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Distillability and partial transposition in bipartite systems

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We study the distillability of a certain class of bipartite density operators which can be obtained via depolarization starting from an arbitrary one. Our results suggest that nonpositivity of the partial transpose of a density operator is not a sufficient condition for distillability, when the dimension of both subsystems is higher than 2.

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I. INTRODUCTION

Maximally entangled states represent an essential ingredient in most applications of quantum information [1]. In particular, in quantum communication one can use them for transmitting secret messages between two locally separated parties [2]. In practice, however, states are mixed due to interaction with the environment, and are not usable for those applications, even though they may be entangled. The solution to this problem was presented by Bennett *et al.*, Deutsch *et al.*, and Gisin [3–5], who gave a procedure to ''distill'' maximally entangled states of two qubits out of a set of pairs in certain (mixed) entangled states, by using only local actions and classical communication [6,7]. Later on, the Horodecki family showed that any even infinitesimally entangled state of two qubits (two-level systems) can be distilled into a singlet [8]. They also proved a necessary condition for the state of an arbitrary bipartite system to be distillable, namely, that the partial transpose of the corresponding density operator must be nonpositive [9].

As shown by Peres [10], the positivity of the partial transpose is a necessary condition for separability. In fact, this condition turns out to be a sufficient condition for separability in cases of both qubits (two-level systems), or one qubit and one trit (a three-level system) [11]. A natural question arises: is this condition also sufficient for separability in higher-dimensional systems? Horodecki recently showed [12] that there are in fact states in higher-dimensional systems which have a positive partial transpose, but are nonseparable (see also Refs. [9,13–17]). As a consequence, the positivity of a partial transpose is, except for 2×2 and 2×3 systems, not sufficient for separability. Similarly, the nonpositivity of a partial transpose is necessary for distillability, and is sufficient for 2×2 and 2×3 systems. However, the question remains open whether this condition is sufficient for distillability in higher-dimensional systems.

In this paper we investigate the distillability of high-dimensional systems shared by two parties, Alice and Bob. We introduce a depolarizing superoperator that allows one to reduce an arbitrary density operator with nonpositive partial transposition (NPPT) to one with the same property, but in a standard form that is characterized by a single parameter. We analyze some properties of those operators, and show that for any given finite number of copies there are density operators ρ for which one can never find a subspace of dimension 2 in Alice and Bob's Hilbert spaces, in which ρ still has NPPT. We also present some numerical evidence that indicates that this class of states is independent of the number of copies. All these results suggest that there exist states with NPPT which are not distillable.

This work is organized as follows: In Sec. II we review some of the present knowledge concerning distillability and entanglement, and we introduce the definitions and properties that are needed in order to study the problem of distillability of general density operators. In Sec. III we concentrate on the case in which Alice and Bob have three-level systems, whereas in Sec. IV we generalize our results to the d-level system case. In Sec. V we show the basics of the numerical procedure used to study the distillability of two and three copies. Finally, we summarize our results.

II. ENTANGLEMENT AND DISTILLABILITY

We consider two parties, Alice and Bob, who share several pairs of particles. Each pair is in a state described by the same density operator ρ. We will assume that Alice's (Bob's) particles are d_A-level systems (d_B); that is, the density operator ρ acts on the Hilbert space $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$. We will denote by $\{|1\rangle,|2\rangle,\ldots,|d_A\rangle\}$ an orthonormal basis in $\mathbb{C}^{d_A}$, and analogously for $\mathbb{C}^{d_B}$. We will also use the notation $|i,j\rangle\equiv|i\rangle_A\otimes|j\rangle_B$.

We will assume that Alice and Bob are able to manipulate their particles by only using local actions (operators and measurements) and classical communication. In this case, we say that the density operator ρ is *distillable* if they can produce a maximally entangled state

$$|\Phi_d\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i,i\rangle, \tag{1}$$

where $d=\min(d_A,d_B)$. On the other hand, we say that ρ is separable if it can be prepared from a product state (e.g., $|1,1\rangle$). In this section we will review some of the results derived by Peres and the Horodecki family concerning distillability and entanglement, and will introduce the definitions and properties that are needed in order to study the problem of distillability of general density operators.

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A. Partial transposition

As shown by Peres [10] and the Horodeckis [8,11,12], the partial transpose of a density operator plays an important role in establishing its distillability and entanglement properties. In general, given an operator X acting on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$, we define the partial transpose of X with respect to the first subsystem in the basis $\{|1\rangle,|2\rangle,\ldots,|d_A\rangle\}$, X^{T_A}, as follows:

$$X^{T_A}\equiv\sum_{i,j=1}^{d_A}\sum_{k,l=1}^{d_B}\langle i,k|X|j,l\rangle\ |j,k\rangle\langle i,l|. \qquad (2)$$

In the following we will use a property of this operation, namely, $\mathrm{tr}(YX^{T_A})=\mathrm{tr}(Y^{T_A}X)$.

We say that a self-adjoint operator X has a nonpositive partial transpose if X^{T_A} is not positive; that is, if there exists some $|\Psi\rangle$ such that $\langle\Psi|X^{T_A}|\Psi\rangle<0$. The positivity of the operator ρ^{T_A} gives necessary criteria for the separability and nondistillability of a density operator ρ. In particular, (1) if ρ is separable, then $\rho^{T_A}\geqslant 0$ [10]; and (2) if $\rho^{T_A}\geqslant 0$, then ρ is not distillable [12]. These two necessary conditions turn out to be sufficient for $d_A=2$ and $d_B\leqslant 3$ [8]. However, it has been shown that the first condition is not sufficient for separability for the rest of the cases ($d_A=2$ and $d_B>3$, and $d_A,d_B>2$) [12]. On the other hand, nothing is known about whether the second condition is also sufficient for nondistillability in these cases.

B. Distillability

The problem of distillability of general density operators acting on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$ can be expressed in a simpler form [9]. A density operator is distillable iff, for a certain positive integer N, we can find a state of the form

$$|\Psi\rangle=a|e_1\rangle_A|f_1\rangle_B+b|e_2\rangle_A|f_2\rangle_B, \qquad (3)$$

such that

$$\langle\Psi|(\rho^{\otimes N})^{T_A}|\Psi\rangle<0. \qquad (4)$$

Here $\{|e_1\rangle_A,|e_2\rangle_A\}$ are two orthonormal vectors in $(\mathbb{C}^{d_A})^{\otimes N}$, and $\{|f_1\rangle_B,|f_2\rangle_B\}$ are two orthonormal vectors in $(\mathbb{C}^{d_B})^{\otimes N}$. This condition basically means that if Alice and Bob share N pairs, one just has to find a two-dimensional subspace in the whole Hilbert space of Alice, and another in Bob's such that the projection of $\rho^{\otimes N}$ in such subspaces has NPPT. The reason for this is that if one finds such a subspace, then according to what was exposed in Sec. II A one can distill a maximally entangled state in $\mathbb{C}^2\otimes\mathbb{C}^2$, which can be converted into a maximally entangled state in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$. Conversely, if one can create one of those states, then one can also produce one in $\mathbb{C}^2\otimes\mathbb{C}^2$, and therefore this ensures that Eq. (4) must be fulfilled.

Thus, in practice, one can analyze for each number of copies $N=1,2,\ldots$ whether condition (4) is fulfilled. In order to facilitate this task, we will use the following definitions: If for a given N condition (4) is fulfilled, we will say that ρ is N-distillable. On the other hand, if for a certain N there *does not exist* any $|\Psi\rangle$ satisfying Eq. (4), then we will say that ρ is N-undistillable. Thus ρ is distillable iff there exists an N for which it is N-distillable. Conversely, ρ is nondistillable iff it is N-undistillable, $\forall N$.

C. Distillability in $\mathbb{C}^2\otimes\mathbb{C}^d$

With the properties and definitions given above, one can very easily prove that when $d_A=2$ and $d_B\geqslant 2$ and if ρ has a NPPT then ρ is 1-distillable. The reason for this is that there exists some $|\Psi\rangle$ such that $\langle\Psi|\rho^{T_A}|\Psi\rangle<0$. On the other hand, since $d_A=2$, then the Schmidt decomposition of $|\Psi\rangle$ has at most two terms, and therefore can be written in the form of Eq. (3). Thus in this case a nonpositive partial transpose of ρ is a necessary and sufficient condition for distillability.

D. Depolarization in $\mathbb{C}^d\otimes\mathbb{C}^d$

In this subsection we will introduce some superoperators which will be useful to study if a given density operator is N-distillable. We will also show that given a density operator one can reduce it to a standard form which is characterized by a single parameter [18], and that preserves the distillability properties of the original state. This standard form was already introduced in Ref. [19].

Let us first define some useful projector operators. Given two quantum systems with corresponding Hilbert spaces $\mathbb{C}^d$, we denote the permutation operator by Π_d, and by

$$A_d=(\mathbb{1}-\Pi_d)/2, \quad S_d=\mathbb{1}-A_d=(\mathbb{1}+\Pi_d)/2, \qquad (5)$$

the projector operators onto the antisymmetric and symmetric subspaces of $\mathbb{C}^d\otimes\mathbb{C}^d$, respectively. Note that $\mathrm{tr}(S_d)=d(d+1)/2$ and $\mathrm{tr}(A_d)=d(d-1)/2$. We also define the projector operators

$$P_d=|\phi_d\rangle\langle\phi_d|, \quad Q_d=\mathbb{1}-P_d, \qquad (6)$$

where $|\phi_d\rangle$ is the maximally entangled state defined in Eq. (1). One can easily check that

$$P_d^{T_A}=\frac{1}{d}(\mathbb{1}-2A_d), \quad A_d^{T_A}=\frac{1}{2}(\mathbb{1}-dP_d). \qquad (7)$$

We define the depolarization superoperator $\mathcal{D}$, acting on any given operator X, as follows:

$$\mathcal{D}(X)=A_d\frac{\mathrm{tr}(A_dX)}{\mathrm{tr}(A_d)}+S_d\frac{\mathrm{tr}(S_dX)}{\mathrm{tr}(S_d)}. \qquad (8)$$

This superoperator is a projector, is self–adjoint (on the Hilbert–Schmidt space of operators acting on $\mathbb{C}^d\otimes\mathbb{C}^d$), and preserves the trace. In Appendix A, we show that we can write [19]

$$\mathcal{D}(X)=\int d\mu_U(U\otimes U)X(U\otimes U)^\dagger, \qquad (9)$$

where the integral is extended to all unitary operators acting on $\mathbb{C}^d$ and $\int d\mu_U=1$ [$d\mu$ represents the standard invariant

Haar measure on the group SU(d)]. We will later use the partial transpose of $\mathcal{D}(X)$, and to this aim we define the superoperator

$$\mathcal{E}(X)\equiv[\mathcal{D}(X^{T_A})]^{T_A}=\int d\mu_U(U^*\otimes U)X(U^*\otimes U)^\dagger \tag{10a}$$

$$=P_d\,\mathrm{tr}(P_dX)+Q_d\frac{\mathrm{tr}(Q_dX)}{\mathrm{tr}(Q_d)}, \tag{10b}$$

where U^* denotes complex conjugation in the basis in which the partial transposition is defined. This superoperator is also a projector, self-adjoint, and preserves the trace. Note that for any unitary operator V acting on $\mathbb{C}^d$ we have

$$(V\otimes V)\mathcal{D}(X)(V\otimes V)^\dagger=\mathcal{D}(X), \tag{11a}$$

$$(V^*\otimes V)\mathcal{E}(X)(V^*\otimes V)^\dagger=\mathcal{E}(X). \tag{11b}$$

Form (9) shows that the superoperators $\mathcal{D}$ and $\mathcal{E}$ can be implemented by means of local operations. In particular, it shows that any density operator ρ can be transformed, using local operations, to the form

$$\mathcal{D}(\rho)=\rho_\alpha=\frac{1}{N(\alpha)}(S_d+\alpha A_d), \tag{12}$$

where α is such that $\mathrm{tr}(A_d\rho)=\mathrm{tr}(A_d\rho_\alpha)$, and $N(\alpha)=\mathrm{tr}(S_d)+\alpha\,\mathrm{tr}(A_d)$ is a normalization constant. That is, one can depolarize any density operator to the one-parameter family [Eq. (12)], while keeping the weight in the antisymmetric subspace. We will be more interested in the partial transpose of ρ_α, which is given by

$$[\mathcal{D}(\rho)]^{T_A}=\mathcal{E}(\rho^{T_A})=\frac{1}{M(\beta)}(Q_d-\beta P_d), \tag{13}$$

where $M(\beta)=\mathrm{tr}(Q_d)-\beta$ is a normalization constant, and the relationship between α and β is $\beta=[(\alpha-1)(d-1)-2]/(\alpha+1)<d-1$. Note that since

$$\rho_\alpha=\mathcal{D}(\rho_\alpha),\quad \rho_\alpha^{T_A}=\mathcal{E}(\rho_\alpha^{T_A}), \tag{14}$$

we have that

$$(U\otimes U)\rho_\alpha(U\otimes U)^\dagger=\rho_\alpha, \tag{15a}$$

$$(U^*\otimes U)\rho_\alpha^{T_A}(U^*\otimes U)^\dagger=\rho_\alpha^{T_A} \tag{15b}$$

for any unitary operator U.

Using the properties derived above, one can easily check that

$$\rho_\alpha \text{ is separable}\Leftrightarrow\rho_\alpha^{T_A}\geqslant 0\Leftrightarrow\beta\leqslant 0. \tag{16}$$

The last equivalence follows directly from Eq. (13). For the first one we have ($\Rightarrow$): see Sec. II A; ($\Leftarrow$). We have that, for $\beta=0$ [i.e., $\alpha=\alpha_0\equiv(d+1)/(d-1)=\mathrm{tr}(S_d)/\mathrm{tr}(A_d)$],

$$\rho_{\alpha_0}^{T_A}\propto Q_d\propto\mathcal{E}(|0,1\rangle\langle 0,1|), \tag{17}$$

which is obviously positive and separable, in which case the same holds for ρ_{α_0}. For $\beta<0$ ($\alpha<\alpha_0$) we can always obtain ρ_α by adding the identity operator (which is separable) to ρ_{α_0} [this is due to the fact that $\mathrm{tr}(S_d)>\mathrm{tr}(A_d)$].

Thus, for $\beta>0$ ($\alpha>\alpha_0$), ρ_α is nonseparable. One can easily check that this condition is equivalent to

$$\mathrm{tr}(A_d\rho_\alpha)>1/2. \tag{18}$$

This last form allows us to show that for any given density operator ρ with NPPT, one can always transform it, using local actions, to the form of Eq. (12), such that it still has NPPT [18]. Let us show this. Suppose that for a given $|\Psi\rangle$, $\langle\Psi|\rho^{T_A}|\Psi\rangle<0$. We can write $|\Psi\rangle=\Sigma_{i=1}^{n\leqslant d}c_i|u_i,v_i\rangle$ where $|u_i\rangle_{i=1}^d$ and $|v_i\rangle_{i=1}^d$ form an orthonormal basis. The operator ρ can be transformed by local operations to $\rho_s\propto(A^\dagger\otimes B^\dagger)\rho(A\otimes B)$, with $\mathrm{tr}(\rho_s^{T_A}P_n)<0$ (by simply taking $A=\Sigma_{i=1}^n|u_i^*\rangle\langle i|/c_i+\Sigma_{i=n+1}^d|u_i^*\rangle\langle i|$ and $B=\Sigma_{i=n+1}^d|v_i\rangle\langle i|$). Using that $0>\mathrm{tr}(\rho_s^{T_A}P_n)=\mathrm{tr}(\rho_sP_n^{T_A})$, and Eq. (7), we immediately obtain that $\mathrm{tr}(A_d\rho_s)\geqslant\mathrm{tr}(A_n\rho_s)\geqslant 1/2$. Since $\mathcal{D}$ conserves this quantity, we obtain that $\mathcal{D}(\rho_s)$ has a negative partial transpose.

As pointed out by Horodecki and Horodecki [18], the problem of distillability can be reduced to a study of density operators of the form of Eq. (12). If we find that all those operators with NPPT are distillable, we will have shown that NPPT is a necessary and sufficient condition for distillability. Conversely, if we find that there exists an operator of the form of Eq. (12) which has NPPT, but is not distillable, we will have shown that such a condition is not sufficient. In the following sections we will show that there are density operators of the standard form with NPPT which are not N-distillable for certain values of N. As we have seen in Sec. II B, we can study this by checking whether there exist vectors of the form of Eq. (3) fulfilling condition (4).

III. *N*-DISTILLABILITY IN $\mathbb{C}^3\otimes\mathbb{C}^3$

We consider the case $d_A=d_B=3$, and a density operator of the form

$$\rho_\alpha=\frac{1}{N(\alpha)}(S+\alpha A), \tag{19}$$

where we have omitted the superindices $d=3$, and $N(\alpha)=6+3\alpha$. According to the discussion in Sec. II D, we just have to consider $\alpha\geqslant\alpha_0=2$, since otherwise ρ_α is separable. For the partial transpose we also have

$$\rho_\beta^{T_A}=\frac{1}{M(\beta)}(Q-\beta P), \tag{20}$$

where $\beta=(2\alpha-4)/(\alpha+1)$, with $2\geqslant\beta\geqslant 0$ and $M(\beta)=8-\beta$.

A. 1-distillability

We look for a vector of the form of Eq. (3), such that Eq. (4) is fulfilled. Choosing U such that $U|e_{1,2}\rangle = |1,2\rangle$, and using property (15b), we see that we can restrict ourselves to the subspace spanned by $\{|1\rangle,|2\rangle\}$ in Alice's Hilbert space. Defining by $\mathbb{1}_2$ the projection operator into this subspace and by $\mathbb{1}_3$ the identity operator in $\mathbb{C}^3$, we obtain, after projecting $\rho_\beta^{T_A}$ onto such a subspace,

$$\mathbb{1}_2^A \rho_\beta^{T_A} \mathbb{1}_2^A \propto \mathbb{1}_2^A \otimes \mathbb{1}_3^B - \frac{2(1+\beta)}{3} P_2, \tag{21}$$

which is positive iff $\beta \leqslant 1/2$. Thus, we obtain that ρ_α is 1-distillable iff $\beta > 1/2$ (or, equivalently, $\alpha > 3$).

B. 2-distillability

Let us now consider two pairs, in a state ρ_α. We will show that for $\beta \leqslant 1/4$ the state ρ_α is 2-undistillable. For any state $|\Psi\rangle$ of the form of Eq. (3) we have

$$\begin{aligned}\langle\Psi|Q^{(1)} \otimes (Q^{(2)} - P^{(2)}/2)|\Psi\rangle \\ = \mathrm{tr}_2[\mathrm{tr}_1(|\Psi\rangle\langle\Psi|Q^{(1)})(Q^{(2)} - P^{(2)}/2)],\end{aligned} \tag{22}$$

where the superscripts 1 and 2 refer to the first and second pair, respectively. Using the fact that Q is separable, and therefore that it can be written as $Q = \Sigma_i c_i |a_i, b_i\rangle\langle a_i, b_i|$ with $c_i > 0$, we have

$$\mathrm{tr}_1(|\Psi\rangle\langle\Psi|Q^{(1)}) = \sum_i c_i |\Psi_i\rangle\langle\Psi_i|, \tag{23}$$

where $|\Psi_i\rangle = \langle a_i, b_i|\Psi\rangle$ is a state acting on the second pair which itself has the form of Eq. (9). Thus, according to the results of Sec. III A we have that $\langle\Psi_i|(Q^{(2)} - P^{(2)}/2)|\Psi_i\rangle \geqslant 0$ and therefore $\langle\Psi|Q^{(1)} \otimes (Q^{(2)} - P^{(2)}/2)|\Psi\rangle \geqslant 0$. In the same way we have that $\langle\Psi|(Q^{(1)} - P^{(1)}/2) \otimes Q^{(2)}|\Psi\rangle \geqslant 0$, i.e.,

$$0 \leqslant \langle\Psi|\left(Q^{(1)} - \frac{P^{(1)}}{4}\right) \otimes \left(Q^{(2)} - \frac{P^{(2)}}{4}\right) - P^{(1)} \otimes \frac{P^{(2)}}{16}|\Psi\rangle. \tag{24}$$

Using the fact that $P^{(1)} \otimes P^{(2)} \geqslant 0$ we obtain the desired result. Note that our results do not imply that for $1/4 \leqslant \beta < 1/2$ ρ_α is 2-distillable. In fact, as shown in Sec. V, numerical calculations indicate that it is 2-undistillable.

C. *N*-distillability

We consider now N pairs, in a state ρ_α. We will show that for $\beta \leqslant 4^{-N}$ the state ρ_α is N-undistillable. For any state $|\Psi\rangle$ of the form of Eq. (3), one can check the following relations: (i) $\langle\Psi|P^{\otimes k}|\Psi\rangle \leqslant 2/3^k$, (ii) $\langle\Psi|Q^{\otimes N-k}P^{\otimes k}|\Psi\rangle \leqslant 2/3^k$, and (iii) $\langle\Psi|Q^{\otimes N}|\Psi\rangle \geqslant 1/3^N$. To show (i), one uses $P_d^{\otimes k} = P_{d^k}$ and property (15b), from which it follows that the projection into the subspace spanned by $\{|1\rangle,|2\rangle\}$ gives the maximum value for $\langle\Psi|P^{\otimes k}|\Psi\rangle$. From (i) we immediately obtain (ii) by using that $\langle\Psi|QX|\Psi\rangle \leqslant \langle\Psi|\mathbb{1}X|\Psi\rangle$ for all positive operators X. Relation (iii) can be obtained by using (i) and the separability of Q in a similar way as in Sec. III B. Combining (ii) and (iii) we find

$$\langle\Psi|a_k Q^{\otimes N} - \sum_{\mathrm{perm}} Q^{\otimes N-k}P^{\otimes k}|\Psi\rangle \geqslant 0, \tag{25}$$

with $a_k = 2\binom{N}{k}3^{N-k}$, and the sum runs over all possible permutations of the pairs. By summing Eq. (25) for all odd k and using that $\Sigma_{k\ \mathrm{odd}} a_k = 4^N - 2^N \equiv \tilde{\beta}_N^{-1}$, one finds, for $\beta_N \leqslant \tilde{\beta}_N$,

$$\begin{aligned}0 &\leqslant \langle\Psi|Q^{\otimes N} - \tilde{\beta}_N \sum_{k\ \mathrm{odd}} \sum_{\mathrm{perm}} Q^{\otimes N-k}P^{\otimes k}|\Psi\rangle \\ &\leqslant \langle\Psi|Q^{\otimes N} - \sum_{k\ \mathrm{odd}} \beta_N^k \sum_{\mathrm{perm}} Q^{\otimes N-k}P^{\otimes k}|\Psi\rangle \\ &\leqslant \langle\Psi|(Q - \beta_N P)^{\otimes N}|\Psi\rangle.\end{aligned} \tag{26}$$

We used that $\max_k(\beta_N^k) \leqslant \tilde{\beta}_N$ (for $\beta_N \leqslant \tilde{\beta}_N \leqslant 1$) to obtain the first inequality (line 2), while we added all positive terms (even k) in the second step (line 3). This is already the desired bound, i.e. for $0 \leqslant \beta \leqslant 4^{-N} \leqslant \tilde{\beta}_N \equiv 1/(4^N - 2^N)$, the state ρ_α is N-undistillable. Again, this does not mean that ρ_α is N-distillable for $4^{-N} \leqslant \beta \leqslant 1/2$. In Appendix C we present a better bound for β.

IV. *N*-DISTILLABILITY IN $\mathbb{C}^D \otimes \mathbb{C}^D$

We consider the case $d_A = d_B = d$, and a density operator of the form of Eq. (12), with the partial transposition given by Eq. (13). Similar techniques as in the $d=3$ case can also be used to obtain bounds for arbitrary d. One finds for example that ρ_α is 1-distillable iff $\beta > d/2 - 1$. We also obtain that ρ_α is 2-undistillable for $\beta \leqslant (d-2)/4$ and N-undistillable for $\beta \leqslant \min(\tilde{\beta}_N, \tilde{\beta}_N^{1/N})$, with $\tilde{\beta}_N = (d-2)^N/[(d+1)^N - (d-1)^N]$. Note that the minimum is required here, since—differently from the $d=3$ case—one can have that $\tilde{\beta}_N \geqslant 1$. In this case one has to chose $\beta_N \leqslant \tilde{\beta}^{1/N}$, which implies $\max_k(\beta_N^k) \leqslant \tilde{\beta}_N$ to ensure that the first inequality in Eq. (26) remains valid.

Furthermore, there is an interesting relation between the states ρ_α for different d. Imagine we would like to convert a single copy of a state ρ_α in $\mathbb{C}^d \otimes \mathbb{C}^d$ to a state in $\mathbb{C}^k \otimes \mathbb{C}^k$ ($k < d$) in an optimal way, i.e., to obtain a new α which is as large as possible. We show here that whenever we convert a state ρ_α to some lower dimension, there will always be some states which lose the negativity of their partial transposition. In order to prove this, we consider vectors $|\Psi_k\rangle$ with k Schmidt coefficients, and show that $\langle\Psi_d|\rho_\alpha^{T_A}|\Psi_d\rangle < 0$, while $\langle\Psi_k|\rho_\alpha^{T_A}|\Psi_k\rangle > 0$ $\forall$ $|\Psi_k\rangle$. Due to property (15b), one can restrict oneself to the subspace spanned by $\{|1\rangle \ldots |k\rangle\}$. Let us denote the identity operator in this subspace by $\mathbb{1}_k$. One finds after projecting $\rho_\alpha^{T_A}$ onto $\mathbb{1}_k$ in A and B,

$$\mathbb{1}_k^A\otimes\mathbb{1}_k^B\rho_\alpha^{T_A}\mathbb{1}_k^A\otimes\mathbb{1}_k^B\propto\mathbb{1}_k^A\otimes\mathbb{1}_k^B-\frac{k(1+\beta)}{d}P_k, \tag{27}$$

which is positive iff $\beta\leqslant(d/k)-1$, while $\rho_\alpha^{T_A}$ before the projection was positive iff $\beta\leqslant 0$. Thus all states with $0<\beta\leqslant(d/k)-1$ lose the negativity of their partial transposition after the optimal projection onto a k-dimensional subspace. The new β_k can be calculated from β_d of the initial state by $\beta_k=(k/d)(\beta_d+1)-1$.

Finally, let us consider N copies of ρ_α of dimension d, which can be viewed as a state in $\mathbb{C}^{\otimes d^N}\otimes\mathbb{C}^{\otimes d^N}$. With $\mathrm{tr}(A_d\rho_\alpha)\equiv\lambda_d$, one finds that the state $\mathcal{D}(\rho_\alpha^{\otimes N})\equiv\tilde{\rho}_\alpha$—the state in the high-dimensional Hilbert space after depolarization—has $\lambda_{d^N}\equiv\mathrm{tr}(A_{d^N}\tilde{\rho}_\alpha)=[1-(1-2\lambda_d)^N]/2$. One checks that for $\lambda_d>1/2$ (i.e., ρ_α is inseparable) we have $\lambda_{d^N}\leqslant\lambda_d\forall N$, which simply means that the weight in the antisymmetric subspace decreases when going to more copies. Note that using this notation, we have that the state ρ_α is separable for $\lambda\leqslant 1/2$, while it is 1-distillable for $\lambda>[3(d-1)]/[2(2d-1)]$, which tends to $\frac{3}{4}$ for $d\to\infty$.

V. NUMERICAL PROCEDURES

In general, we are interested in showing there exists a $|\Psi\rangle$ of the form of Eq. (3) for which condition (4) is fulfilled, i.e.,

$$\lambda\equiv\langle\Psi|R|\Psi\rangle<0, \tag{28}$$

where we have defined $R=(Q-\beta P)^{\otimes N}$, with $0<\beta\leqslant 1/2$. In order to check this we can minimize λ with respect to $|e_{1,2}\rangle$, $|f_{1,2}\rangle$, and a while keeping the normalization and orthogonality relations. One can readily check that the minimization implies

$$\langle e_1|R|\Psi\rangle=\lambda_0 a|f_1\rangle, \tag{29a}$$

$$\langle e_2|R|\Psi\rangle=\lambda_0 b|f_2\rangle, \tag{29b}$$

$$\langle f_1|R|\Psi\rangle=\lambda_0 a|e_1\rangle, \tag{29c}$$

$$\langle f_2|R|\Psi\rangle=\lambda_0 b|e_2\rangle. \tag{29d}$$

Note that the operator ${}_A\langle e_1|R|e_1\rangle_A$ is strictly positive, and therefore invertible. The reason for that is that for any $|f\rangle_B$, we have $\langle e,f|R|e,f\rangle=\langle e^*,f|R^{T_A}|e^*,f\rangle>0$ since according to Eq. (12) we can always write $R^{T_A}=c\mathbb{1}+B$ (where $B\geqslant 0$ and $c>0$). On the other hand, $a,b\neq 0$ since otherwise $|\Psi\rangle$ would be a product vector and therefore $\lambda\geqslant 0$. Thus we can use Eq. (29a) to write

$$|f_1\rangle=\frac{b}{a}\,\frac{1}{\lambda_0-\langle e_1|R|e_1\rangle}\langle e_1|R|e_2\rangle\ \ |f_2\rangle, \tag{30}$$

which, after substituting in Eq. (29b), gives

$$\langle e_2|F(\lambda_0)|e_2\rangle\ \ |f_2\rangle=\lambda_0|f_2\rangle, \tag{31}$$

where we have defined

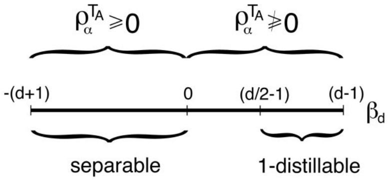

FIG. 1. Separability and 1-distillability properties of ρ_α [Eq. (12)].

$$F(\lambda_0)=R-R|e_1\rangle\frac{1}{\langle e_1|R|e_1\rangle-\lambda_0}\langle e_1|R. \tag{32}$$

The normalization of $|f_1\rangle$ gives

$$\left|\frac{a}{b}\right|^2=\langle e_2,f_2|\left[\frac{1}{\langle e_1|R|e_1\rangle-\lambda_0}\right]^2|e_2,f_2\rangle. \tag{33}$$

Thus the problem is reduced to showing whether Eq. (31) possesses solutions for $\lambda_0<0$. In that case, we can find $a,b,|f_1\rangle$ using the other equations. On the other hand, if we have that

$$\langle e_2|F(0)|e_2\rangle\geqslant 0 \tag{34}$$

for all $|e_2\rangle$, then we will have that there exists no solution with $\lambda_0<0$. This is so since $F(-|\lambda_0|)-F(0)\geqslant 0$.

We have made a systematic search of the states $|e_{1,2}\rangle$ which minimize the minimum eigenvalue of $\langle e_2|F(0)|e_2\rangle\geqslant 0$ for $d=3$ and $N=2$ and 3. Note that for $N=2$ copies we can further simplify the numerical search by using the symmetries of the problem, which imply that we can choose $|e_1\rangle=\Sigma_{i=1}^3 c_i|i,i\rangle$ with $c_i\geqslant 0$.

In both cases we have found that this minimum eigenvalue is $\geqslant 0$ for $\beta\leqslant 1/2$, which strongly indicates that ρ_α is 3-undistillable (and hence also 2-undistillable) for $\beta\leqslant 1/2$. This is exactly the same bound that we had obtained analytically for 1-undistillability.

VI. CONCLUSIONS

We have shown that in order to study the distillability properties of bipartite d-level systems, it is sufficient to consider only the one-parameter class of states ρ_α [Eq. (12)]. By investigating the distillability properties of this family of states, we found strong indications that this family provides examples for nondistillable states with nonpositive partial transposition. In particular, we found that for any given number of copies N there exist N-undistillable states which have NPPT. Guided by the results of the numerical investigations, we conjecture that for $d=3$ the states ρ_α are nondistillable for $\beta\leqslant 1/2$, while they have NPPT for $\beta>0$ (see also Fig. 1).

Note added: Recently we became aware of the results of DiVincenzo *et al.* [20], in which they also found evidences for the existence of nondistillable states with NPPT.

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APPENDIX A: INTEGRAL REPRESENTATION OF $\mathcal{D}(X)$

Here we show that the superoperator $\mathcal{D}$ as defined in Eq. (8) can also be written in the form of Eq. (9). Here, for convenience, we restrict ourselves to operators X, which are density operators ρ, but exactly the same line of arguments holds for arbitrary self adjoint-operators X. As shown in Appendix B, we have that the depolarization superoperator $\mathcal{D}$ can be implemented by a finite sequence of bilocal operations [Eq. (B1)] of the form $U\otimes U$. Furthermore we have that the projector onto the antisymmetric subspace is invariant under unitary operations of the form $U\otimes U$, i.e., $U\otimes U A_d U^\dagger\otimes U^\dagger = A_d$, which can be easily seen by using that $A_d=(1-\Pi_d)/2$. From this property automatically follows that ρ_α [Eq. (12)] is also invariant under unitary operations of the form $U\otimes U$, since $\rho_\alpha \propto (\mathbb{1}+\tilde{\alpha}A_d)$. It is now straightforward to show Eq. (9) by using, for any V,

$$\int d\mu_U (U\otimes U)\rho(U\otimes U)^\dagger = \int d\mu_{U'}(U'\otimes U')(V\otimes V)\rho(V\otimes V)^\dagger(U'\otimes U')^\dagger, \tag{A1}$$

with $U'V=U$. Taking p_k such that $\Sigma_k p_k=1$, we can write

$$\begin{aligned}\int d\mu_U(U\otimes U)\rho(U\otimes U)^\dagger &= \sum_k p_k \int d\mu_{U'}(U'\otimes U')(U_k\otimes U_k) \\ &\quad \times\rho(U_k\otimes U_k)^\dagger(U'\otimes U')^\dagger \\ &= \int d\mu_{U'}(U'\otimes U')\rho_\alpha(U'\otimes U')^\dagger = \rho_\alpha = \mathcal{D}(\rho),\end{aligned} \tag{A2}$$

where we used Eq. (A1) in the first equality, while the second equality follows from Eq. (B1), and we finally used the invariance of ρ_α under operations of the form $U\otimes U$. This already shows that Eq. (9) is fulfilled, i.e., $\mathcal{D}(\rho)=\int d\mu_U (U\otimes U)\rho(U\otimes U)^\dagger$.

APPENDIX B: DEPOLARIZATION

We are going to show now that an arbitrary state ρ can be depolarized to the standard form [Eq. (12)] by a finite sequence of bilocal (random) operations without changing the weight in the antisymmetric subspace $\mathcal{H}_a(d)$, i.e., we show that there exist unitary operators U_k and probabilities p_k such that

$$\sum_k p_k U_k\otimes U_k \rho U_k^\dagger \otimes U_k^\dagger = \mathcal{D}(\rho) = \rho_\alpha \tag{B1}$$

with $\mathrm{tr}(A_d\rho)=\mathrm{tr}(A_d\rho_\alpha)$. We first introduce a basis of $\mathbb{C}^d\otimes\mathbb{C}^d$,

$$\begin{aligned}|\varphi_{ij}^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|i\rangle_A|j\rangle_B \pm |j\rangle_A|i\rangle_B), \\ |\chi_k\rangle &= |k\rangle_A|k\rangle_B,\end{aligned} \tag{B2}$$

with $i<j$ and $(i,j,k)\in\{1,\ldots,d\}$. Note that $|\varphi_{ij}^-\rangle$ provides a basis of the antisymmetric subspace $\mathcal{H}_a(d)$, while $\{|\varphi_{ij}^+\rangle,|\chi_k\rangle\}$ is a basis of the symmetric subspace $\mathcal{H}_s(d)$. The projectors into the symmetric-antisymmetric subspace can thus be written

$$\begin{aligned}A_d &= \sum_{i,j=1(i<j)}^{d} |\varphi_{ij}^-\rangle\langle\varphi_{ij}^-|, \\ S_d &= \sum_{i,j=1(i<j)}^{d} |\varphi_{ij}^+\rangle\langle\varphi_{ij}^+| + \sum_{k=1}^{d}|\chi_k\rangle\langle\chi_k|.\end{aligned} \tag{B3}$$

Let us write ρ in basis (B2). In order to prove Eq. (B1), we divide the depolarization procedure into three steps.

(i) We show that one can make ρ diagonal in basis (B2) without changing the diagonal elements.

(ii) We prove that the antisymmetric subspace $\mathcal{H}_a(d)$ can be mixed up, i.e., one can equalize the coefficients of $|\varphi_{ij}^-\rangle\langle\varphi_{ij}^-|$ without changing the weight in $\mathcal{H}_a(d)$.

(iii) Finally we show that the symmetric subspace can also be completely mixed up without changing the weight in $\mathcal{H}_s(d)$.

These three steps together ensure that ρ can be depolarized to the standard form ρ_α.

1. Diagonalizing ρ

By mixing we understand in the following that a certain operation U is (randomly) performed with probability p by Alice and Bob, while with probability $1-p$ no operation is performed. The resulting density operator after this mixing operation reads

$$\rho_{\mathrm{new}} = p(U\otimes U\rho U^\dagger\otimes U^\dagger) + (1-p)\rho. \tag{B4}$$

We define the operation U_l as

$$U_l|k\rangle = e^{i\pi\delta_{kl}}|k\rangle, \tag{B5}$$

i.e., the state $|l\rangle$ picks up a minus sign while all others remain unchanged. Let us perform a sequence of d mixing operations, using p$=\frac{1}{2}$ and U_l with $l=\{1,\ldots,d\}$ respectively. One can easiliy check that all diagonal elements remain unchanged, while all off-diagonal elements of the form $|\varphi_{ij}^\pm\rangle\langle\chi_k|$ and $|\varphi_{ij}^\pm\rangle\langle\varphi_{i'j'}^\pm|$ for $(i\neq i', j\neq j')$ are eliminated.

Let us now define the operation $\hat{U}_l$, which introduces a phase i for state $|l\rangle$ while it leaves all other states unchanged.

Again performing the same sequence of d mixing operations as described above, but with U_l replaced by $\hat{U}_l$, one can check that all off-diagonal elements of the form $|\chi_i\rangle\langle\chi_j|$ are eliminated.

We finally define the operation U_{kl}, which simply swaps states $|k\rangle$ and $|l\rangle$ while leaving all other states untouched. Now performing a sequence of mixing operations using all possible combinations $k<l\in\{1,\ldots,d\}$ for U_{kl} and $p=1/2$, one gets rid of the remaining off-diagonal elements of the form $|\varphi^-_{ij}\rangle\langle\varphi^+_{ij}|$ without changing the diagonal ones. Thus ρ can be made diagonal in basis (B2) by a sequence of bilocal operations.

2. Mixing of $\mathcal{H}_a(d)$

Let $\mathcal{H}_{i_0}$ be the subspace spanned by $\{|\varphi^-_{i_0 j}\rangle\}$ for fixed i_0. In a first step we will show that one can depolarize all subspaces $\mathcal{H}_{i_0}$ independently, while in a second step we prove that these subspaces can be mixed with each other.

To depolarize $\mathcal{H}_1$, one just has to keep state $|1\rangle$ and perform (randomly) one of the cyclic permutations of states $\{|2\rangle,\ldots,|d\rangle\}$, each with probability $p=1/(d-1)$. Similary, one depolarizes $\mathcal{H}_2$ by keeping states $\{|1\rangle,|2\rangle\}$ and performing, with probability $p=1/(d-2)$ one of the cyclic permutations of the states $\{|3\rangle,\ldots,|d\rangle\}$. Since $\mathcal{H}_1$ is already depolarized, it is not affected by this operation. One can continue in the same way until one has depolarized $\mathcal{H}_{d-1}$. So the antisymmetric part of the density operator now has the form

$$A_d\rho A_d=\sum_{i=1}^{d-1} a_i \sum_{j=i+1}^{d} |\varphi^-_{ij}\rangle\langle\varphi^-_{ij}|. \tag{B6}$$

The second step starts by mixing of $\mathcal{H}_{d-1}$ with $\mathcal{H}_{d-2}$, i.e., equalizing the coefficients a_{d-1} and a_{d-2}. To achieve this, both Alice and Bob swap states $|d-1\rangle$ and $|d-2\rangle$ with probability $p_s=\frac{2}{3}$, or both apply the identity operator with probability $1-p_s$. If one now depolarizes $\mathcal{H}_{d-1}$ and $\mathcal{H}_{d-2}$ independently as described in step 1, one finds that $|\varphi^-_{(d-1)j}\rangle\langle\varphi^-_{(d-1)j}|$ and $|\varphi^-_{(d-2)j}\rangle\langle\varphi^-_{(d-2)j}|$ now all have the same weight, i.e., the coefficients are equal. Thus the subspace $\mathcal{H}_{d-1}$ is completely mixed with $\mathcal{H}_{d-2}$. One now continues by mixing the subspaces $\mathcal{H}_{d-3}$ with $\{\mathcal{H}_{d-2},\mathcal{H}_{d-1}\}$ and so on, until one reaches $\mathcal{H}_1$.

We now investigate one particluar step in this procedure, namely, the mixing of $\mathcal{H}_k$ with $\{\mathcal{H}_{k+1},\mathcal{H}_{k+2},\ldots,\mathcal{H}_{d-1}\}$. Both Alice and Bob swap states $|k\rangle$ and $|k+1\rangle$ with probability $p_s=(d-k)/(d-k+1)$, or both apply the identity operator with probability $1-p_s$. After this one depolarizes $\mathcal{H}_j (j\leqslant k)$ independently, then mixes $\mathcal{H}_{d-1}$ with $\mathcal{H}_{d-2}$ as described above. Next one mixes $\mathcal{H}_{d-3}$ with $\{\mathcal{H}_{d-2},\mathcal{H}_{d-1}\}$, and continues in this way until one has mixed $\mathcal{H}_k$ and $\{\mathcal{H}_{k+1},\mathcal{H}_{k+2},\ldots,\mathcal{H}_{d-1}\}$. It can be checked that after this procedure all weight factors $\tilde{a}_j$ are equal for $(j\leqslant k)$. Thus once one has mixed $\mathcal{H}_1$ with $\{\mathcal{H}_2,\mathcal{H}_3,\ldots,\mathcal{H}_{d-1}\}$, one has achieved that the whole antisymmetric subspace is completely depolarized, i.e. it can be written as $\alpha_d A_d$.

3. Mixing of $\mathcal{H}_s(d)$

Note that the depolarization of the antisymmetric subspace also mixes the subspace spanned by $\{|\varphi^+_{ij}\rangle\}$ in a similar way. Here we now show that one can also depolarize the subspace spanned by $\{|\chi_k\rangle\}$, and also finally that these two subspaces together can be mixed up. To prove the first step, let us define the operation $\tilde{U}_l$ as follows:

$$\tilde{U}_l|k\rangle=|(k+l)\mathrm{mod}(d)\rangle. \tag{B7}$$

Now performing $\tilde{U}_l\otimes\tilde{U}_l, l=\{1,\ldots,d\}$, each with probability p$=1/d$, ensures that the subspace spanned by $\{|\chi_k\rangle\}$ is completely depolarized, i.e., that ρ now has the form

$$\rho=aA_d+b\sum_{i,j=1(i<j)}^{d}|\varphi^+_{ij}\rangle\langle\varphi^+_{ij}|+c\sum_{k=1}^{d}|\chi_k\rangle\langle\chi_k|. \tag{B8}$$

For the second step, we define the unitary operation T:

$$T|j\rangle=\frac{1}{\sqrt{d}}\sum_{k=1}^{d}e^{i2\pi(j-1)(k-1)/d}|k\rangle. \tag{B9}$$

One can check that if we perform the operation $T\otimes T$ with probability $p_T=d/(d+1)$, and the identity operation with probability $p_I=1-p_T$, the diagonal elements of the symmetric subspace will each be identical to $[bd(d-1)+2cd]/[d(d+1)]$. The introduced off-diagonal elements can be eliminated using the procedure explained above. Note that the antisymmetric subspace is not affected by this kind of operation, and will thus remain untouched. Thus, finally, we manage to show that ρ can be converted to the standard form [Eq. (12)] by a sequence of local operations [Eq. (B1)]. The weight in $\mathcal{H}_a(d)$ was not affected by any of the mixing operations used, which ensures that $\mathrm{tr}(A_d\rho)=\mathrm{tr}(A_d\rho_\alpha)$.

APPENDIX C: BETTER BOUND FOR β

In this appendix we prove that for any vector $|\Psi\rangle$ of the form of Eq. (3),

$$\langle\Psi|(Q-\beta P)^{\otimes N}|\Psi\rangle\geqslant\frac{1}{3^N}\left(1-\frac{\beta}{\beta_N}\right), \tag{C1}$$

for $\beta\leqslant\beta_N$, where

$$\beta_N=\frac{x_N}{3^{N/3}N^{1/3}}, \tag{C2}$$

and $x_N=x^*[1-O(1/N)]$ as $N\rightarrow\infty$, where $x^*=3(1-3^{-1/3})^{1/3}$, and $O(1/N)$ denotes a quantity of the order of $1/N$.

The proof is by induction. For $N=1$, we have $\beta_1=1/2$. Assuming that the statement holds for $N-1$, we observe that

$$\langle\Psi|Q(Q-\beta P)^{\otimes N-1}|\Psi\rangle\geqslant\frac{1}{3^N}\left(1-\frac{\beta}{\beta_{N-1}}\right), \tag{C3}$$

and the same holds for all possible permutations with respect to the copies. Adding the left-hand sides, dividing the sum by N, and rearranging various terms, we find, for large N,

$$\langle\Psi|\left(Q-\frac{N-1}{N}\beta P\right)^{\otimes N}|\Psi\rangle\geqslant\frac{1}{3^N}\left(1-\frac{\beta}{\beta_{N-1}}\right)-\frac{N\beta^3}{27}[1+O(1/N)]-O(\beta^5). \tag{C4}$$

For $\beta\simeq O(\beta_N)$, the last correction term in the above inequality can be safely neglected. It is easy to check that the right-hand side has zero at $\beta=\beta_N$, i.e., that

$$\langle\Psi|(Q-\beta P)^{\otimes N}|\Psi\rangle\geqslant\frac{1}{3^N}\left(1-\frac{\beta}{\beta_N}\right) \tag{C5}$$

for $\beta\leqslant\beta_N$; the statement thus holds for every N. The above result provides a better bound for N-undistillability: the states ρ_α such that

$$\beta\leqslant\frac{x^*}{3^{N/3}N^{1/3}}[1-O(1/N)] \tag{C6}$$

are N-undistillable.

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