Electron-atom scattering in a circularly polarized laser field

Aurelia Cionga,[1] Fritz Ehlotzky,[2,*] and Gabriela Zloh[1]
[1]*Institute for Space Sciences, P. O. Box MG-23, R-76900 Bucharest, Romania*
[2]*Institute for Theoretical Physics, University of Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria*
(Received 29 November 1999; published 17 May 2000)

We consider electron-atom scattering in a circularly polarized laser field at sufficiently high electron energies, permitting us to describe the scattering process by a first-order Born approximation. Assuming the radiation field has sufficiently moderate intensities, the laser dressing of the hydrogen target atom in its ground state will be treated in second-order perturbation theory. Within this approximation scheme, it is shown that the nonlinear differential cross sections of free-free transitions do not depend on the *dynamical phase* ϕ of the radiative process nor on the *helicity* of the circularly polarized laser light. Relations to the corresponding results for linear laser polarization are established.

PACS number(s): 34.80.Qb, 34.50.Rk, 32.80.Wr

I. INTRODUCTION

Since the early theoretical work of Bunkin and Fedorov [1] and Kroll and Watson [2] and the experiments by Weingartshofer and Jung [3] a considerable amount of work has been devoted to the investigation of electron-atom scattering in the presence of a powerful laser field. Reviews on this topic can be found on the theoretical side in a survey by one of the present authors [4] and on the experimental situation in a summary given by Mason [5]. Further details can also be found in the books by Mittleman [6] and by Faisal [7], as well as in the work by Gavrila [8]. Initially the atomic target was described by a static potential but starting with the work of Gersten and Mittleman [9] the laser dressing of the target was taken into account, treating the radiation-atom interaction perturbatively. Along the same lines work was published by Zon [10], Beilin and Zon [11] and, in particular, by Joachain and co-workers in several consecutive papers [12–15] as well as by Maquet and co-workers [16–18]. In all these investigations a linearly polarized (LP) laser field was considered. More recently, it became of interest to analyze in some detail the case of a circularly polarized (CP) laser field [19–25] to find out, in particular, whether for CP the nonlinear scattering cross sections depend explicitly on the *dynamical phase* ϕ and the *helicity* of the radiation field.

It is the purpose of the present work to investigate free-free transitions on a hydrogen atom for a CP laser field. The scattering process is treated in the first-order Born approximation and the target dressing by the radiation field is taken into account in second-order perturbation theory. It will be explicitly shown that in this case the nonlinear differential cross sections depend neither on the *dynamical phase* ϕ nor on the *helicity* of the radiation field. In Sec. II we shall start our investigations by considering free-free transitions in a CP laser field on a laser-dressed model potential in order to define the essential parameters of the process. We shall then investigate in Sec. III in greater detail and generality the effects of atomic dressing evaluating first- and second-order radiative corrections to the bound state. Section IV will be devoted to a discussion of our numerical results for the angular distribution and the frequency dependence of the nonlinear signals in electron-hydrogen scattering in a CP laser field. Comparison will be made between these signals for CP and those for LP fields and the main differences encountered will be analyzed. The final section will summarize our findings. Atomic units will be used throughout our investigation.

II. SCATTERING ON A POTENTIAL

We consider free-free transitions for scattering of an electron by the potential

$$V(\vec{r},t)=V(r)+\alpha_s\frac{\vec{r}\cdot\vec{\mathcal{E}}(t)}{r^3}, \tag{1}$$

which may describe a hydrogen atom in a laser field. $V(r)$ denotes the potential

$$V(r)=-e^{-2r}\left(1+\frac{1}{r}\right) \tag{2}$$

and α_s is the static polarizability ($\alpha_s=4.5$ a.u. for hydrogen in its ground state). The second term in Eq. (1) describes approximately the interaction between the electron and the atomic dipole moment induced by the field. An effective laser-dressed potential of the form Eq. (1) was already used by several authors [10,26,27] for LP fields.

For CP the electric field is given in the dipole approximation by

$$\vec{\mathcal{E}}(t)=i\frac{\mathcal{E}_0}{2}[\exp(-i\omega t)\vec{\varepsilon}-\exp(i\omega t)\vec{\varepsilon}^*]$$

$$\equiv\frac{\mathcal{E}_0}{\sqrt{2}}(\vec{e}_i\sin\omega t-\vec{e}_j\cos\omega t) \tag{3}$$

with the polarization vector defined by

*Author to whom correspondence should be addressed. Electronic address: Fritz.Ehlotzky@uibk.ac.at

$$\vec{\varepsilon}=\frac{\vec{e}_i+i\vec{e}_j}{\sqrt{2}}, \tag{4}$$

$\mathcal{E}_0$ is the amplitude and ω the frequency of the electric field; $\vec{e}_i$ and $\vec{e}_j$ are unit vectors along two orthogonal directions in the polarization plane.

In the first-order Born approximation, the *S*-matrix element corresponding to the scattering of the electron on the potential Eq. (1) is

$$S_{if}^{B1}=-i\int_{-\infty}^{\infty}dt\langle\chi_{\vec{k}_f}(\vec{r},t)|V(\vec{r},t)|\chi_{\vec{k}_i}(\vec{r},t)\rangle. \tag{5}$$

$\chi_{\vec{k}_{i,f}}(\vec{r},t)$ are Volkov solutions, which describe the projectile in the initial and final state, respectively. Since the Volkov state is written in the velocity gauge, while the electron-dipole interaction in Eq. (1) is written in the length gauge, a gauge factor would have to be introduced for consistency reasons. In the present approximation, however, the gauge factors drop out in Eq. (5).

For an electron of kinetic energy E_k and momentum $\vec{k}$, the Volkov solution reads

$$\chi_{\vec{k}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\exp\{-iE_kt+i\vec{k}\cdot\vec{r}-i\vec{k}\cdot\vec{\alpha}(t)\}, \tag{6}$$

where $\vec{\alpha}(t)$ represents the classical oscillation of the electron in the electric field $\vec{\mathcal{E}}(t)$. In the case of the above CP laser field this quiver motion is given by

$$\vec{\alpha}(t)=\frac{\alpha_0}{\sqrt{2}}(\vec{e}_i\sin\omega t-\vec{e}_j\cos\omega t), \tag{7}$$

with $\alpha_0=\mathcal{E}_0/\omega^2$. The Fourier expansion of Eq. (6) leads to the following series in terms of ordinary Bessel functions, J_N,

$$\exp\left[\frac{-i\alpha_0}{\sqrt{2}}\vec{k}\cdot(\vec{e}_i\sin\omega t-\vec{e}_j\cos\omega t)\right]$$
$$=\exp\{-i\mathcal{R}_k\sin(\omega t-\phi_k)\}=\sum_N J_N(\mathcal{R}_k)$$
$$\times\exp(-iN\omega t)\exp(iN\phi_k), \tag{8}$$

which is obtained using Graf's addition theorem [28]. Accordingly, the following notations were introduced

$$\mathcal{R}_k=\frac{\alpha_0}{\sqrt{2}}\sqrt{(\vec{k}\cdot\vec{e}_i)^2+(\vec{k}\cdot\vec{e}_j)^2}\equiv\alpha_0|\vec{\varepsilon}\cdot\vec{k}|, \tag{9}$$

$$\sin\phi_k=\frac{\vec{k}\cdot\vec{e}_j}{\sqrt{(\vec{k}\cdot\vec{e}_i)^2+(\vec{k}\cdot\vec{e}_j)^2}}, \tag{10}$$

$$\cos\phi_k=\frac{\vec{k}\cdot\vec{e}_i}{\sqrt{(\vec{k}\cdot\vec{e}_i)^2+(\vec{k}\cdot\vec{e}_j)^2}}. \tag{11}$$

The last two equations lead to

$$\phi_k=\operatorname{arctg}\left(\frac{\vec{k}\cdot\vec{e}_j}{\vec{k}\cdot\vec{e}_i}\right)+l\pi \tag{12}$$

with l an integer. We stress that the correct values of l should satisfy **both** Eqs. (10) and (11) in order to be consistent with a proper use of Graf's addition theorem. By means of the *dynamical phase* ϕ_k defined above we get

$$\exp(i\phi_k)=\frac{\vec{k}\cdot\vec{\varepsilon}}{|\vec{k}\cdot\vec{\varepsilon}|}. \tag{13}$$

Writing down the expansion Eq. (8) for the momentum transfer of the scattered electron, $\vec{q}=\vec{k}_i-\vec{k}_f$, one can perform the time integration in Eq. (5) to obtain

$$S_{if}^{B1}=\frac{i}{2\pi}\sum_N \delta(E_f-E_i-N\omega)f_N^{B1}, \tag{14}$$

where

$$f_N^{B1}=e^{iN\phi_q}\left\{J_N(\mathcal{R}_q)f_{el}^{B1}(q)-\alpha_s\frac{\mathcal{E}_0}{q}\left[e^{-i\phi_q}\frac{\vec{\varepsilon}\cdot\vec{q}}{q}J_{N-1}(\mathcal{R}_q)\right.\right.$$
$$\left.\left.-e^{i\phi_q}\frac{\vec{\varepsilon}^*\cdot\vec{q}}{q}J_{N+1}(\mathcal{R}_q)\right]\right\}. \tag{15}$$

J_N denotes a Bessel function of order N and f_{el}^{B1} is the elastic transition amplitude in the first-Born approximation for the static potential Eq. (2)

$$f_{el}^{B1}(q)=2(q^2+8)/(q^2+4)^2, \tag{16}$$

$\mathcal{R}_q$ and ϕ_q are defined according to Eqs. (9)–(11) using $\vec{q}$ instead of $\vec{k}$.

In the presence of the radiation field the scattered electron may gain or loose energy equal to $N\omega$, such that $E_f=E_i+N\omega$, where $E_{i(f)}$ is the initial (final) energy of the projectile and N is the net number of photons exchanged (absorbed or emitted) by the colliding system and the CP field. The energy spectrum of the scattered electrons therefore consists of the elastic term, corresponding to $N=0$, and of a number of sidebands, each pair of sidebands corresponding to the same value of $|N|$. For free-free transitions involving N photons one can write down the differential cross section in terms of the scattering amplitude f_N^{B1} as

$$\frac{d\sigma_N^{\mathrm{CP}}}{d\Omega}=\frac{k_f}{k_i}|f_N^{B1}|^2. \tag{17}$$

Using Eq. (13) to rewrite the scattering amplitude Eq. (15) one gets by elementary vector algebra the following form of the differential cross section

$$\frac{d\sigma_N^{\mathrm{CP}}}{d\Omega}=\frac{k_f}{k_i}\left|J_N(\mathcal{R}_q)f_{el}^{B1}(q)-2\alpha_s\mathcal{E}_0\frac{|\vec{\varepsilon}\cdot\vec{q}|}{q^2}J_N'(\mathcal{R}_q)\right|^2, \tag{18}$$

where J_N' is the first derivative of the Bessel function with respect to its argument. It satisfies the relation

$$J_N'(\mathcal{R}_q)=\frac{1}{2}[J_{N-1}(\mathcal{R}_q)-J_{N+1}(\mathcal{R}_q)]. \tag{19}$$

In this form of the differential cross section it is apparent that the laser assisted signals in a CP field are neither sensitive to the *dynamical phase*, ϕ_q, nor to the *helicity* of the photon.

As in the case of linear polarization, the main limitation of the formula Eq. (18) is its failure to describe the well known decreasing of the target dressing with the increasing of the scattering angle [16]. An improvement, suggested by Milošević *et al.* [29], consists in replacing the static polarizability, α_s, by the so-called dynamical polarizability

$$\alpha_d=\frac{\alpha_s}{(1+q^2/4)^3}. \tag{20}$$

This permits use of Eq. (18) at higher scattering angles. Despite the above limitation, for low frequencies and small scattering angles, Eq. (18) might be useful as a starting point for the corresponding investigation of many electron targets for which other methods will likely be prohibitively difficult to employ.

III. ELECTRON-ATOM SCATTERING

We assume that at moderate laser field intensities, one can describe the field-atom interaction by time-dependent perturbation theory [12]. We shall use in the following *second-order perturbation theory* to describe the hydrogen ground state in the presence of a CP field. According to Florescu *et al.* [30], one can write down an approximate solution for an electron bound to a Coulomb potential in the presence of an electromagnetic field as follows:

$$|\Psi_1(t)\rangle=e^{-iE_1t}[|\psi_{1s}\rangle+|\psi_{1s}^{(1)}\rangle+|\psi_{1s}^{(2)}\rangle], \tag{21}$$

where $|\psi_{1s}\rangle$ is the unperturbed ground state of hydrogen, of energy E_1, and $|\psi_{1s}^{(1),(2)}\rangle$ denote first- and second-order corrections, respectively. According to Refs. [30] and [31] these corrections can be written in terms of the linear response

$$|\vec{w}_{1s}(\Omega)\rangle=-G_C(\Omega)\vec{P}|\psi_{1s}\rangle, \tag{22}$$

and of the quadratic response

$$|w_{ij,1s}(\Omega',\Omega)\rangle=G_C(\Omega')P_iG_C(\Omega)P_j|\psi_{1s}\rangle. \tag{23}$$

Here $G_C(\Omega)$ is the Coulomb Green's function and $\vec{P}$ the momentum operator of the bound electron. For a monochromatic field there are four values of the argument of the Green functions necessary in order to write down the approximate solution Eq. (21), namely,

$$\Omega^{\pm}=E_1\pm\omega,\quad \Omega'^{\pm}=E_1\pm 2\omega. \tag{24}$$

On the other hand, as in the case of potential scattering in Sec. II, the interaction between the CP field and the projectile can be treated exactly using the Volkov-type solution Eq. (6).

We restrict our considerations to high scattering energies where the first-Born approximation in terms of the scattering potential is reliable. Neglecting exchange effects, we describe this interaction by a static potential, $V(r,R)$, and the scattering matrix element is then given by

$$S_{if}^{B1}=-i\int_{-\infty}^{+\infty}dt\langle\chi_{\vec{k}_f}(t)\Psi_1(t)|V|\chi_{\vec{k}_i}(t)\Psi_1(t)\rangle, \tag{25}$$

where Ψ_1 and $\chi_{\vec{k}_{i,f}}$ are taken from Eqs. (21) and (6).

The differential cross sections for a process in which N photons are involved can be written as

$$\frac{d\sigma_N^{\mathrm{CP}}}{d\Omega}=(2\pi)^4\frac{k_f(N)}{k_i}|T_N^{\mathrm{CP}}|^2, \tag{26}$$

where the transition matrix element, related to the S-matrix element Eq. (25), has the following general structure

$$T_N^{\mathrm{CP}}=\exp(iN\phi_q)[T_N^{(0)}+T_N^{(1)}+T_N^{(2)}]. \tag{27}$$

The first term,

$$T_N^{(0)}=J_N(\mathcal{R}_q)\langle\psi_{1s}|F(\vec{q})|\psi_{1s}\rangle, \tag{28}$$

relates to the Bunkin-Fedorov formula [1], in which the dressing of the target is neglected. In this case T_N^{CP} reduces to $T_N^{\mathrm{CP}}=\exp(iN\phi_q)T_N^{(0)}$ and the ordinary Bessel function, $J_N(\mathcal{R}_q)$, contains all the field dependences of the transition matrix elements. $F(\vec{q})$ is the form factor operator

$$F(\vec{q})=\frac{1}{2\pi^2q^2}[\exp(i\vec{q}\cdot\vec{r})-1]. \tag{29}$$

The other two terms in Eq. (27) are due to the dressing of the atomic state in the CP field and are discussed in the next two subsections.

A. First-order dressing of the target

The second term in Eq. (27), $T_N^{(1)}$, is connected to the first-order corrections to the atomic state: *one* of the N photons exchanged between the field and the colliding system interacts with the bound electron. This photon may be emitted or absorbed and therefore, once the integration over the coordinates of the projectile was performed, the general structure of $T_N^{(1)}$ is given by

$$T_N^{(1)}=-\frac{\alpha_0\omega}{2}[e^{-i\phi_q}J_{N-1}(\mathcal{R}_q)\mathcal{M}_{at}^{(I)}(\Omega^+)+e^{i\phi_q}J_{N+1}(\mathcal{R}_q)\mathcal{M}_{at}^{(I)}(\Omega^-)]. \tag{30}$$

The transition matrix element $\mathcal{M}_{at}^{(I)}(\Omega^{\pm})$ is related to the exchange of one photon between the atomic electron and the field. Their expressions read in terms of the linear response Eq. (22) for absorption

$$\mathcal{M}_{at}^{(I)}(\Omega^{+})=\langle\psi_{1s}|F(\vec{q})|\vec{\varepsilon}\cdot\vec{w}_{1s}(\Omega^{+})\rangle+\langle\vec{\varepsilon}^{*}\cdot\vec{w}_{1s}(\Omega^{-})|F(\vec{q})|\psi_{1s}\rangle \tag{31}$$

and emission

$$\mathcal{M}_{at}^{(I)}(\Omega^{-})=\langle\psi_{1s}|F(\vec{q})|\vec{\varepsilon}^{*}\cdot\vec{w}_{1s}(\Omega^{-})\rangle+\langle\vec{\varepsilon}\cdot\vec{w}_{1s}(\Omega^{+})|F(\vec{q})|\psi_{1s}\rangle, \tag{32}$$

respectively. Using Eqs. (8) and (10)–(12) of Ref. [32], one gets

$$\mathcal{M}_{at}^{(I)}(\Omega^{+})=-\frac{\vec{\varepsilon}\cdot\vec{q}}{2\pi^{2}q^{3}}\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q),\quad \mathcal{M}_{at}^{(I)}(\Omega^{-})=-\frac{\vec{\varepsilon}^{*}\cdot\vec{q}}{2\pi^{2}q^{3}}\mathcal{J}_{1,0,1}(\tau^{-},\tau^{+},q). \tag{33}$$

The parameters $\tau^{\pm}$ are related to the parameters $\Omega^{\pm}$ defined in Eq. (24) by

$$\tau^{\pm}=1/\sqrt{-2\Omega^{\pm}}. \tag{34}$$

An analytic expression for $\mathcal{J}_{1,0,1}$ can be obtained from Eqs. (17)–(22) in Ref. [32]. One has

$$\mathcal{J}_{1,0,1}(\tau^{\pm},\tau^{\mp},q)=\mathcal{J}_{1,0,1}^{a}(q,\tau^{\pm})-\mathcal{J}_{1,0,1}^{b}(q,\tau^{\mp}), \tag{35}$$

where

$$\mathcal{J}_{1,0,1}^{a}(q,\tau)=\mathcal{J}_{1,0,1}^{b}(q,\tau)=-\frac{16}{q}\frac{1}{(1+\tau)^{4}}\frac{\tau}{2-\tau}\times\mathrm{Re}\left\{a^{3}F_{1}(2-\tau,1,3,3-\tau;\xi,\zeta)-\frac{ia^{2}}{q}F_{1}(2-\tau,2,2,3-\tau;\xi,\zeta)\right\}. \tag{36}$$

The foregoing equation is written for frequencies below the ionization threshold, where $\tau^{\pm}$ are real. $F_{1}(a,b,b',c;x,y)$ is the Appell function of two variables, defined in Ref. [33] and the following notations are used

$$a=\frac{2\tau}{1+\tau+iq\tau}, \tag{37}$$

$$\xi=-\frac{1-\tau}{1+\tau},\quad \zeta=\frac{1-\tau}{1+\tau}\frac{1-\tau-iq\tau}{1+\tau+iq\tau}. \tag{38}$$

Our expressions in Eq. (33) are equivalent with the ones based on Eqs. (18a)–(18c) of Dubois *et al.* [16] and Eqs. (11)–(12) of Dubois and Maquet [17], respectively.

By means of Eqs. (33) and (13) one can write down

$$T_{N}^{(1)}=\frac{\alpha_{0}\omega}{4\pi^{2}q^{2}}\frac{|\vec{\varepsilon}\cdot\vec{q}|}{q}[J_{N-1}(\mathcal{R}_{q})\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q)+J_{N+1}(\mathcal{R}_{q})\mathcal{J}_{1,0,1}(\tau^{-},\tau^{+},q)], \tag{39}$$

which leads to the following transition matrix element

$$T_{N}^{\mathrm{CP}}=\frac{\exp(iN\phi_{q})}{2\pi^{2}q^{2}}\left\{-\frac{q^{2}}{2}f_{el}^{B1}(q)J_{N}(\mathcal{R}_{q})+\alpha_{0}\omega\frac{|\vec{\varepsilon}\cdot\vec{q}|}{q}J_{N}'(\mathcal{R}_{q})\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q)\right\}. \tag{40}$$

To obtain the last expression, we used the following identity

$$\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q)=-\mathcal{J}_{1,0,1}(\tau^{-},\tau^{+},q). \tag{41}$$

In this framework, where the first-order radiation correction to the ground state is taken into account only, the differential cross section for a process in which N photons are exchanged between the colliding system and the CP laser field is given by

$$\frac{d\sigma_{N}^{\mathrm{CP}}}{d\Omega}=\frac{k_{f}}{k_{i}}\left|f_{el}^{B1}(q)J_{N}(\mathcal{R}_{q})-2\alpha_{0}\omega\frac{|\vec{\varepsilon}\cdot\vec{q}|}{q^{3}}J_{N}'(\mathcal{R}_{q})\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q)\right|^{2}. \tag{42}$$

We conclude from this result: (*i*) the *dynamical phase* ϕ_{q} drops out and hence has no effect on the differential cross section and (*ii*) due to the appearance of the modulus $|\vec{\varepsilon}\cdot\vec{q}|$, the *helicity* of the photon is not a relevant parameter.

For the purpose of future reference, we also write down the corresponding differential cross section for linear polarization

$$\frac{d\sigma_{N}^{\mathrm{LP}}}{d\Omega}=\frac{k_{f}}{k_{i}}\left|f_{el}^{B1}(q)J_{N}(\vec{\alpha}_{0}\cdot\vec{q})-2\alpha_{0}\omega\frac{\vec{e}\cdot\vec{q}}{q^{3}}J_{N}'(\vec{\alpha}_{0}\cdot\vec{q})\mathcal{J}_{1,0,1}(\tau^{+},\tau^{-},q)\right|^{2}. \tag{43}$$

To avoid possible confusion, the linear polarization vector was denoted by $\vec{e}$. One can see that, apart from the arguments of the Bessel functions ($\mathcal{R}_{q}$ instead of $\vec{\alpha}_{0}\cdot\vec{q}$), the only difference between Eqs. (42) and (43) concerns the angular parts ($|\vec{\varepsilon}\cdot\vec{q}|$ instead of $\vec{e}\cdot\vec{q}$).

Based on the low frequency limit in Eq. (35), we also mention that the transition matrix element (40) leads to the expression

$$\frac{d\sigma_N^{\rm CP}}{d\Omega} \simeq \frac{k_f}{k_i}\left| f_{el}^{B1}(q) J_N(\mathcal{R}_q) - \frac{192}{(q^2+4)^3} \times \left(1+\frac{8}{q^2+4}\right)\mathcal{E}_0 \frac{|\vec{\varepsilon}\cdot\vec{q}|}{q^2} J_N'(\mathcal{R}_q)\right|^2, \quad (44)$$

which can be immediately compared with Eq. (2.31a) of Byron *et al.* [14], evaluated for linear polarization. In addition, for small scattering angles, where $q \ll 1$, the quantity in front of $\mathcal{E}_0$ may be approximated by $9 \simeq 2\alpha_s$. This shows that one may consider Eq. (18) as the low frequency limit of the differential cross section Eq. (42), valid at small scattering angles.

Our Eqs. (42) and (43) emphasize the importance of the geometrical relation between the momentum transfer of the scattering electron and the polarization vector. To make the discussion clear, we shall choose the quantization axis, Oz, along the direction of the initial momentum of the projectile and the axis Oy in the scattering plane.

It is worthwhile to point out here the correspondence between the three most frequently considered scattering geometries for LP laser light, namely,

LP1: $\vec{e}$ parallel to the initial momentum, $\vec{e}||Oz$;

LP2: $\vec{e}$ orthogonal to the initial momentum but in the scattering plane, $\vec{e}||Oy$;

LP3: $\vec{e}$ parallel to the momentum transfer, $\vec{e}||\vec{q}$, and the following configurations involving CP,

CP1: $\vec{\varepsilon}=(\vec{e}_z+i\vec{e}_x)/\sqrt{2}$, when the laser beam is propagating in the scattering plane,

CP2: $\vec{\varepsilon}=(\vec{e}_x+i\vec{e}_y)/\sqrt{2}$, when the laser beam is parallel to the direction of the initial momentum, $\vec{k}_i$,

CP3: $\vec{\varepsilon}=(\vec{e}_y+i\vec{e}_z)/\sqrt{2}$ lies in the scattering plane (yOz) and the laser beam propagates on the Ox direction.

We mention that for **CP1** and **CP2** there is only one component of the CP vector in the scattering plane, while for **CP3** both components are active. One can immediately see that the following relation holds

$$|\vec{\varepsilon}_j\cdot\vec{q}|^2 = |\vec{e}_j\cdot\vec{q}|^2/2, \quad (45)$$

where the index j refers to the above enumerations.

Once we have established these correspondences, we can make a couple of remarks concerning the relations between laser assisted signals in CP and LP laser fields. We shall always refer to LP and CP, which are connected to each other by Eq. (45).

The first remark concerns the difference between the laser assisted signals in LP and CP in the case of the elastic term $(N=0)$. For low frequencies, in the forward direction, the laser assisted signal is smaller in CP than in LP and the difference is given by

$$\frac{d\sigma_0^{\rm LP}}{d\Omega} - \frac{d\sigma_0^{\rm CP}}{d\Omega} \simeq \alpha_s \frac{\mathcal{E}_0^2}{\omega^2}\frac{|\vec{e}_j\cdot\vec{q}|^2}{q^2} \quad (46)$$

in any of the three related configurations.

The second remark is more general and it is valid for any photon frequency. For weak laser fields at any scattering angles and for moderate laser intensities at small scattering angles, i.e., whenever the arguments of the Bessel functions are small, the following relation exists for $|N| \geq 1$:

$$\frac{d\sigma_N^{\rm CP}}{d\Omega} \simeq \frac{1}{2^{|N|}}\frac{d\sigma_N^{\rm LP}}{d\Omega}. \quad (47)$$

This relation is of particular interest since for $N=\pm 1$ one can recover in this way the perturbative limit given by

$$\frac{d\sigma_{\pm 1}^{\rm CP}}{d\Omega} = \alpha_0^2 \frac{k_f}{k_i}\frac{|\vec{q}\cdot\vec{\varepsilon}|^2}{4}\left| f_{el}^{B1}(q) \mp \frac{2\omega}{q^3}\mathcal{J}_{1,0,1}(\tau^+,\tau^-,q)\right|^2. \quad (48)$$

The same expression can be obtained by using Eqs. (14)–(16) of Ref. [32]. One should keep in mind that in that paper the photon energy was expressed in Rydbergs and that Eq. (7) of the same paper (devoted to excitation processes) should be modified for free-free transitions by putting $f_{el} = \mathcal{J}_{10}(q) - 1 = -q^2 f_{el}^{B1}/2$. In the weak field limit we find out that, on account of the relations Eq. (45), the laser assisted signal involving one CP photon (absorbed/emitted) will be always one-half of the corresponding signal for LP. For higher intensities, the deviations from this relation appear as a signature of nonlinear dynamics.

B. Second-order corrections to the ground state

We shall show in this section that by adding second- or higher-order terms in the expansion (21) we get no change in our main conclusion that neither the dynamical phase ϕ_q nor the photon helicity are relevant parameters in free-free transitions at high scattering energies.

If the second-order correction, $|\psi_{1s}^{(2)}\rangle$ in Eq. (21), is added to the wave function that describes the ground state of hydrogen in the laser field, we get the third contribution to the transition matrix element in Eq. (27). After integration over the coordinates of the projectile, this contribution reads

$$T_N^{(2)} = \frac{\alpha_0^2\omega^2}{4}\{e^{-2i\phi_q}J_{N-2}(\mathcal{R}_q)\mathcal{M}_{at}^{(II)}(\Omega'^+,\Omega^+) + e^{2i\phi_q}J_{N+2}(\mathcal{R}_q)\mathcal{M}_{at}^{(II)}(\Omega'^-,\Omega^-) + J_N(\mathcal{R}_q) \times[\tilde{\mathcal{M}}_{at}^{(II)}(E_1,\Omega^-) + \tilde{\mathcal{M}}_{at}^{(II)}(E_1,\Omega^+)]\}. \quad (49)$$

In this expression *two* of the N photons exchanged between the field and the colliding system interact with the bound electron. $\mathcal{M}_{at}^{(II)}(\Omega'^\pm,\Omega^\pm)$ are related to the absorption (upper signs) or emission (lower signs) of *both* photons. Written in terms of the quantities given by Eqs. (22)–(23) $\mathcal{M}_{at}^{(II)}$ has the form

$$\mathcal{M}_{at}^{(II)}(\Omega'^{\pm},\Omega^{\pm}) = \sum_{j,l=1}^{3} \varepsilon_j \varepsilon_l [\langle \psi_{1s} | F(\vec{q}) | w_{lj,1s}(\Omega'^{\pm},\Omega^{\pm}) \rangle + \langle w_{j,1s}(\Omega^{\mp}) | F(\vec{q}) | w_{l,1s}(\Omega^{\pm}) \rangle + \langle w_{lj,1s}(\Omega'^{\mp},\Omega^{\mp}) | F(\vec{q}) | \psi_{1s} \rangle]. \quad (50)$$

We stress that the complex conjugate of the polarization vector $\vec{\varepsilon}$ must be taken in Eq. (50) when $\mathcal{M}_{at}^{(II)}(\Omega'^{-},\Omega^{-})$, related to emission, is computed for a CP laser field. For this polarization $\vec{\varepsilon}^2 = 0$ and the angular behavior of $\mathcal{M}_{at}^{(II)}$ is determined by

$$\mathcal{M}_{at}^{(II)} = \frac{(\vec{\varepsilon}\cdot\vec{q})^2}{2\pi^2 q^4} \mathcal{T}_1(\tau'^{+},\tau'^{-};\tau^{+},\tau^{-},q), \quad (51)$$

where $\mathcal{T}_1$ depends not only on q and $\tau^{\pm}$ but also on

$$\tau'^{\pm} = 1/\sqrt{-2\Omega'^{\pm}} \quad (52)$$

and can be expressed in terms of a series of hypergeometric functions, as shown in the Appendix.

The other two atomic matrix elements in Eq. (49), $\tilde{\mathcal{M}}_{at}^{(II)}$, are related to the processes in which one photon is absorbed and the other is emitted. They can be constructed by using Eq. (50) with the tensor $\tilde{w}_{lj}$ (instead of the tensor w_{lj}), which is also defined in Ref. [30]. Their angular behavior is different, namely,

$$\tilde{\mathcal{M}}_{at}^{(II)} = \frac{1}{2\pi^2 q^2}\left[\frac{|\vec{\varepsilon}\cdot\vec{q}|^2}{q^2}\tilde{\mathcal{T}}_1 + |\vec{\varepsilon}|^2 \tilde{\mathcal{T}}_2 \right]. \quad (53)$$

We point out that the radial integrals $\tilde{\mathcal{T}}_1$ and $\tilde{\mathcal{T}}_2$ must be computed for $\Omega' = E_1$. A special work [34] will be devoted to their analytic evaluation since a number of technical difficulties that are related to their singular behavior must be discussed in detail.

Similar to the case of the first-order correction in Sec. III A, one can express $T_N^{(2)}$ as

$$T_N^{(2)} = \frac{\alpha_0^2 \omega^2}{8\pi^2 q^2}\left\{ \frac{|\vec{\varepsilon}\cdot\vec{q}|^2}{q^2}\{\mathcal{T}_1[J_{N+2}(\mathcal{R}_q) + J_{N-2}(\mathcal{R}_q)] + \tilde{\mathcal{T}}_1 J_N(\mathcal{R}_q)\} + \tilde{\mathcal{T}}_2 J_N(\mathcal{R}_q) \right\}. \quad (54)$$

Then, due to the structure of the transition matrix element including second-order laser dressing of the target,

$$T_N^{\mathrm{CP}} = -\frac{1}{4\pi^2}\left\{ f_{el}^{B1} J_N(R_q) - 2\alpha_0\omega \frac{|\vec{\varepsilon}\cdot\vec{q}|}{q^3}\mathcal{J}_{1,0,1} J_N'(R_q) - \alpha_0^2\omega^2 \frac{|\vec{\varepsilon}\cdot\vec{q}|^2}{2q^4}\{\mathcal{T}_1[J_{N+2}(\mathcal{R}_q) + J_{N-2}(\mathcal{R}_q)] + \tilde{\mathcal{T}}_1 J_N(\mathcal{R}_q)\} - \frac{\alpha_0^2\omega^2}{2q^2}\tilde{\mathcal{T}}_2 J_N(\mathcal{R}_q) \right\}, \quad (55)$$

one can immediately say that the differential cross section is again helicity independent and the dynamical phase is not a relevant parameter.

Moreover, based on angular momentum algebra considerations, one can argue that any contribution to the transition matrix element due to the jth order perturbative corrections to the atomic state will only contain terms proportional to $|\vec{\varepsilon}\cdot\vec{q}|^p|\vec{\varepsilon}|^{2s}$, where p and s are positive integers such that $p + 2s = j$. Therefore, as long as the scattering is treated in a first-order Born approximation, the helicity will remain an unobservable parameter.

It is interesting to note that the weak field limit of the differential cross section for two CP photon absorption/emission has a simple angular dependence given by $|\vec{q}\cdot\vec{\varepsilon}|^4$. Indeed, one finds

$$\frac{d\sigma_{\pm 2}^{\mathrm{CP}}}{d\Omega} = \alpha_0^4 \frac{k_f}{k_i}\frac{|\vec{q}\cdot\vec{\varepsilon}|^4}{2^6}\left| f_{el}^{B1} - \frac{4\omega}{q^3}\mathcal{J}_{1,0,1} - \frac{4\omega^2}{q^4}\mathcal{T}_1 \right|^2. \quad (56)$$

On the contrary, for LP fields where $\vec{e}^2 = 1$, the atomic matrix element $\mathcal{M}_{at}^{(II)}$ has a different angular behavior given by

$$\mathcal{M}_{at}^{(II)} = \frac{1}{2\pi^2 q^2}\left[\frac{(\vec{e}\cdot\vec{q})^2}{q^2}\mathcal{T}_1 + \mathcal{T}_2 \right]. \quad (57)$$

The amplitudes $\mathcal{T}_1$ and $\mathcal{T}_2$ depend on the momentum transfer of the scattered electron and on the four parameters $\tau'^{\pm}$ and $\tau^{\pm}$, given in Eqs. (34) and (52). Finally, in the weak field domain, this leads to the following expressions for the differential cross sections for two photon absorption/emission in LP fields

$$\frac{d\sigma_{\pm 2}^{\mathrm{LP}}}{d\Omega} = \alpha_0^4 \frac{k_f}{k_i}\frac{1}{2^6}\left| (\vec{q}\cdot\vec{e})^2 \left(f_{el}^{B1} - \frac{4\omega}{q^3}\mathcal{J}_{1,0,1} - \frac{4\omega^2}{q^4}\mathcal{T}_1 \right) - \frac{4\omega^2}{q^2}\mathcal{T}_2 \right|^2. \quad (58)$$

IV. RESULTS AND DISCUSSION

In this section we illustrate our results by considering the numerical evaluation of the nonlinear differential cross sections for the elastic term $(N=0)$ and the next two sidebands $(N=1$ and $N=2)$. We focus our discussion on the geometries denoted earlier by **CP3** and **LP3** because in these geometries the coupling between the laser field and the collid-

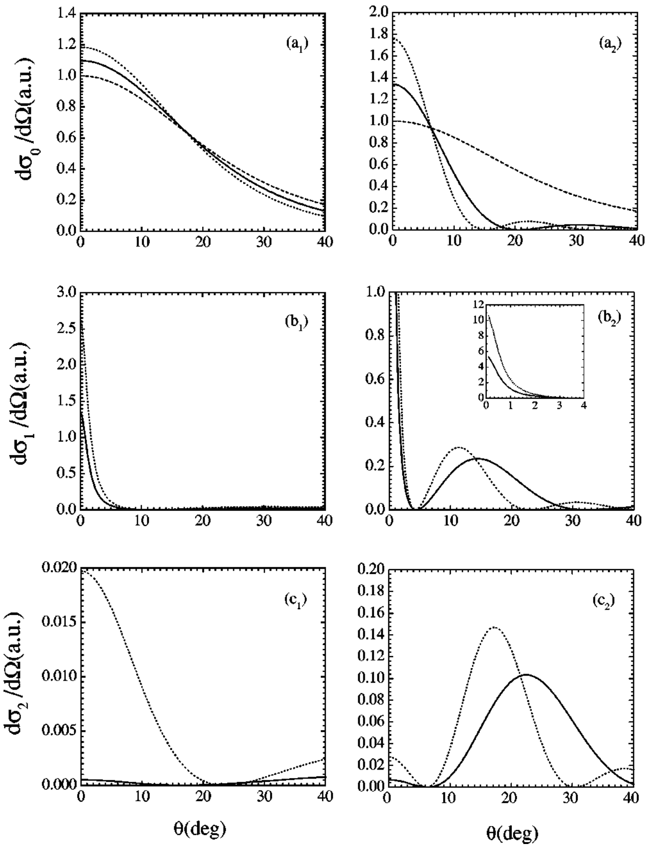

FIG. 1. $d\sigma_N/d\Omega$ are presented as a function of the scattering angle, θ, for a laser field $\mathcal{E}_0=10^8$ V/cm for the geometry **CP3** (full line) and **LP3** (dotted line). The initial projectile energy is $E_i=100$ eV. Panels a_1, b_1, and c_1 on the left-hand side refer to $\omega=5$ eV and the right-hand panels correspond to $\omega=2$ eV. In panels a_1 and a_2 are also presented the field-free differential cross sections (dashed lines).

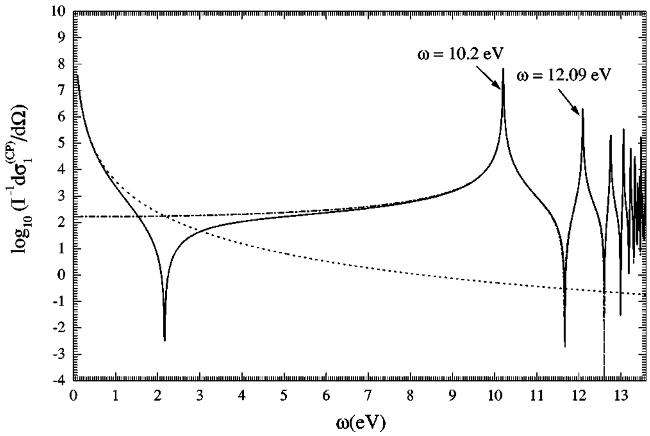

FIG. 2. $d\sigma_1/d\Omega$ normalized with respect to the field intensity, is shown as a function of the laser frequency for the geometry **CP3** at the scattering angle $\theta=5°$ and initial projectile energy $E_i=100$ eV. Also plotted are the corresponding data of the Bunkin-Fedorov formula (dotted line) and the atomic contribution of the first-order dressing (dotted-dashed line).

ing system is particularly strong.

The angular distributions of the scattered electrons with final energies given by $E_f=E_i+N\omega$ are shown in Fig. 1 for the three values of N: 0, 1, and 2. We have chosen two frequencies in the optical domain, namely, $\omega=5$ eV and $\omega=2$ eV, and our results are evaluated for the laser field strength $\mathcal{E}_0=10^8$ V/cm and the initial scattering energy $E_i=100$ eV.

In the left-hand panels a_1, b_1, and c_1 of Fig. 1 the laser frequency is $\omega=5$ eV and the quiver amplitude, α_0, takes the value 0.58 a.u., corresponding to the perturbative regime. In panel a_1, at small scattering angles where the dressing of the target is important, the differential cross section $d\sigma_0/d\Omega$ exceeds the field-free signal for both linear and circular polarizations. The nonlinear signals, $d\sigma_1/d\Omega$, belonging to the final energy $E_f=105$ eV are presented in panel b_1. Here we recognize that for small arguments of the Bessel functions the assisted signals for CP have half the value of the signals for LP. Finally, in panel c_1 we find large differences between the CP and LP signals. To understand this different behavior we focus on the dominant contributions to the differential cross sections, given by (56) and (58), respectively. Moreover, we note that only $\mathcal{T}_2$ of Eq. (58) has a pole at $\tau'=2$ while it does not appear in Eq. (56). Hence it becomes clear that the enhancement of the LP signal originates in a two-photon virtual transition between the ground state and the first excited state ($E_2-E_1=10.2$ eV). Considerations of angular momentum algebra can be used to show that such transitions are forbidden if the two photons have circular polarization.

In Fig. 1 (panels a_2, b_2, and c_2) we also present the angular distributions for the second laser frequency, $\omega=2$ eV. In this case the amplitude of the quiver motion takes the value $\alpha_0=3.6$ a.u. and the nonlinear dynamical behavior becomes apparent. Therefore the angular distributions are considerably different from those of the previous case. We point out that our formula (46) reproduces quite well the differences between LP and CP signals for $N=0$, since here second-order corrections are of minor importance. For $N=1$ the CP signals are again one-half of the LP ones at small scattering angles, as is shown in the window inserted in panel (b_2). With increasing scattering angle, the argument of the Bessel functions increases and nonlinear contributions become important. The present frequency, $\omega=2$ eV, is too small for establishing a two-photon resonance and the chosen intensity is not strong enough for higher-order contributions. Therefore LP and CP signals remain comparable for $N=2$. We think that the differences between our data for $\omega=2$ eV and those published earlier for the same parameters [20] are due to spurious phase effects present in the calculations of that work.

We shall next discuss the resonance structure of the first ($N=1$) and of the second ($N=2$) sidebands considering a sufficiently small scattering angle, $\theta=5°$, such that the target dressing effects are relevant.

In Fig. 2 the resonance structures of $d\sigma_1/d\Omega$ are shown for one photon absorption in the geometry **CP3**. We restrict ourselves to the weak field domain and we normalize the signals with respect to the intensity of the laser field. The differential cross sections exhibit a number of resonance

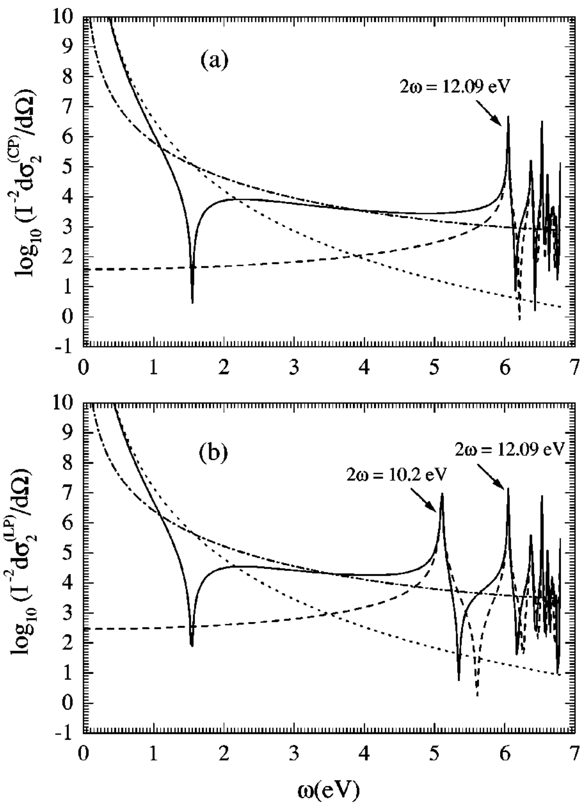

FIG. 3. (a) $d\sigma_2/d\Omega$ normalized with respect to I^2, is shown as a function of ω for the geometry **CP3** at the scattering angle $\theta = 5°$ and initial projectile energy $E_i = 100$ eV. Also plotted are the results of the Bunkin-Fedorov formula (dotted line) and the atomic contribution due to first-order (dotted-dashed line) and second-order dressing (long dashed line). (b) Same as in (a), but for the geometry **LP3**.

peaks corresponding to $\omega = |E_1|(1-n^{-2})$, where $|E_1|$ is the binding energy of the ground state and n is the principal quantum number. They correspond to the poles in the analytic expression of $\mathcal{J}_{1,0,1}$ in Eq. (35). At very low frequencies the major contribution stems from the Bunkin-Fedorov term (dotted curve). The first minimum of the differential cross sections, close to $\omega = 2$ eV, comes from an interference between the atomic and the electronic term; the other minima, located between two consecutive resonances, are related to the contribution of the first-order dressing correction to the ground state in Eq. (21). Since the relation, Eq. (47), holds in the perturbative domain, the resonance structures for the related scattering geometry, **LP3**, are obtained by a vertical upshift of these curves by a factor $\log_{10}(2)$.

The frequency dependences of the next sideband ($N=2$) exhibit two series of resonances. One-photon resonances are located above 10.2 eV as discussed earlier. In addition, a second series of resonances, located between 5.1 and 6.8 eV, is predicted. It corresponds to *two*-photon virtual transitions to excited states. This further series of resonances is related to second-order corrections to the ground state (21) and is presented in our Fig. 3. The panel *a* refers to the geometry **CP3**, while the other one to the geometry **LP3**. As discussed before, the resonance located at $\omega = 5.1$ eV is present for LP only. One can explicitly see in our figures that the dressing effects are increasing for increasing frequencies of the laser beam.

V. SUMMARY AND CONCLUSIONS

In the present work we have investigated scattering of electrons by hydrogen atoms in the presence of a circularly polarized laser field. For comparison, we also considered linearly polarized laser light. Since we assumed the scattered electrons to have initially some 100 eV kinetic energy, we were permitted to treat the scattering process in first-order Born approximation. The laser dressing of the atomic target was treated in second-order perturbation theory, while that of the scattering electron was described by a Volkov solution. Within this approximation scheme, we were able to show that the nonlinear cross sections $d\sigma_N^{\mathrm{CP}}/d\Omega$ neither depend on the dynamical phase ϕ_q, contrary to what was predicted by earlier work on this topic [20,22], nor is there any indication of circular dichroism. In our derivation of the above findings we devoted particular attention to the proper definition of the phases in Graf's addition theorem of Bessel functions, basing our considerations on the corresponding definitions in Watson's book [28]. This was outlined, in particular, in Sec. II of this work. As we found out, it is very crucial to make a careful analysis of the phase relations in the above treatment, for otherwise quite easily spurious phase dependences can creep in to finally simulate circular dichroism in the process studied above. Besides, we took advantage of our analysis to also make a comparison between nonlinear electron-atom scattering in a circularly and a linearly polarized laser beam of equal frequency and intensity. Among other differences between these two cases, we were able to show that for weak fields, at any scattering angles, and for moderate fields, at small scattering angles, $d\sigma_N^{\mathrm{CP}}/d\Omega$ are always smaller than $d\sigma_N^{\mathrm{LP}}/d\Omega$. Moreover, the resonance structures of the two cross sections are different, in particular, there are more resonances in the linear than in the circular case. Although one can qualitatively understand these differences by using angular momentum considerations, we have explicitly shown in the Appendix how the additional resonance in the case of linear polarization comes about.

Nevertheless, we should stress that a possible phase dependence may occur if the scattering process is treated beyond the first-order Born approximation [6]. In this case the appearance of imaginary parts of the scattering amplitude may lead to phase dependences and eventually circular dichroism in the scattering of electrons by atoms in circularly polarized laser light. In a forthcoming paper we shall show that, for a particular laser configuration, circular dichroism due to the target dressing can be predicted for high scattering energies, choosing the laser frequency and the scattering geometry in an appropriate way.

ACKNOWLEDGMENTS

This work has been supported by the Jubilee Foundation of the Austrian National Bank under project number 6211.

APPENDIX

In order to evaluate analytically the atomic matrix element $\mathcal{M}_{at}^{II}$ in Eq. (50) we use the expressions of the linear and quadratic response given in Refs. [31] and [30], respectively. The exponential in the form factor formula (29) is written in the standard way as an expansion in spherical harmonics. After integration over the angular coordinates of the bound electron, one can write down the following expression of the matrix element for two-photon absorption/emission

$$\begin{aligned}\mathcal{M}_{at}^{(II)}(\Omega'^{\pm},\Omega^{\pm}) = &-\frac{(\vec{s}\cdot\vec{q})^2}{8\pi^2 q^4}[\,{}_2\mathcal{I}_{10}^{21}(\tau'^{\mp},\tau^{\mp},q)\\ &+{}_2\mathcal{I}_{10}^{21}(\tau'^{\pm},\tau^{\pm},q)+{}_2\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)]\\ &+\frac{\vec{s}^2}{24\pi^2 q^2}[\,{}_0\mathcal{I}_{10}^{01}(\tau^{\mp},\tau^{\mp},q)\\ &+{}_2\mathcal{I}_{10}^{21}(\tau^{\mp},\tau^{\mp},q)+{}_0\mathcal{I}_{10}^{01}(\tau^{\pm},\tau^{\pm},q)\\ &+{}_2\mathcal{I}_{10}^{21}(\tau^{\pm},\tau^{\pm},q)+{}_0\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)\\ &+{}_2\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)].\end{aligned} \tag{A1}$$

Here the upper signs correspond to absorption and the lower ones to emission processes. If the polarization vector $\vec{s}$ is complex, its complex conjugate should be taken in order to compute two-photon emission. Using the previous equation, one can write down in a straightforward manner the general structure of $\mathcal{M}_{at}^{(II)}$ as

$$\mathcal{M}_{at}^{(II)} = \frac{1}{2\pi^2 q^2}\left[\frac{(\vec{s}\cdot\vec{q})^2}{q^2}\mathcal{T}_1+\vec{s}^2\mathcal{T}_2\right], \tag{A2}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ denote the following two combinations of radial integrals

$$\begin{aligned}\mathcal{T}_1 = &-\frac{1}{4}[\,{}_2\mathcal{I}_{10}^{21}(\tau'^{\mp},\tau^{\mp},q)+{}_2\mathcal{I}_{10}^{21}(\tau'^{\pm},\tau^{\pm},q)\\ &+{}_2\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)],\end{aligned} \tag{A3}$$

$$\begin{aligned}\mathcal{T}_2 = &\frac{1}{12}[\,{}_0\mathcal{I}_{10}^{01}(\tau^{\mp},\tau^{\mp},q)+{}_2\mathcal{I}_{10}^{21}(\tau^{\mp},\tau^{\mp},q)+{}_0\mathcal{I}_{10}^{01}(\tau^{\pm},\tau^{\pm},q)\\ &+{}_2\mathcal{I}_{10}^{21}(\tau^{\pm},\tau^{\pm},q)+{}_0\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)\\ &+{}_2\mathcal{I}_{101}(\tau^{\mp},\tau^{\pm},q)].\end{aligned} \tag{A4}$$

As a consequence, $\mathcal{T}_1$ and $\mathcal{T}_2$ depend on the momentum transfer, q, and on the four parameters $\tau'^{\pm}$ and $\tau^{\pm}$.

The four radial integrals present in Eq. (A1) are defined by

$${}_0\mathcal{I}_{101}(\Omega',\Omega,q) \equiv \int dr r^2 \mathcal{B}_{101}(\Omega',r) j_0(qr)\mathcal{B}_{101}(\Omega,r), \tag{A5}$$

$${}_2\mathcal{I}_{101}(\Omega',\Omega,q) \equiv \int dr r^2 \mathcal{B}_{101}(\Omega',r) j_2(qr)\mathcal{B}_{101}(\Omega,r), \tag{A6}$$

$${}_0\mathcal{I}_{10}^{01}(\Omega',\Omega,q) \equiv \int dr r^2 \mathcal{B}_{10}^{01}(\Omega',\Omega,r) j_0(qr) R_{10}(r), \tag{A7}$$

$${}_2\mathcal{I}_{10}^{21}(\Omega',\Omega,q) \equiv \int dr r^2 \mathcal{B}_{10}^{21}(\Omega',\Omega,r) j_2(qr) R_{10}(r), \tag{A8}$$

where $j_l(qr)$ are spherical Bessel functions of the order l. R_{10} is the radial function of the hydrogen ground state, $\mathcal{B}_{101}$ is defined by Eq. (32) of Ref. [31], and $\mathcal{B}_{10}^{01}$, $\mathcal{B}_{10}^{21}$ by Eqs. (43)–(44) of Ref. [30].

We have obtained analytic expressions for the radial integrals in Eqs. (A5)–(A8). The first two integrals, ${}_0\mathcal{I}_{101}$ and ${}_2\mathcal{I}_{101}$, are related to the linear response. They may be expressed, in terms of Appell functions, all of which depend on the same variables, namely,

$$\xi_1 = \frac{1-\tau'}{2}, \quad \zeta_1 = \frac{(1-\tau')\tau}{\tau+\tau'-iq\tau\tau'}. \tag{A9}$$

Note that, given the definitions (A5)–(A6), these two radial integrals are symmetric with respect to the parameters τ and τ'. We present below the expression for the first integral

$$\begin{aligned}{}_2\mathcal{I}_{101}(\tau',\tau,q) = &\frac{8\tau\tau'}{q(2-\tau')(1+\tau)}\left[\frac{2}{1+\tau'}\right]^{2+\tau'}\sum_{m=0}^{\infty} a'_m(\tau)\\ &\times\left[\frac{1-\tau}{\tau}\right]^m \sum_{p=0}^{2}\frac{(p+1)(p+2)}{(2q)^p}\frac{(3+m-p)!}{(2-p)!}\\ &\times \mathrm{Re}\{i^{p-3}\chi_1^{4+m-p}F_1(2-\tau',-1-\tau',4+m\\ &-p,3-\tau';\xi_1,\zeta_1)\}.\end{aligned} \tag{A10}$$

and that for the combination ${}_0\mathcal{I}_{101}(\tau',\tau,q)+{}_2\mathcal{I}_{101}(\tau',\tau,q)$

$$\begin{aligned}&{}_0\mathcal{I}_{101}(\tau',\tau,q)+{}_2\mathcal{I}_{101}(\tau',\tau,q)\\ &= -\frac{24\tau\tau'}{q^2(2-\tau')(1+\tau)}\left[\frac{2}{1+\tau'}\right]^{2+\tau'}\\ &\times\sum_{m=0}^{\infty} a'_m(\tau)\left[\frac{1-\tau}{\tau}\right]^m (m+1)!\\ &\times \mathrm{Re}\left\{\sum_{p=1}^{2}\left(\frac{i}{q}\right)^{2-p}(m+p)^{p-1}\chi_1^{1+m+p}\right.\\ &\left.\times F_1(2-\tau',-1-\tau',1+m+p,3-\tau',\xi_1,\zeta_1)\right\}.\end{aligned} \tag{A11}$$

Here the following notations have been used:

$$\chi_1=\frac{\tau\tau'}{\tau+\tau'-iq\tau\tau'}, \tag{A12}$$

$$a'_m(\tau)=\frac{1}{m!}\,\frac{1}{2-\tau+m}\,{}_2F_1\left(1,-1-\tau,3-\tau+m,\frac{\tau-1}{\tau+1}\right), \tag{A13}$$

where ${}_2F_1$ denotes the Gauss function. We stress that all the definitions and notations used in this paper for the hypergeometric functions are those given in Ref. [33]. To avoid possible confusion, be reminded that according to Eq. (A1) the parameters τ and τ' take the values $\tau^{\pm}$ given by Eq. (34).

The other two radial integrals are related to the quadratic response; they are expressed as a series of hypergeometric functions F_2 instead of Appell functions F_1. Although all the functions F_2, which are involved, depend also on the same variables, namely,

$$\xi_2=\frac{\tau-\tau'}{2\tau},\qquad \zeta_2=\frac{\tau+\tau'}{\tau(1-\tau'+iq\tau')}, \tag{A14}$$

the integrals ${}_0\mathcal{I}^{01}_{10}$ and ${}_2\mathcal{I}^{21}_{10}$ are no more symmetric with respect to the parameters τ and τ', than those that take the values in Eqs. (34) and (52), respectively. The expressions for the integral ${}_2\mathcal{I}^{21}_{10}(\tau',\tau,q)$ as well as that for the combination ${}_0\mathcal{I}^{01}_{10}(\tau',\tau,q)+{}_2\mathcal{I}^{21}_{10}(\tau',\tau,q)$ are written below

$${}_2\mathcal{I}^{21}_{10}(\tau',\tau,q)=\frac{2}{q}(1+\tau)\left[\frac{\tau'(\tau+\tau')}{\tau(\tau'+1)}\right]^5\sum_{m=0}^{\infty}d'_m(\tau',\tau)\times\left[\frac{\tau'(\tau-1)}{\tau(\tau'+1)}\right]^m\frac{(6)_m}{3-\tau'+m}\times\sum_{p=0}^{2}\frac{(p+1)(p+2)(3-p)}{(2q\tau')^p}\times\mathrm{Re}\{i^{p-1}\chi_2^{4-p}F_2(6+m,1,4-p,4-\tau'+m,6;\xi_2,\zeta_2)\} \tag{A15}$$

and

$${}_0\mathcal{I}^{01}_{10}(\tau',\tau,q)+{}_2\mathcal{I}^{21}_{10}(\tau',\tau,q)=\frac{24\tau\tau'}{q(1+\tau)}\left[\frac{\tau'(\tau+\tau')}{\tau(1+\tau')}\right]^3\sum_{m=0}^{\infty}c'_m(\tau',\tau)\frac{(5)_m}{2-\tau'+m}\left[\frac{\tau'(\tau-1)}{\tau(\tau'+1)}\right]^m\mathrm{Re}\{i\chi_2^3F_2(5+m,1,3,3-\tau'+m,5;\xi_2,\zeta_2)\}+\frac{6(1+\tau)}{q^2\tau'}\left[\frac{\tau'(\tau+\tau')}{\tau(\tau'+1)}\right]^5\sum_{m=0}^{\infty}d'_m(\tau',\tau)\frac{(6)_m}{3-\tau'+m}\left[\frac{\tau'(\tau-1)}{\tau(\tau'+1)}\right]^m\times\mathrm{Re}\left\{\sum_{p=1}^{2}\left(\frac{i}{2q\tau'}\right)^{p-2}\chi_2^{1+p}F_2(6+m,1,1+p,4-\tau'+m,6;\xi_2,\zeta_2)\right\}-\frac{12\tau\tau'}{q(1+\tau)}\left[\frac{\tau'(\tau+\tau')}{\tau(1+\tau')}\right]^3\sum_{m=0}^{\infty}b'_m(\tau',\tau)\left[\frac{\tau'(\tau-1)}{\tau(\tau'+1)}\right]^m\mathrm{Re}\left\{i\chi_2^3\left[2\frac{\tau+\tau'}{\tau}\frac{(5)_m}{3-\tau'+m}\times F_2(5+m,1,3,4-\tau'+m,5;\xi_2,\zeta_2)-\chi_2^{-1}\frac{(4)_m}{3-\tau'+m}F_2(4+m,1,2,4-\tau'+m,4;\xi_2,\zeta_2)+\chi_2^{-1}\frac{(4)_m}{1-\tau'+m}F_2(4+m,1,2,2-\tau'+m,4;\xi_2,\zeta_2)\right]\right\}, \tag{A16}$$

where $(n)_m$ denotes the Pochhammer symbol and the following notations have been used:

$$\chi_2=\frac{1}{\tau'-1-iq\tau'} \tag{A17}$$

and

$$b'_m(\tau',\tau)=\frac{1}{m!}\,\frac{1}{2-\tau+m}\,{}_2F_1(1,-1-\tau,3-\tau+m,\delta_2), \tag{A18}$$

$$c'_m(\tau',\tau)=\frac{1}{m!}\left[2\frac{\tau'(\tau-1)}{\tau(\tau'+1)}\frac{1}{3-\tau+m}\times{}_2F_1(1,-1-\tau,4-\tau+m,\delta_2)-\frac{\tau-\tau'}{\tau}\frac{1}{2-\tau+m}\times{}_2F_1(1,-1-\tau,3-\tau+m,\delta_2)\right], \tag{A19}$$

$$d'_m(\tau',\tau)=\frac{1}{m!}\left[-\left(\frac{\tau-1}{\tau+1}\right)^2\frac{1}{4-\tau+m}{}_2F_1(1,-1-\tau,5-\tau+m,\delta_2)+\frac{1}{2-\tau+m}{}_2F_1(1,-3-\tau,3-\tau+m,\delta_2)\right], \tag{A20}$$

with

$$\delta_2=\frac{(\tau-1)(\tau-\tau')}{(\tau+1)(\tau+\tau')}. \tag{A21}$$

The expressions of the radial integrals in Eqs. (A10)–(A11) and (A15)–(A16) were written down for real values of the parameters $\tau^{\pm}$ and $\tau'^{\pm}$.

The *one*-photon resonances discussed in Sec. IV are related to the poles of the four radial integrals (A5)–(A8). They occur for $\tau^{+}=n$, which corresponds to $\omega=|E_1|(1-n^{-2})$ with $n\geqslant 2$. *Two*-photon resonances are related to poles of ${}_0\mathcal{I}_{10}^{01}$ and ${}_2\mathcal{I}_{21}^{01}$, only. They occur for $\tau'^{+}=n$ and they correspond to $2\omega=|E_1|(1-n^{-2})$. The integral ${}_0\mathcal{I}_{10}^{01}$ has poles for any value of *n*, while we should note that ${}_2\mathcal{I}_{21}^{01}$ has poles only for $\tau'^{+}\geqslant 3$. This explains the absence of a resonance at $\omega=5.1$ eV in the frequency dependence of the nonlinear signal for $N=2$ if the laser field is circularly polarized, see Fig. 3(a), because only ${}_2\mathcal{I}_{21}^{01}$ enters into the expression for $\mathcal{T}_1$.

[1] F. V. Bunkin and M. V. Fedorov, Zh. Eksp. Theor. Fiz. **49**, 1215 (1965) [Sov. Phys. JETP **22**, 884 (1966)].
[2] N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1973).
[3] A. Weingartshofer and C. Jung, in *Multiphoton Ionization of Atoms*, edited by S. L. Chin and P. Lambropoulos (Academic Press, New York, 1984), p. 155.
[4] F. Ehlotzky, A. Jaroń, and J. Z. Kamiński, Phys. Rep. **297**, 63 (1998).
[5] N. J. Mason, Rep. Prog. Phys. **56**, 1275 (1993).
[6] M. H. Mittleman, *Introduction to the Theory of Laser-Atom Interactions* (Plenum Press, New York, 1993).
[7] F. H. M. Faisal, *Theory of Multiphoton Processes* (Plenum Press, New York, 1987).
[8] M. Gavrila, in *Collision Theory for Atoms and Molecules*, edited by F. Gianturco (Plenum, New York, 1989).
[9] J. I. Gersten and M. H. Mittleman, Phys. Rev. A **13**, 123 (1976).
[10] B. A. Zon, Zh. Eksp. Teor. Fiz. **73**, 128 (1977) [Sov. Phys. JETP **46**, 65 (1977)].
[11] E. L. Beilin and B. A. Zon, J. Phys. B **16**, L159 (1983).
[12] Jr. F. W. Byron and C. J. Joachain, J. Phys. B **17**, L295 (1984).
[13] P. Francken and C. J. Joachain, Phys. Rev. A **35**, 1590 (1987).
[14] Jr. F. W. Byron, P. Francken, and C. J. Joachain, J. Phys. B **20**, 5487 (1987).
[15] P. Francken and C. J. Joachain, J. Opt. Soc. Am. B **7**, 554 (1990).
[16] A. Dubois, A. Maquet, and S. Jetzke, Phys. Rev. A **34**, 1888 (1986).
[17] A. Dubois and A. Maquet, Phys. Rev. A **40**, 4288 (1989).
[18] G. Kracke, J. S. Briggs, A. Dubois, A. Maquet, and V. Veniard, J. Phys. B **27**, 3241 (1994).
[19] M. H. Mittleman, J. Phys. B **26**, 2709 (1993).
[20] P. Feinstein and M. Maquet, J. Phys. B **27**, 5563 (1994).
[21] D. Khalil, O. El Akramine, A. Makhoute, A. Maquet, and R. Taïeb, J. Phys. B **31**, 1115 (1998).
[22] M. Bouzidi, A. Makhoute, and M. N. Hounkonnou, Eur. Phys. J. D **5**, 159 (1999).
[23] O. El Akramine, A. Makhoute, D. Khalil, A. Maquet, and R. Taïeb, J. Phys. B **32**, 2783 (1999).
[24] A. Makhoute, D. Khalil, A. Maquet, C. J. Joachain, and R. Taïeb, J. Phys. B **32**, 3255 (1999).
[25] A. Cionga and G. Buică, Laser Phys. **8**, 164 (1998).
[26] S. Geltman, Phys. Rev. A **51**, R34 (1995).
[27] S. Varró and F. Ehlotzky, Phys. Lett. A **203**, 203 (1995).
[28] G. N. Watson, *Theory of Bessel Functions*, 2nd. edition (Cambridge University Press, Cambridge, 1962), p. 359.
[29] D. B. Milošević, F. Ehlotzky, and B. Piraux, J. Phys. B **30**, 4347 (1997).
[30] V. Florescu, A. Halasz, and M. Marinescu, Phys. Rev. A **47**, 394 (1993).
[31] V. Florescu and T. Marian, Phys. Rev. A **34**, 4641 (1986).
[32] A. Cionga and V. Florescu, Phys. Rev. A **45**, 5282 (1992).
[33] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques. Polynomes d'Hermite* (Gauthier-Villars, Paris, 1926).
[34] A. Cionga and G. Zloh (unpublished).