltonian

$$\begin{aligned}H&=\langle T\rangle+\langle V\rangle\\&=p^2+2\lambda\delta^2-\int_{\phi_c-\pi}^{\phi_c+\pi}RR''d\phi+\langle V(\phi)\rangle+E_0p.\end{aligned} \tag{8}$$

Thus $p(t)$ is the momentum associated with the center-of-mass motion $\phi_c(t)=\int_{\phi_c-\pi}^{\phi_c+\pi}R^2(\phi-\phi_c)\phi d\phi$, and $\delta(t)$ is the conjugate momentum of the width of the wave function, which is proportional to $\lambda(t)=\frac{1}{2}\int_{\phi_c-\pi}^{\phi_c+\pi}R^2\phi^2d\phi$. The $\langle\ \rangle$ means $\int_{\phi_c-\pi}^{\phi_c+\pi}|\Psi(\phi-\phi_c)|^2\cdots d\phi$, and the prime stands for $\partial/\partial\phi$. The mean value of the population imbalance between the two traps is $N(t)=\langle\Psi(\phi,t)\hat{N}\Psi(\phi,t)\rangle=p(t)$, the relative phase is $\phi(t)=\phi_c(t)$, and the corresponding mean square deviations are $\sigma_N^2(t)=\langle\hat{N}^2\rangle-\langle\hat{N}\rangle^2=\langle T(t)\rangle$ and $\sigma_\phi^2(t)=2\lambda(t)$.

The functional form of $R(\phi,t)$ depends on the value of Γ. For $\Gamma\gg 1$, $R\{[\phi-\phi_c(t)]/\lambda(t)\}$ can be well approximated by a Gaussian:

$$R(\phi,t)=(4\pi\lambda)^{-1/4}e^{-(1/8\lambda)(\phi-\phi_c)^2} \tag{9}$$

with the caveat that during the dynamics its width $2\sigma_\phi(t)=2\sqrt{2\lambda(t)}\ll 2\pi$.

The equations of motion become

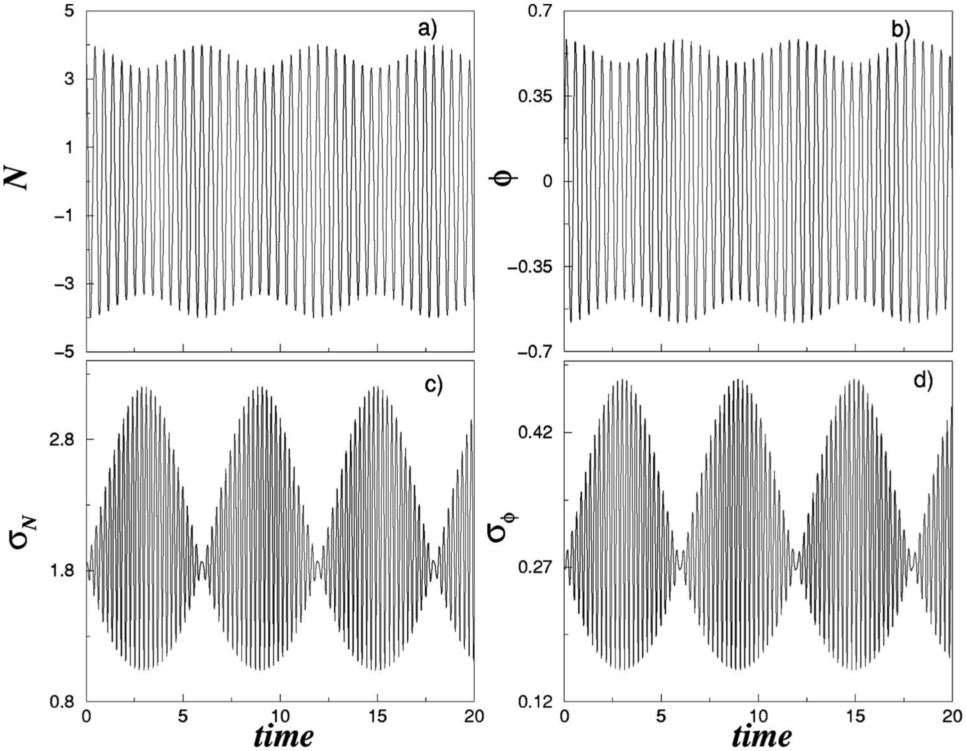

FIG. 1. The population imbalance N, the relative phase ϕ, and the corresponding fluctuations σ_N and σ_ϕ as a function of time. The initial conditions are $N=4$, $\phi=0$, $\Gamma=100$, $\sigma_N=0$, $\sigma_\phi=0.26$, and $E_0=0$.

$$\dot{N}=-\Gamma \sin \phi e^{-\sigma_\phi^2/2}, \tag{10a}$$

$$\dot{\phi}=2N+E_0, \tag{10b}$$

$$\dot{\sigma}_\phi=2\sigma_\phi \delta, \tag{10c}$$

$$\dot{\delta}=-2\delta^2+\frac{1}{2\sigma_\phi^4}-\Gamma \cos \phi e^{-\sigma_\phi^2/2} \tag{10d}$$

with the total (conserved) energy and the relative population dispersion

$$H=N^2+\sigma_N^2-\Gamma \cos \phi e^{-\sigma_\phi^2/2}+E_0 N, \tag{11}$$

$$\sigma_N=\frac{1}{2\sigma_\phi}\sqrt{1+4\sigma_\phi^4\delta^2}. \tag{12}$$

The canonically conjugate dynamical variables are N,ϕ, as in the classical Josephson Hamiltonian, and the pair $\sigma_\phi^2/2$, $\delta=(1/\sigma_\phi)\sqrt{\sigma_N^2-(1/4\sigma_\phi^2)}$, characterizing the respective quantum fluctuations. As expected, $\sigma_N\sigma_\phi \geqslant \frac{1}{2}$ during the dynamics. The classical Josephson equations are recovered from Eqs. (10a) and (10b) in the limit $\sigma_\phi \to 0$. We will discuss more about the transition from the quantum to the classical regimes in the following.

The variational ground-state energy of Eq. (2) is given by

$$E_{gs}=\frac{1}{4\sigma_{\phi,s}^2}-\Gamma e^{-\sigma_{\phi,s}^2/2}-\frac{E_0^2}{4}=\sigma_{N,s}^2-\Gamma e^{-1/8\sigma_{N,s}^2}-\frac{E_0^2}{4}, \tag{13}$$

where $\sigma_{\phi,s},\sigma_{N,s}$ are the solutions of

$$2\sigma_{\phi,s}^4 e^{-\sigma_{\phi,s}^2/2}=e^{-1/8\sigma_{N,s}^2}/8\sigma_{N,s}^2=\frac{1}{\Gamma}. \tag{14}$$

The stationary results were discussed in Ref. [10], where Eqs. (13) and (14) were obtained minimizing the ground-state energy with a Gaussian trial wave function in the case $E_0=0$. Linearizing Eq. (10) for small amplitude ϕ oscillations, we have $\ddot{\phi}=-2\Gamma e^{-\sigma_{\phi,s}^2/2}\phi$. The condensate atoms oscillate coherently with a frequency (unscaled):

$$\omega_q=\sqrt{E_c E_J}e^{-\sigma_{\phi,s}^2/4} \tag{15}$$

where the classical Josephson relation gives $\omega_c=\sqrt{E_c E_J}$. The quantum fluctuations renormalize the Josephson plasma frequency, with $\omega_q/\omega_c=\exp(-\sigma_{\phi,s}^2/4)=(\sigma_{\phi,s}^2\sqrt{2\Gamma})^{-1}$. Notice that in the linear regime, the current-phase equations (10a) and (10b) are effectively decoupled from the dynamics of the respective fluctuation equations (10c) and (10d). On the contrary, for large amplitude ϕ oscillations, Eqs. (10) cannot be decoupled. In this case the exponential factor modulates the amplitude and the frequencies of the oscillations, inducing partial collapses and revivals. This can be seen in Figs. 1(a)–1(d), where we show the population imbalance, the relative phase, and the respective mean square deviations as a function of time.

Above the critical point ($N=0,\phi=\pi/2$), the phase $\phi(t)$ starts running, Fig. 2(b), and the system is set into a macroscopic quantum self-trapping (MQST) mode [15,16]. The width of the wave function grows and the amplitude of oscillations ‘‘collapses,’’ Fig. 2(a). In the deep MQST regime, when $N(t)\simeq N(t=0)$, the phase diffuses as $\sigma_\phi^2(t)\simeq\sigma_{\phi,s}^2+(E_c^2/4\sigma_{\phi,s}^2)t^2$, Fig. 2(d), regardless of the initial value of $N(t=0)$ [32]. The relative population oscillations collapse with a lifetime $\tau\simeq 2E_J^{-1/4}E_c^{-3/4}$, while the $\sigma_N^2(t)$ tends to a constant value, Fig. 2(c). However, since the total number of

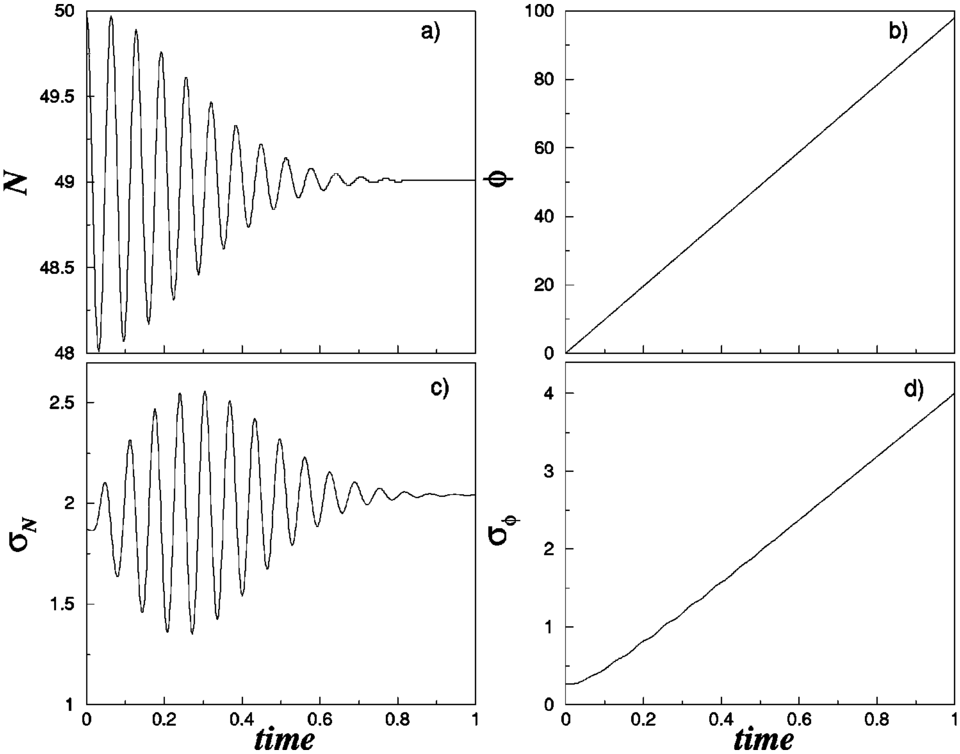

FIG. 2. $N, \phi, \sigma_N, \sigma_\phi$ as a function of time. The initial conditions are the same as in Fig. 1 except $N(t=0)=50$.

condensate atoms is finite, the phase can eventually revive partially or completely. This can be seen rewriting the wave function in the N representation: $\Psi(\phi,t) = \Sigma_N a_N \Phi_N(\phi) e^{-i/\hbar E_N t}$, where E_N are the eigenenergies of Eq. (2). For instance, in the limit $E_c N_T \ll E_J$, the eigenvalue spectrum is approximately linear $E_N \sim E_J / N_T N$ (for a discussion of the spectrum in various regimes, see, e.g., [16]), and the revival time is $\tau_R \sim h N_T / E_J$. More generally, the occurrence of a complete or partial revival, or the complete destruction of it, depends on the detailed eigenspectrum of the Hamiltonian. We note that Eq. (10) cannot describe the revival after a complete collapse since the Gaussian ansatz Eq. (9) (and, consequently, the semiclassical approximation underlying it) breaks down when $\sigma_\phi \simeq \pi$.

Equations (10) admit, as a dynamical solution, a quite peculiar oscillation mode, with zero relative phase and population imbalance, but oscillating fluctuations, according to

$$N(t)=0, \tag{16a}$$

$$\phi(t)=0, \tag{16b}$$

$$\sigma_N^2(t) = \frac{1}{4\sigma_\phi^2} + \frac{\dot{\sigma}_\phi^2}{4}, \tag{16c}$$

$$\ddot{\sigma}_\phi(t) = \frac{1}{\sigma_\phi^3} - 2\Gamma \sigma_\phi e^{-\sigma_\phi^2/2}, \tag{16d}$$

with initial conditions $N(t=0)=0, \phi(t=0)=0; E_0=0$ and arbitrary $\sigma_\phi(t=0)$. This collective oscillation mode can be experimentally observed by lowering or increasing the height of the barrier of a BJJ ensemble in thermodynamic equilibrium. This corresponds to changing Γ in Eq. (16) from its initial value. The temporal evolution of the ensemble averaged observables and the respective mean square deviations can be calculated by tracing the dynamical $N(t), \phi(t)$ trajectories of each junction.

Classical limit. Increasing the number of atoms, $\sigma_\phi \rightarrow 0$ as $\Gamma^{-1/4} \sim N_T^{-(1/4)(\alpha+\beta)}$, and $\sigma_N / N_T \sim \Gamma^{1/2}/N_T \sim N_T^{(1/2)(\alpha+\beta)-1} \rightarrow 0$, if $\frac{1}{2}(\alpha+\beta)-1<0$. Moreover, for a given initial value $N(t=0)$, the amplitude of the "particle" oscillations in the ϕ potential of Eq. (2), decreases as $\phi_{max} \sim \Gamma^{-1/2} \sim N_T^{-(1/2)(\alpha+\beta)}$. Then Eqs. (10a) and (10b) decouple from Eqs. (10c) and (10b), and the time evolution of the mean values of current and phase become independent of the corresponding dispersions. In the MQST regime, the collapse time (and, consequently, the time over which the semiclassical predictions are reliable), increases as $\tau \sim N_T^{(1/4)(3\beta-\alpha)}$. In this framework the classical limit emerges naturally [6]. A similar result displaying the asymptotic approach towards the classical limit upon increase of the number of condensate atoms has also been found in [17].

Numerical estimates. Following the analytical estimations of the Josephson coupling energy and the on-site energy given for two weakly coupled condensates in [22], we have

$$\Gamma \simeq 1.7 N_T \frac{a_0}{a_s} \frac{\exp(-S)}{\tanh(S/2)},$$

with a_0, a_s the trap length and the scattering length, respectively, and with $S \sim (1/\hbar)\sqrt{2m\sigma_B^2(V_0-\mu)}$. σ_B is the width of the barrier, V_0 its height, and μ the chemical potential. For typical traps and condensates, and taking sodium atoms $a_0 \sim 10^4$ Å, $a_s \sim 50$ Å, and $\sigma_B \sim 5$ μm. With a height of the barrier such that $(V_0-\mu) \sim 30$ nK, we have $S \approx 8$ and $\Gamma \approx 80$ for $N_T \sim 1000$. By varying the width and/or the height of the barrier, and the total number of condensate atoms, one can let the system span from the $\Gamma \ll 1$ to the $\Gamma \gg 1$ limits. The temperature should be small compared to the Josephson coupling energy [19,22] to avoid destroying the quantum fluctuations. Damping effects are also reduced by decreasing the total number of atoms. We conclude noting that Eqs. (10) can be easily generalized to describe interwell tunneling in an array of trapped condensates, recently observed in [3]. In that experiment, the average population *per site* is of the order of thousand condensate atoms, a regime where quantum fluctuations can play an important role. This problem, along with the effects of temperature and damping, certainly deserves further studies.

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[31] Clearly, Eq. (2) can be studied numerically, in terms of the eigenstates of the Mathieu equation or by direct integration; but this would obscure the subtle relations between observables in the semiclassical regime.

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