

Evidence for bound entangled states with negative partial transpose

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We exhibit a two-parameter family of bipartite mixed states ρ_{bc} , in a $d \otimes d$ Hilbert space, which are negative under partial transposition (NPT), but for which we conjecture that no maximally entangled pure states in $2 \otimes 2$ can be distilled by local quantum operations and classical communication (LQ+CC). Evidence for this undistillability is provided by the result that, for certain states in this family, we cannot extract entanglement from any arbitrarily large number of copies of ρ_{bc} using a projection on $2 \otimes 2$. These states are canonical NPT states in the sense that any bipartite mixed state in any dimension with NPT can be reduced by LQ+CC operations to a NPT state of the ρ_{bc} form. We show that the main question about the distillability of mixed states can be formulated as an open mathematical question about the properties of composed positive linear maps.

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I. INTRODUCTION

Maximally entangled quantum states, when their two halves are shared between two parties, are a uniquely valuable resource for various information-processing tasks. Used in conjunction with a quantum communications channel, they can increase the classical data-carrying capacity of that channel, in some cases by an arbitrarily large factor [1]. Possession of maximally entangled states can ensure perfect privacy of communication between the two parties by the use of quantum cryptography [2]. These states can facilitate the rapid performance of certain forms of distributed computations [3]. Of course, maximally entangled states are the key resource in quantum teleportation [4]. On the other hand, the surreptitious establishment of entanglement between two parties can thwart the establishment of trust between parties via bit commitment [5].

How can two parties come into the possession of a shared maximally entangled state? If the storage and transportation of quantum particles were perfect, then the state could have been synthesized in some laboratory long in the past and given to Alice and Bob (our personified parties) for storage until needed. In practice no such perfect infrastructure exists. Since the most interesting scenarios for the use of quantum entanglement are in cases where Alice and Bob are remote from one another, we will consider the long-distance transportation of quantum states needed to establish the shared entanglement to be difficult and imperfect, while the local processing of quantum information (unitary transformations, measurement) we will assume, for the sake of analysis, to be essentially perfect.

Under these assumptions, when we wish to assess whether a given physical setup is or is not useful for entanglement-assisted information processing, our analysis focuses on the mixed quantum state ρ in the hands of Alice and Bob after the difficult transportation step. We enquire whether $\rho^{\otimes n}$ can be transformed, by local quantum operations and classical communication (LQ+CC) operations, to a supply of maximally entangled states. Here the $\otimes n$ notation

indicates that n copies of the state ρ are available, and we will be concerned with asymptotic results as n is taken to infinity. LQ+CC operations (sometimes called LOCC in the literature) are obtained by any arbitrary sequence of *local* quantum operations (appending ancillae, performing unitary operations, discarding ancillae) supplemented by *classical communication* between Alice and Bob.

An interesting fact about this possibility for the *distillation* of entanglement is that it is neither rare nor ubiquitous; a finite fraction of the set of all possible bipartite mixed states ρ can be successfully distilled [6], and a finite fraction cannot [7]. Much work has been focused on whether ρ falls into a distillable or undistillable class, and this paper is primarily a contribution to this classification task. Before describing our contributions, we will give a brief review of previous results on classifying states according to their distillability.

Multipartite density matrices ρ are considered *unentangled* if there exists a decomposition of ρ into an ensemble of pure product states; for the bipartite case this means that we can write

$$\rho = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i|. \quad (1.1)$$

These are also referred to as separable states. It is clear that separable states are never distillable. However, the converse proposition, that entangled states are always distillable, is false in general, although true for density matrices in $2 \otimes 2$ and $2 \otimes 3$ Hilbert spaces [8]. This became clear shortly after the introduction by Peres [9] of a computationally simple criterion for separability, the *partial transposition* test. The partial-transpose operation, denoted as $I \otimes T$ when the transpose is applied to Bob's Hilbert space, is specified by the action

$$\langle ij|(I \otimes T)(\rho)|kl\rangle = \langle il|\rho|kj\rangle. \quad (1.2)$$

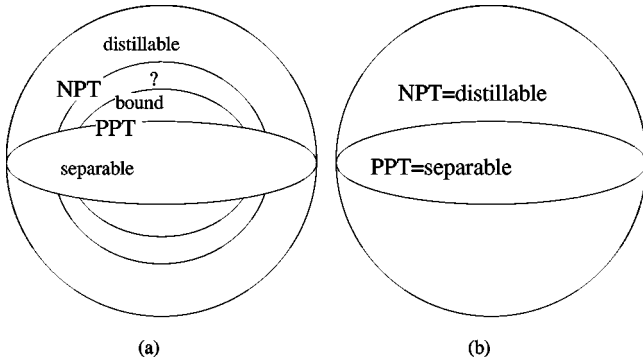


FIG. 1. Layout of the set of all mixed states. (a) General case for arbitrary Hilbert space dimension $m \otimes n$. The “?” region, that of bound or undistillable NPT states, is the subject of this paper. This region is known to contain no states for $2 \otimes n$. (b) Simplified situation for a dimension of $2 \otimes 2$ and $2 \otimes 3$ for which it is known that all PPT states are separable and all NPT states are distillable.

While application of T to Alice’s Hilbert space will lead to identical results, we will always apply it to Bob’s space in this paper. Here $|i\rangle$ and $|k\rangle$ indicate an orthonormal basis on Alice’s Hilbert space, and $|j\rangle$ and $|l\rangle$ the same for Bob. It is easy to show that separable states are positive under partial transpose, that is, that the matrix

$$(I \otimes T)(\rho) = \rho^{PT} \quad (1.3)$$

is a positive semidefinite operator, denoted by $\rho^{PT} \geq 0$. [Equation (1.3) introduces the partial transposition (PT) notation that we will use throughout this paper.] This positivity property is abbreviated as PPT; states for which $\rho^{PT} \neq 0$ are called NPT states. It was soon recognized [10] that the set of PPT density operators ρ is larger than the set of unentangled states (except in $2 \otimes 2$ and $2 \otimes 3$); see Fig. 1. It was also discovered that all PPT states, even those which are inseparable, are not distillable. The existence of such states, in which entanglement is present (since entanglement is required to synthesize the states) but cannot be reextracted in pure form, was a surprising observation, indicating the possibility of a fundamentally new form of irreversibility in physics. States having this property are said to possess *bound entanglement*.

The introduction of the PPT-NPT classification suggested a conjecture about distillability, namely, that all states with NPT would possess distillable entanglement, and it is the purpose of the present paper to explore this conjecture. While no rigorous results have been obtained concerning this conjecture, we will introduce a two-parameter family of NPT states for which we obtain evidence that the conjecture is false. That is, we consider it likely that the family of states we introduce below has only bound entanglement, despite being NPT.

We have been able to recast the question about the distillability property of the ρ_{bc} states, or of any NPT states, as a question about the 2-positivity properties of certain positive linear maps [11]. These maps arise because there is a one-to-one correspondence between mixed states on $d \otimes d$ and completely positive linear maps \mathcal{S} on d dimensions which,

when applied to half a maximally entangled state $|\Psi^+\rangle$, produces the mixed state ρ . Then, the NPT property is related to the map $T \circ \mathcal{S}$, and its compositions $(T \circ \mathcal{S})^{\otimes n}$. The open question about distillability can be posed compactly as a question concerning the mathematical properties of these maps. This approach also permits us to consider the question of whether distillability is an additive property, that is, whether the amount of distillable entanglement of $\rho_1 \otimes \rho_2$ is just the sum of the two separately. Horodecki *et al.* [12] gave some evidence of a kind of undistillability involving single copies of PPT bound entangled states. In the positive-map language, the most general questions about nonadditivity can be compactly framed. This shows that further developments of the theory of positive maps will be very desirable in settling some of the fundamental questions about the entanglement properties of quantum states.

This paper develops in the following way: Section II introduces the canonical states ρ_{bc} , and shows the LQ+CC mapping that produces them. Section III considers the distillability of any state ρ by application of the basic criterion of whether it remains entangled when projected into $2 \otimes 2$. Section III A considers a single copy of the ρ_{bc} state, establishing the ρ_{bc} for which there exist such projections into $2 \otimes 2$. Section III B takes up the much harder case of multiple copies, with Sec. IV proving the result that, for some $\rho_{bc}^{\otimes n}$ states, no entanglement remains upon projection into $2 \otimes 2$ even for arbitrarily large n . Section V recasts the question about distillability in terms of the 2 positivity of linear maps isomorphic to the mixed states.

II. A CANONICAL SET OF NPT DENSITY MATRICES

The desired, but too-ambitious, program would be to assess the distillability of all NPT states. We will attempt this assessment only for a specific subset of the NPT states parametrized by two real numbers. This subset will, however, have a specific relation to the set of NPT states, in that there is a LQ+CC operation that will map the general NPT state onto one parameter in our two-parameter family. This LQ+CC operation preserves the NPT property. Thus, if we could exhibit a protocol for the distillation of our two-parameter family, this would suffice to show that all NPT states were distillable. However, our canonical two-parameter family has properties which make distillation quite hard for certain ranges of parameters, suggesting that in fact some portion of the full set of NPT states is *not* distillable.

Our canonical states, with real parameters b and c , are written

$$\begin{aligned} \rho_{bc} = & a \sum_{i=0}^{d-1} |ii\rangle\langle ii| + b \sum_{i,j=0,i<j}^{d-1} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| \\ & + c \sum_{i,j=0,i<j}^{d-1} |\psi_{ij}^+\rangle\langle\psi_{ij}^+|. \end{aligned} \quad (2.1)$$

Here

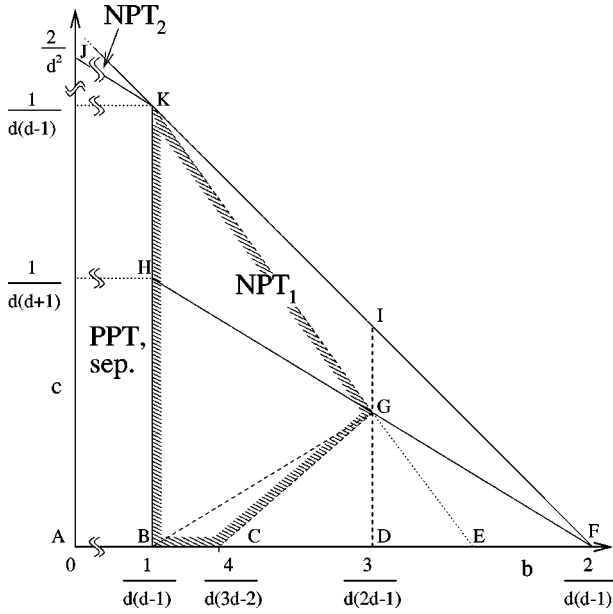


FIG. 2. The relevant region of the bc parameter space for the states ρ_{bc} . All NPT states can be brought by LQ+CC action into the region NPT_1 , triangle BFK . For general dimension, region $CFKG$ is distillable by projection on one copy, and region $BCGK$ is pseudo-one-copy undistillable. In $3 \otimes 3$ we have strong evidence that region $BCGK$ is pseudo-two-copy undistillable. We conjecture that the entire region $BCGK$ is undistillable by any means. All states in the PPT region $ABKJ$ are separable; that is, there are no bound PPT states among the ρ_{bc} set.

$$|\psi_{ij}^{\pm}\rangle = \frac{1}{\sqrt{2}} (|ij\rangle \pm |ji\rangle). \quad (2.2)$$

The states live in a $d \otimes d$ Hilbert space. The parameter a in Eq. (2.1) is not independent, because of the unit trace condition it is related to b and c by

$$da + (b+c)d(d-1)/2 = 1. \quad (2.3)$$

The range of interest for the parameters b and c is shown in Fig. 2. As we will show in Sec. III, the state is NPT in two triangular regions of parameter space; one of these regions, NPT_2 , which will not be of much interest to us (all these states are distillable), lies above the straight line KJ , and is defined by the inequality $c > 2/d^2 + b(d-2)/2$. The region NPT_1 , about which we will have much more to say, lies in the region BFK and is defined by $b > 1/[d(d-1)]$. Region $ABKJ$ contains PPT states; in Sec. III B we prove that all these states are also separable.

To show that ρ_{bc} represents a canonical set, we will exhibit a procedure involving only LQ+CC operations that will convert any NPT density matrix ρ , that is, one satisfying the condition

$$\langle \psi | (\mathbf{1} \otimes T)(\rho) | \psi \rangle < 0 \quad (2.4)$$

for some state $|\psi\rangle$, to one of the ρ_{bc} form having NPT. We will take the Hilbert space dimension to be $n \otimes m$; that is, we will not restrict Alice's and Bob's dimensions to be the

same. Here is the sequence of LQ+CC operations that will reduce the general NPT state ρ to ρ_{bc} .

(i) *Rotation to the Schmidt basis.* We write $|\psi\rangle$ of Eq. (2.4) as

$$|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |\alpha_i\rangle \otimes |\beta_i\rangle. \quad (2.5)$$

Here $d \leq \min(n, m)$. Let $U_A |\alpha_i\rangle = |i\rangle$ and $U_B |\beta_i\rangle = |i\rangle$, or

$$|\psi\rangle = U_A^\dagger \otimes U_B^\dagger |\phi\rangle, \quad (2.6)$$

where

$$|\phi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle. \quad (2.7)$$

We define $\rho^{(i)} = U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger$. Equation (2.4) can be rewritten as

$$\langle \phi | (\mathbf{1} \otimes T^U)(\rho^{(i)}) | \phi \rangle < 0, \quad (2.8)$$

where T^U is transposition in a rotated basis determined by U_B . The negativity of the expression [Eq. (2.8)] does not depend on the basis in which T is performed; therefore, we will replace T^U by T again in the remainder.

(ii) *Local filtering* (see Ref. [6]). We define the state $|\Phi^+\rangle$ as

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle. \quad (2.9)$$

The filter operation W on Alice's Hilbert space is defined by the equation

$$W^\dagger \otimes \mathbf{1} |\Phi^+\rangle = |\phi\rangle. \quad (2.10)$$

We apply this local filter to the state $\rho^{(i)}$ to obtain $\rho^{(ii)}$:

$$\rho^{(ii)} = \frac{(W \otimes \mathbf{1}) \rho^{(i)} (W^\dagger \otimes \mathbf{1})}{\text{Tr}(W^\dagger W \otimes \mathbf{1}) \rho^{(i)}}. \quad (2.11)$$

Equation (2.4) implies that

$$\langle \Phi^+ | (\mathbf{1} \otimes T)(\rho^{(ii)}) | \Phi^+ \rangle = \text{Tr} |\Phi^+ \rangle \langle \Phi^+ | (\mathbf{1} \otimes T)(\rho^{(ii)}) < 0. \quad (2.12)$$

We now use that $\text{Tr}[A^\dagger T(B)] = \text{Tr}[T^\dagger(A^\dagger) B]$ and $T^\dagger = T$ to rewrite this NPT condition in a form which will be convenient below:

$$\text{Tr} H \rho^{(ii)} < 0, \quad (2.13)$$

with

$$H = (\mathbf{1} \otimes T)(|\Phi^+\rangle \langle \Phi^+|). \quad (2.14)$$

This Hermitian operator H can be written in its eigenbasis,

$$\begin{aligned}
 H = & \frac{1}{d} \sum_{i=0}^{d-1} |ii\rangle\langle ii| - \frac{1}{d} \sum_{i,j=0,i<j}^{d-1} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| \\
 & + \frac{1}{d} \sum_{i,j=0,i<j}^{d-1} |\psi_{ij}^+\rangle\langle\psi_{ij}^+|, \quad (2.15)
 \end{aligned}$$

where

$$|\psi_{ij}^\pm\rangle = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle). \quad (2.16)$$

(iii) *Project into $d \otimes d$.* Since $|\Phi^+\rangle$, and H , have support only a $d \otimes d$ -dimensional subspace of the Hilbert space, Alice and Bob can project locally onto this subspace and leave the NPT condition [Eq. (2.12)], or Eq. (2.14), unchanged. We call the resulting NPT density matrix in $d \otimes d$ $\rho^{(iii)}$.

(iv) *Diagonal twirl.* Alice and Bob perform a equal mixture of identical unitary operations, which are diagonal in the Schmidt basis given by the vectors $|i\rangle$, giving state $\rho^{(iv)}$. This unitary operation is

$$(U_{A,B}(\{\theta_i\}))_{i,j} = \delta_{ij} e^{i\theta_i}. \quad (2.17)$$

The phases θ_i are chosen randomly over a uniform distribution from 0 to 2π , independently for each i . This leaves the operators

$$|ij\rangle\langle ji|, |ij\rangle\langle ij|, |ii\rangle\langle ii| \quad (2.18)$$

invariant. This operation therefore leaves the eigenvectors of H and thus H itself invariant. Thus it follows that

$$\begin{aligned}
 \text{Tr } H \rho^{(iv)} = & \text{Tr} \int d\{\theta\} U^\dagger(\{\theta\}) \otimes U^\dagger(\{\theta\}) H U(\{\theta\}) \\
 & \otimes U(\{\theta\}) \rho^{(iii)} = \text{Tr } H \rho^{(iii)} < 0. \quad (2.19)
 \end{aligned}$$

The ‘‘twirled’’ density matrix $\rho^{(iv)}$ has the form

$$\begin{aligned}
 \rho^{(iv)} = & \sum_{i=0}^{d-1} \alpha_i |ii\rangle\langle ii| + \sum_{i,j=0,i \neq j}^{d-1} \beta_{ij}^1 |ij\rangle\langle ij| \\
 & + \sum_{i,j=0,i \neq j}^{d-1} \beta_{ij}^2 |ij\rangle\langle ji|. \quad (2.20)
 \end{aligned}$$

Note that the coefficients in these sums are all in general distinct, with β_{ij}^1 not necessarily equal to β_{ji}^1 and similarly for β_{ij}^2 .

(v) *Symmetrize by permutation.* Alice and Bob carry out identical, randomly chosen, unitary transformations which are drawn uniformly from all possible permutation operations over the elements of the Schmidt basis $|i\rangle$. This ensures that in the new density matrix $\rho^{(v)}$ the α_i coefficients, for all i , become equal to a single number a , all the β_{ij}^1 become equal [we call this constant $(c+b)/2$], and all the β_{ij}^2 become equal [call this constant $(c-b)/2$]. So we obtain

$$\begin{aligned}
 \rho^{(v)} = & a \sum_{i=0}^{d-1} |ii\rangle\langle ii| + \frac{c+b}{2} \sum_{i,j=0,i \neq j}^{d-1} |ij\rangle\langle ij| \\
 & + \frac{c-b}{2} \sum_{i,j=0,i \neq j}^{d-1} |ij\rangle\langle ji|. \quad (2.21)
 \end{aligned}$$

But comparing with Eq. (2.1), we note that we have arrived at the desired canonical form

$$\rho^{(v)} = \rho_{bc}. \quad (2.22)$$

As the Hermitian matrix H of Eqs. (2.14) and (2.15) is again invariant under this symmetrization, we note that the NPT property is again preserved:

$$\text{Tr } H \rho^{(v)} = \text{Tr } H \rho_{bc} < 0. \quad (2.23)$$

We may summarize the foregoing line of argument as a theorem.

Theorem 1. Let ρ be a bipartite density matrix on $n \otimes m$ with the property that $\rho \neq 0$. The density matrix ρ can be converted by local operations and classical communication to a density matrix ρ_{bc} on $d \otimes d$ with $d \leq \min(n, m)$ characterized by two real parameters b and c such that $\rho_{bc} \neq 0$. This density matrix ρ_{bc} is

$$\rho_{bc} = a \sum_{i=0}^{d-1} |ii\rangle\langle ii| + b \sum_{i,j,i<j}^{d-1} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| + c \sum_{i,j,i<j}^{d-1} |\psi_{ij}^+\rangle\langle\psi_{ij}^+|, \quad (2.24)$$

with

$$da + (b+c)d(d-1)/2 = 1. \quad (2.25)$$

It is easy to see from the form of H that these transformations carry all NPT states ρ into a ρ_{bc} sitting in the NPT₁ region of Fig. 2. This is why the NPT₂ region will not be of concern to us. We note that it is possible to follow the five-step reduction above with another LQ+CC operation, resulting in a canonical NPT density operator characterized by just a single real parameter.

(vi) *Full twirl.* Alice and Bob perform an equal mixture of identical unitary operations drawn uniformly (with the Haar measure) from the entire group $U(d)$. It is straightforward to show that the resulting density matrix $\rho^{(vi)}$ has the same form as above [Eq. (2.24)]:

$$\begin{aligned}
 \rho^{(vi)} = & a' \sum_{i=0}^{d-1} |ii\rangle\langle ii| + b' \sum_{i,j,i<j}^{d-1} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| \\
 & + c' \sum_{i,j,i<j}^{d-1} |\psi_{ij}^+\rangle\langle\psi_{ij}^+|, \quad (2.26)
 \end{aligned}$$

with

$$b' = b, \quad (2.27)$$

$$c' = \frac{2}{d(d+1)} - \frac{d-1}{d+1} b, \quad (2.28)$$

and a' given by the same constraint as in Eq. (2.25). Thus $\rho^{(vi)}$ depends only on the single parameter b ; it is the same Werner density matrix studied recently by Horodecki and Horodecki [6],

$$\rho_W = \frac{1}{d^3 - d} [(d - \phi)\mathbf{1} + (d\phi - 1) dH]; \quad (2.29)$$

note that H of Eq. (2.14) is proportional to the ‘‘swap’’ operator

$$dH|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle. \quad (2.30)$$

This ‘‘full twirl’’ carries all the states in the *BFK* region of Fig. 2 onto the line *FH*, without changing the value of b .

Of course, if it were possible to prove that all the NPT states of the one-parameter form ρ_W were distillable, then all NPT states would be distillable through the reductions we have developed above. In fact we conjecture, as Horodecki and Horodecki did previously (Sec. VIII, Ref. [6]), that some of these NPT states are undistillable. Under these circumstances, it is desirable to provide evidence for the undistillability for the widest class of states possible, and in this paper we will concentrate on providing such evidence for the two-parameter family of canonical states ρ_{bc} , more particularly, for those lying near the line segment *BK* in Fig. 2. All of the results we develop will, of course, also apply to the restricted one-parameter family ρ_W as well.

III. TOOLS FOR THE STUDY OF DISTILLABILITY

In this section we will explore all the known tools at our disposal for analyzing the distillability of states. For some of the ρ_{bc} states we believe that no distillation protocol exists; evidence of this is provided by the last result of this section, that for some ρ_{bc} states, any successful distillation protocol, if it exists, must act on some very large number n of copies of the state; we show that n must diverge along an entire boundary *BK* in Fig. 2.

Much of the discussion of distillation strategies will need the notion of the *Schmidt rank* of a pure state in an ensemble decomposition of density matrix ρ . We first define this term.

Definition 1. A bipartite pure state $|\psi\rangle$ has Schmidt rank k if the state can be written in the Schmidt polar form as

$$|\psi\rangle = \sum_{i=1}^k \sqrt{\lambda_i} |a_i\rangle \otimes |b_i\rangle, \quad (3.1)$$

with $\langle a_i | a_j \rangle = \delta_{ij}$ and $\langle b_i | b_j \rangle = \delta_{ij}$.

The distillation of the ρ_{bc} states (or more particularly, of the ρ_W subset of these states) was already considered in Ref. [6]. There, a distillation protocol was developed based on the positive linear map $\Lambda_c : \rho \rightarrow \text{Tr} \rho \mathbf{1} - \rho$. In Sec. V we will discuss other aspects of the relation between the theory of positive maps and the distillability of mixed states. For all states ρ for which $(\mathbf{1} \otimes \Lambda_c)(\rho) \not\geq 0$, it was shown how to distill them by converting these states to a different canonical density-matrix form introduced by Werner.

However, all the states ρ_{bc} remain positive under the action of Λ_c , so long as the dimension $d > 2$, because $(\mathbf{1} \otimes \Lambda_c)(\rho_{bc}) \propto \rho_{b'c'}$, where $b' = (c + b)/2 - b/(d - 1)$ and $c' = (c + b)/2 - c/(d - 1)$. (Positivity under the action of Λ_c was already known for ρ_W [6].) Thus, the simple distillation procedure studied in Ref. [6] will not work for these states. Thus, to study the distillability of these states, we need to consider the more general necessary and sufficient condition developed by Horodecki *et al.*

Lemma 1 (Horodecki *et al.* [13]): A density matrix $\rho \in m_A \otimes m_B$ is distillable if and only if there exists a finite n and projections $P_A : \mathcal{H}_{m_A}^{\otimes n} \rightarrow \mathcal{H}_2$ and $P_B : \mathcal{H}_{m_B}^{\otimes n} \rightarrow \mathcal{H}_2$ such that $\sigma = (P_A \otimes P_B) \rho^{\otimes n} (P_A^\dagger \otimes P_B^\dagger)$ is entangled.

In $2 \otimes 2$, a density matrix σ is entangled if and only if it is NPT. Lemma 1 requires the examinations of projection of the density matrix (or n copies of the density matrix). The following lemma gives a convenient recasting of these properties of projections in terms of properties of the original density matrix itself.

Lemma 2. Let ρ be a density matrix on $m_A \otimes m_B$. Let $P_A : \mathcal{H}_{m_A} \rightarrow \mathcal{H}_2$ be a projection and also $P_B : \mathcal{H}_{m_B} \rightarrow \mathcal{H}_2$. There exist P_A and P_B such that $P_A \otimes P_B \rho P_A^\dagger \otimes P_B^\dagger$ is entangled if and only if

$$\rho_{2 \otimes m_B} = P_A \otimes \mathbf{1}_B \rho P_A^\dagger \otimes \mathbf{1}_B \quad (3.2)$$

has the property that

$$\rho_{2 \otimes m_B}^{PT} \not\geq 0. \quad (3.3)$$

Equation (3.3) is equivalent to the condition that there exists a state $|\phi\rangle$ that has Schmidt rank 2 and

$$\langle \phi | (\mathbf{1} \otimes T)(\rho) | \phi \rangle < 0. \quad (3.4)$$

Proof. If the density matrix $\rho_{2 \otimes m_B}$ is not positive semidefinite under partial transposition, then there exists a Schmidt rank-2 vector $|\psi\rangle$, written in its Schmidt basis as

$$|\psi\rangle = \sqrt{\lambda_1} |a_0, b_0\rangle + \sqrt{\lambda_2} |a_1, b_1\rangle, \quad (3.5)$$

such that

$$\langle \psi | \rho_{2 \otimes m_B}^{PT} | \psi \rangle < 0. \quad (3.6)$$

[The state $|\psi\rangle$ cannot be a product vector since, if it were, $\langle \psi | \rho_{2 \otimes m_B}^{PT} | \psi \rangle = \text{Tr} |\psi\rangle \langle \psi| \rho_{2 \otimes m_B}^{PT} = \text{Tr} (|\psi\rangle \langle \psi|)^{PT} \rho_{2 \otimes m_B} \geq 0$.]

We note that the projector P_A in Eq. (3.2), consistent with Eq. (3.5), has the form $P_A = |a_0\rangle \langle a_0| + |a_1\rangle \langle a_1|$. Note also that the state $|\psi\rangle$ is invariant under the projector $P_B = P_B^\dagger = |b_0\rangle \langle b_0| + |b_1\rangle \langle b_1|$:

$$(\mathbf{1}_A \otimes P_B^\dagger) |\psi\rangle = |\psi\rangle. \quad (3.7)$$

Plugging Eqs (3.7) and (3.2) into Eq. (3.6),

$$\begin{aligned}
 & \langle \psi | (\mathbf{1}_A \otimes P_B) [(P_A \otimes \mathbf{1}_B) \rho (P_A^\dagger \otimes \mathbf{1}_B)]^{PT} (\mathbf{1} \otimes P_B^\dagger) | \psi \rangle \\
 &= \langle \psi | [(P_A \otimes P_B^*) \rho (P_A^\dagger \otimes P_B^T)]^{PT} | \psi \rangle \\
 &< 0.
 \end{aligned} \tag{3.8}$$

Therefore, the state $(P_A \otimes P_B^*) \rho (P_A^\dagger \otimes P_B^T)$ on $2 \otimes 2$ is entangled.

Conversely, if the density matrix $\rho_{2 \otimes m_B}$ is positive semidefinite under partial transposition for all P_A , meaning that $\rho_{2 \otimes m_B}$ is either separable or has bound entanglement, then there does not exist a P_B such that $(P_A \otimes P_B) \rho (P_A^\dagger \otimes P_B^\dagger)$ is entangled, because then it could be distilled.

Finally, by rewriting Eq. (3.8) as

$$\langle \psi | (P_A \otimes P_B) \rho^{PT} (P_A^\dagger \otimes P_B^\dagger) | \psi \rangle < 0, \tag{3.9}$$

we note that $|\phi\rangle = (P_A^\dagger \otimes P_B^\dagger) |\psi\rangle$ is the state needed for Eq. (3.4). ■

Note that an easy consequence of lemma 2 is that all NPT states in $2 \otimes n$ for any n are distillable.

A. Single copy

The real difficulty in applying lemma 1 is that it requires an examination of an arbitrary number of copies n of the state to be distilled. We will therefore first develop a set of strong results for the special case of $n = 1$, then we will move on to obtain some results for the much more difficult case of arbitrary n . We begin with some terminology.

Definition 2. We say that density matrix ρ is *pseudo one-copy undistillable* if, for all Schmidt rank-2 states $|\phi\rangle$, $\langle \phi | \rho^{PT} | \phi \rangle \geq 0$. Then, by lemma 2, there exists no $2 \otimes 2$ projection of ρ that is inseparable. We say ρ is *pseudo- n -copy undistillable* if and only if $\rho^{\otimes n}$ is pseudo-one-copy undistillable.

We will establish which states ρ_{bc} are pseudo-one-copy undistillable and which are distillable. The partial transpose of ρ_{bc} reads

$$\begin{aligned}
 \rho_{bc}^{PT} &= a \sum_{i=0}^{d-1} |ii\rangle\langle ii| + \frac{c-b}{2} \sum_{i,j=0;i \neq j}^{d-1} |ii\rangle\langle jj| \\
 &+ \frac{c+b}{2} \sum_{i,j=0;i \neq j}^{d-1} |ij\rangle\langle ij|.
 \end{aligned} \tag{3.10}$$

The eigendecomposition of ρ_{bc}^{PT} is

$$\rho_{bc}^{PT} = \lambda_0 |\Phi_0\rangle\langle \Phi_0| + \lambda_1 \sum_{i=1}^{d-1} |\Phi_i\rangle\langle \Phi_i| + \lambda_2 \sum_{i,j=0;i \neq j}^{d-1} |ij\rangle\langle ij|, \tag{3.11}$$

with

$$|\Phi_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i2\pi jk/d} |jj\rangle, \tag{3.12}$$

which we refer to as the “ e -dit eigenstates” in analogy with “ e bit,” because they are the maximally entangled states in

$d \otimes d$ having a “dit” ($\log_2 d$ bits) of entanglement. Correspondingly, we refer to the $|ij\rangle$ states with $i \neq j$ as the “product eigenstates.” The eigenvalues λ_i are given by

$$\lambda_0 = (d-1) \left(\frac{1}{d(d-1)} - b \right) \quad (< 0 \text{ in NPT}_1), \tag{3.13}$$

$$\lambda_1 = \frac{1}{d} - \frac{d}{2}c - \frac{d-2}{2}b \quad (> 0 \text{ in NPT}_1), \tag{3.14}$$

$$\lambda_2 = \frac{1}{2}(c+b) \geq 0. \tag{3.15}$$

The negative eigenvalue λ_0 is independent of c , showing why the PPT-NPT boundary is a vertical line (BK in Fig. 2). Notice that the eigenvectors of ρ_{bc}^{PT} are independent of parameters b and c .

We now specialize to the state for which the positive eigenvalues are all equal, $\lambda_1 = \lambda_2$, and therefore

$$c = \frac{2}{d(d+1)} - \frac{d-1}{d+1}b. \tag{3.16}$$

These are precisely the Werner states ρ_W of Eq. (2.29) above, the states along the line FH in Fig. 2. We take advantage of the fact that lemma 2 does not require normalized states to write the partial transpose of these states in the simple unnormalized form

$$\sigma^{PT}(\lambda) = \lambda I - (\lambda + 1) |\Phi_0\rangle\langle \Phi_0|, \tag{3.17}$$

with $\lambda = \lambda_1 / (-\lambda_0)$. We will show that for $\lambda \geq 2/(d-2)$, $\min_{|\psi^2\rangle} \langle \psi^2 | \sigma^{PT} | \psi^2 \rangle \geq 0$ and that for $\lambda < 2/(d-2)$, $\min_{|\psi^2\rangle} \langle \psi^2 | \sigma^{PT} | \psi^2 \rangle < 0$, with the minimum taken over all Schmidt rank-2 vectors. Thus $\lambda = 2/(d-2)$, corresponding to $b = 3/(d(2d-1))$ and $c = 1/(d(2d-1))$ (the point G in Fig. 2) is the transition point separating distillable Werner states (line segment FG) from those which are pseudo-one-copy undistillable (line segment GH). To establish this we first need to prove the following lemma.

Lemma 3. In $d \otimes d$, the overlap of a Schmidt rank-2 state with a maximally entangled state is at most $\sqrt{2/d}$. In other words, if $|v\rangle$ has Schmidt rank 2 and $|\Psi\rangle$ is a maximally entangled state, then

$$|\langle \Psi | v \rangle| \leq \sqrt{2/d}. \tag{3.18}$$

Proof. In its Schmidt basis, $|\Psi\rangle = (\sum_{i=0}^{d-1} |ii\rangle) / \sqrt{d}$. Since $|v\rangle$ is Schmidt rank 2, it may be written in its Schmidt decomposition as $|v\rangle = \sqrt{\mu_1} |e_1\rangle |e_2\rangle + \sqrt{\mu_2} |e_3\rangle |e_4\rangle$, with $\mu_1 + \mu_2 = 1$. The overlap then is

$$\begin{aligned}
\langle \Psi | v \rangle &= \frac{\sqrt{\mu_1}}{\sqrt{d}} \sum_{i=0}^{d-1} \langle i | e_1 \rangle \langle i | e_2 \rangle + \frac{\sqrt{\mu_2}}{\sqrt{d}} \sum_{i=0}^{d-1} \langle i | e_3 \rangle \langle i | e_4 \rangle \\
&= \frac{\sqrt{\mu_1}}{\sqrt{d}} \sum_{i=0}^{d-1} \langle i | e_1 \rangle \langle e_2^* | i \rangle + \frac{\sqrt{\mu_2}}{\sqrt{d}} \sum_{i=0}^{d-1} \langle i | e_3 \rangle \langle e_4^* | i \rangle \\
&= \frac{\sqrt{\mu_1}}{\sqrt{d}} \langle e_2^* | e_1 \rangle + \frac{\sqrt{\mu_2}}{\sqrt{d}} \langle e_4^* | e_3 \rangle, \tag{3.19}
\end{aligned}$$

where $|e_i^*\rangle$ is the vector obtained by complex conjugation of the components of $|e_i\rangle$ in the Schmidt basis of the state $|\Psi\rangle$. Thus we have

$$|\langle \Psi | v \rangle| \leq \frac{\sqrt{\mu_1} + \sqrt{\mu_2}}{\sqrt{d}}. \tag{3.20}$$

Maximizing with constraint $\mu_1 + \mu_2 = 1$ gives the desired result. \blacksquare

Now we are ready for the main result.

Theorem 2. Given $\sigma(\lambda)$ whose partial transpose is given in Eq. (3.17), we have (i) if $\lambda \geq 2/(d-2)$, then σ is not pseudo-one-copy distillable; and (ii) if $\lambda < 2/(d-2)$, then σ is pseudo-one-copy distillable.

Proof. We start with the first part. Let $|v\rangle$ be any Schmidt rank-2 vector. Then

$$\begin{aligned}
\langle v | \sigma^{PT} | v \rangle &= \lambda - (\lambda + 1) |\langle v | \Phi_0 \rangle|^2 \\
&\geq \lambda - 2(\lambda + 1)/d \\
&\geq \frac{d-2}{d} \left(\lambda - \frac{2}{d-2} \right), \tag{3.21}
\end{aligned}$$

where we have used lemma 3. This is greater than or equal to zero for $\lambda \geq 2/(d-2)$, showing the first part of the result. For the second part, consider $|v\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. We have $\langle v | \sigma | v \rangle = ((d-2)/2)(\lambda - 2/(d-2))$, which is less than zero for $\lambda < 2/(d-2)$, proving the second part of the result. From this it is a simple matter to completely characterize the one-copy undistillability of the ρ_{bc} states.

Proposition 1. The states ρ_{bc} are pseudo-one-copy undistillable in the region of parameter space $BCGK$ in Fig. 2 [14].

Proof. Since any state in the region is a convex linear combination of the states B , C , G , and K , it suffices to show that the partial transpose of each of these four states has a positive expectation value with respect to any Schmidt rank-2 vector (lemma 2). This is obviously true for the PPT states B and K , and it is true for state G by theorem 2. To show it for C , which has parameters $b = 4/(d(3d-2))$, $c = 0$, we note that the partial transpose of the state C can be written

$$\begin{aligned}
\rho^{PT} \left(b = \frac{4}{d(3d-2)}, c = 0 \right) \\
= \frac{2d-1}{3d-2} \rho_G^{PT} + \frac{2}{d(3d-2)} \sum_{i,j=0,i<j}^{d-1} \Pi_{ij}. \tag{3.22}
\end{aligned}$$

Here ρ_G^{PT} is the partial transpose of the normalized state at point G , and Π_{ij} is the normalized projector $\Pi_{ij} = \frac{1}{2}(|ii\rangle - |jj\rangle)(\langle ii| - \langle jj|)$. The expectation value of the first term on the right-hand side of Eq. (3.22) is positive by theorem 2, and that of the second term is positive because it is a projector. All other states are distillable.

Proposition 2. The states ρ_{bc} are distillable in the region of parameter space $CFKG$ in Fig. 2 [14].

Proof. In the region EFK the partial transpose has a negative expectation value with respect to the Schmidt rank-2 state $|00\rangle + |11\rangle$, and in the region CEG with respect to the state

$$\begin{aligned}
\left(\sum_{j=0}^{d-1} |j\rangle \right) \otimes \left(\sum_{k=0}^{d-1} |k\rangle \right) + \left(\sum_{j=0}^{d-1} e^{2\pi i j/d} |j\rangle \right) \\
\otimes \left(\sum_{k=0}^{d-1} e^{2\pi i k/d} |k\rangle \right). \tag{3.23}
\end{aligned}$$

B. Multiple copies

It has proved to be much harder to obtain definitive results concerning the pseudo- n -copy undistillability of the ρ_{bc} states. But we have accumulated various pieces of evidence, which we will present here, all indicating the likelihood that many of the NPT_1 states are undistillable.

Our attention will focus here on a particular subset of the ρ_{bc} states labeled by c and a small parameter ϵ , which sit just to the right of the line segment BK in Fig. 2:

$$\begin{aligned}
\rho(c, \epsilon) &= \left(\frac{1}{2d} - \frac{d-1}{2} (c + \epsilon) \right) \sum_{i=0}^{d-1} |ii\rangle \langle ii| \\
&+ \left(\frac{1}{d(d-1)} + \epsilon \right) \sum_{i < j} |\psi_{ij}^-\rangle \langle \psi_{ij}^-| + c \sum_{i < j} |\psi_{ij}^+\rangle \langle \psi_{ij}^+|. \tag{3.24}
\end{aligned}$$

The eigenvectors of the partial transpose of this state $\rho(c, \epsilon)^{PT}$ are given in Eqs. (3.11) and (3.12), since these are common to all ρ_{bc} states. The eigenvalues are $\lambda_0 = -(d-1)\epsilon$,

$$\lambda_1 = \frac{1}{2(d-1)} - \frac{d-2}{2} \epsilon - \frac{d}{2} c,$$

and

$$\lambda_2 = \frac{1}{d} \left(\frac{1}{2(d-1)} + \frac{d}{2} \epsilon + \frac{d}{2} c \right).$$

The only properties of these eigenvalues that we will use is that for small, positive ϵ and $0 \leq c < 1/(d(d-1))$, λ_0 is negative and goes to zero as $\epsilon \rightarrow 0$, and λ_1 and λ_2 are strictly positive.

Although we will not need any more properties of the density matrices $\rho(c, \epsilon = 0)$, we can at this point note the interesting fact that they are all separable; in fact, *all* the PPT states of the form ρ_{bc} (the region $ABKJ$ in Fig. 2) are separable.

rable [Eq. (1.1)]. This is established by showing that the density matrices at the extremal points A , B , K , and J are separable; all other states in this region are convex combinations of these. The state at A is proportional to $\sum_i |ii\rangle$, and the one at K is proportional to $\sum_{i \neq j} |ij\rangle$, so these are both obviously separable.

We can also create the state $\rho(c=0, \epsilon=0)$ at point B using separable states. It is easiest to construct this ensemble for the partial transpose of this state [see Eq. (3.10)], which is done by equally mixing the states

$$(-|i\rangle + e^{2\pi ik/3}|j\rangle) \otimes (|i\rangle + e^{-2\pi ik/3}|j\rangle) \quad (3.25)$$

for all pairs $i \neq j$, and $k=0, 1$, and 2 . By mixing these states with equal probabilities, all terms of the forms $|ii\rangle\langle ij|$, $|ij\rangle\langle ii|$ and $|ij\rangle\langle ji|$ for $j \neq i$ cancel out; each of these will come with a factor $\sum_{k=0}^2 e^{\pm 4\pi ik/3} = 0$ or $\sum_{k=0}^2 e^{\pm 2\pi ik/3} = 0$. A term such as $|00\rangle\langle 00|$ will occur $d-1$ times as much as a term $|00\rangle\langle 11|$, which is indeed the correct ratio for $\rho(c=0, \epsilon=0)^{PT}$. The state $\rho(c=0, \epsilon=0)$ itself at point B is obtained from mixing the states

$$(-|i\rangle + e^{2\pi ik/3}|j\rangle) \otimes (|i\rangle + e^{2\pi ik/3}|j\rangle) \quad (3.26)$$

with equal probabilities.

The partial transpose of the state at point J has a simple form [$\lambda_1=0$ in Eq. (3.11)]; it is straightforward to show that ρ^{PT} at J is realized by an equal mixture of the separable states

$$\left(\sum_{j=0}^{d-1} e^{2\pi i k_j/3} |j\rangle \right) \otimes \left(\sum_{j=0}^{d-1} e^{-2\pi i k_j/3} |j\rangle \right), \quad (3.27)$$

where each integer k_0, k_1, \dots, k_{d-1} runs independently over $0, 1$, and 2 . This is clearly not a separable decomposition with the minimal possible number of states.

A few notes about the decomposition for point B : for $d=3$ the state ρ^{PT} at point B has rank 8. This implies that the optimal decomposition of ρ^{PT} , and therefore of ρ itself, needs at least eight states in its decomposition; this despite the fact that the rank of ρ is only 6 (see lemma 1 of Ref. [15]). Thus we have a new example of a state for which the number of states in its minimal decomposition exceeds its rank; however see Ref. [16]. For general d , the number of states in our separable ensemble at B , 3_2^d , which is more than the dimension d^2 for $d > 3$. There are no known prior explicit examples in which the number of members of the optimal ensemble is greater than the dimension; it would be interesting to prove that Eq. (3.26) constitutes a minimal optimal ensemble. The separability of the PPT states permits us to give an extension of proposition 1 indicating that the undistillability of states in this region is linked.

Lemma 4. If the state ρ_{bc} at point G is pseudo- n -copy undistillable, then all states in the region BGK are pseudo- n -copy undistillable.

Proof. First, note that if the state at point G is pseudo- n -copy undistillable, then it is also pseudo- k -copy undistillable for $1 \leq k \leq n$. Since the two extremal points B and K of the convex set of states BGK are separable, the partial trans-

pose of all states in this region can be written as a convex combination [using notation from Eq. (3.22)]

$$\rho^{PT} = a_0 \rho_G^{PT} + \sum_{\alpha} a_{\alpha} \Pi_{\alpha}, \quad (3.28)$$

where Π_{α} are product projectors and $a_{\alpha} \geq 0$. Applying lemma 2, we consider the expectation value of n copies of this state with respect to any Schmidt rank-2 vector $|v\rangle$ [14]:

$$\langle v | \left(a_0 \rho_G^{PT} + \sum_{\alpha} a_{\alpha} \Pi_{\alpha} \right)^{\otimes n} | v \rangle. \quad (3.29)$$

We need to show that this is non-negative; we show this by demonstrating that each term in the tensor product, when expanded out, is not negative. Consider a term containing k ρ_G^{PT} factors and $n-k$ factors involving the projectors Π_{α} . We can apply the $n-k$ projectors to $|v\rangle$; since they are all product projectors, the projected vector $|v'\rangle$ still has Schmidt rank 2 (or 1). So the matrix element of Eq. (3.29) is proportional to

$$\langle v' | (\rho_G^{PT})^{\otimes k} | v' \rangle. \quad (3.30)$$

But if G is pseudo- k -copy undistillable, this matrix element is non-negative. ■

Note that this analysis does not apply to state C , because the projectors Π_{ij} of Eq. (3.22) are not product projectors; therefore, they can increase the Schmidt rank of $|v\rangle$.

For $d=3$ we have performed extensive numerical studies to search for states distillable by projection on two copies in the region BGK . We find none, reinforcing the indication of lemma 4 that an entire region inside the NPT_1 set will prove to be undistillable. Section IV will provide further evidence for this idea.

IV. UNDISTILLABILITY FOR MULTIPLE COPIES

In this section we will obtain our strongest result, which suggests that some of the NPT states ρ_{bc} are not distillable. We will be able to conclude that for any finite n there exists an ϵ such that $\rho(c, \epsilon)^{\otimes n}$ [Eq. (3.24)] is not entangled on any $2 \otimes 2$ subspace, and is therefore one-copy undistillable. This result can have only one of two further implications: (1) For some c , this ϵ asymptotes to some finite value $\bar{\epsilon}(c)$ as $n \rightarrow \infty$. In this case, the NPT states $\rho(c, \epsilon < \bar{\epsilon}(c))$ are absolutely undistillable. (2) For all c , this ϵ goes to zero as $n \rightarrow \infty$. In this case all states immediately to the right of line BK are distillable; thus all ρ_{bc} states with NPT would be distillable, since all such states can be first mixed with some separable ρ_{bc} state (a LQ+CC operation) to bring it to the BK line. However, one might say that the states near BK are ‘‘barely’’ distillable: an arbitrarily large number of copies of the state are required before there is any sign of undistillability of the state. It would be fair to say that these states would still be undistillable in any practical sense.

First, we establish the significance of the null-space properties of $\rho(c, \epsilon=0)$ for the argument. We consider the function

$$f(c, \epsilon, n) = \min_{|\psi^2\rangle} \langle \psi^2 | [\rho^{PT}(c, \epsilon)]^{\otimes n} | \psi^2 \rangle. \quad (4.1)$$

Here the minimum is taken over all Schmidt rank-2 states $|\psi^2\rangle$ in the full $d^n \otimes d^n$ Hilbert space. By lemmas 1 and 2, we know that the sign of $f(c, \epsilon, n)$ determines whether $\rho(c, \epsilon)$ is pseudo- n -copy undistillable. For $\epsilon=0$ the state is separable and therefore $f(c, \epsilon=0, n) \geq 0$ for all n . The question is, does the state become pseudo- n -copy undistillable as $\epsilon \rightarrow 0$? The answer is provided by the result whose proof we outline in a moment, that there is no Schmidt rank-2 vector in the null space of $\rho^{PT}(c, \epsilon=0)^{\otimes n}$ for any n . In other words, for all n and c ,

$$f(c, \epsilon=0, n) > 0. \quad (4.2)$$

Since f is a continuous function of ϵ , there must therefore exist an $\epsilon_0(c, n) > 0$ such that

$$f(c, 0 \leq \epsilon \leq \epsilon_0(c, n), n) \geq 0. \quad (4.3)$$

Thus there is a finite range of $\epsilon > 0$ for every n [and for all c such that $0 \leq c < 1/(d(d-1))$] such that $\rho(c, \epsilon)$ is pseudo- n -copy undistillable.

The only knowledge lacking at this point for a complete demonstration of the undistillability of $\rho(c, \epsilon)$ is the asymptotic behavior of $\epsilon_0(c, n)$ as $n \rightarrow \infty$. If $\epsilon_0(n) \rightarrow 0$ as $n \rightarrow \infty$, then NPT undistillability would not be established; we would merely have shown that distillation becomes difficult as $\epsilon \rightarrow 0$, requiring more and more copies of the state in the distillation protocol. If $\epsilon_0(c, n)$ remains larger than some positive $\bar{\epsilon}(c)$ for all n and for some c , then we would know that all states $\rho(c, 0 \leq \epsilon \leq \bar{\epsilon}(c))$ are absolutely undistillable. Since the signs from our few-copy work are that indeed this threshold remains positive we are led to the conjecture:

Conjecture. States $\rho(c, \epsilon)$ of Eq. (3.24), for sufficiently small positive ϵ , are undistillable.

We can further speculate that the undistillable region will correspond exactly to region $BCGK$, in which the state is pseudo-one-copy and, apparently, pseudo-two-copy undistillable. It may well be that pseudo one-copy undistillability and absolute undistillability are equivalent. Now we present our result about the null-space properties of $\rho(c, \epsilon=0)$ on which the above discussion is based: its null space does not contain any non-zero vectors of Schmidt rank less than 3.

First we set up some notation. Plain roman indices take values from 0 to $d-1$ unless otherwise stated. Let indices with superscript p represent composite indices, e.g., i_1^p represents $(l_1, m_1 | l_1 \neq m_1)$. Let indices with superscript e represent plain indices, e.g., $i_1^e = i_1$. Label the eigenvectors [Eq. (3.11)] as $|i_k^e\rangle = |\Phi_{i_k}\rangle$ and $|i_k^p\rangle = |l_k m_k\rangle$ with $l_k \neq m_k$. The label e stands for “ e -dit eigenstate” and the label p stands for “product eigenstate.” Let us denote n -tuples of indices such as (i_1, i_2, \dots, i_n) by letters in bold font such as \mathbf{i} ; in sums over \mathbf{i} , each i_k runs independently between 0 and $d-1$. Next we prove an important lemma.

Lemma 5. The null space of the partial transpose of density matrix $\rho(c, \epsilon=0)^{\otimes n}$, for all c , $d \geq 3$ and $n \geq 1$, does not contain any nonzero vectors with Schmidt rank less than 3 of the form

$$|\psi^{ee \dots e}\rangle = \sum_{\mathbf{i}=0}^{d-1} a_{\mathbf{i}} |\Phi_{i_1}\rangle \otimes |\Phi_{i_2}\rangle \otimes \dots \otimes |\Phi_{i_n}\rangle. \quad (4.4)$$

Proof. For $n=1$ the result is obvious since Φ_0 is the only vector in the null space and it has Schmidt rank $d \geq 3$. For $n \geq 2$, we first note that the partial trace of the state in Eq. (4.4) is

$$\rho_{\psi} = \text{Tr}_B |\psi^{ee \dots e}\rangle \langle \psi^{ee \dots e}| = \sum_{\mathbf{i}=0}^{d-1} |\tilde{a}_{\mathbf{i}}|^2 |\mathbf{i}\rangle \langle \mathbf{i}|, \quad (4.5)$$

where the coefficients

$$\tilde{a}_{\mathbf{i}} = \frac{1}{d^{(n/2)}} \sum_{\mathbf{k}=0}^{d-1} a_{\mathbf{k}} e^{i2\pi(\mathbf{i} \cdot \mathbf{k})/d} \quad (4.6)$$

are the n -dimensional discrete Fourier transforms of the a 's. Here $\mathbf{i} \cdot \mathbf{k} = i_1 k_1 + i_2 k_2 + \dots + i_n k_n$. Note that ρ_{ψ} is already diagonalized. Since the Schmidt rank of a pure state equals the rank of the partial trace, we require that the rank of ρ_{ψ} be less than 3. Thus at most two coefficients $|\tilde{a}_{\mathbf{i}}|^2$ are nonzero, i.e.

$$\tilde{a}_{\mathbf{i}} = |\alpha| e^{i\phi_{\alpha}} \delta_{\mathbf{i}, \mathbf{x}} + |\beta| e^{i\phi_{\beta}} \delta_{\mathbf{i}, \mathbf{y}}, \quad (4.7)$$

where $\delta_{\mathbf{i}, \mathbf{x}} = \delta_{i_1, x_1} \delta_{i_2, x_2} \dots \delta_{i_n, x_n}$.

Solving for the a 's by doing an inverse Fourier transform, we have

$$a_{\mathbf{i}} = \frac{1}{d^{(n/2)}} (|\alpha| e^{i\phi_{\alpha}} e^{-i2\pi \mathbf{i} \cdot \mathbf{x}/d} + |\beta| e^{i\phi_{\beta}} e^{-i2\pi \mathbf{i} \cdot \mathbf{y}/d}). \quad (4.8)$$

First suppose the Schmidt rank of the vector is exactly 2, in which case both $|\alpha|$ and $|\beta|$ must be nonzero and $\mathbf{x} \neq \mathbf{y}$. Now we start putting constraints on the a 's such that the vector $|\psi^{ee \dots e}\rangle$ is in the null space of $\rho^{PT}(c, \epsilon=0)$. The vector $\psi^{ee \dots e}$ belongs to the null space only if $a_{11 \dots 1} = 0$, because the corresponding eigenvalue λ_1^n is positive at $\epsilon=0$. Now we impose the null space constraint $a_{21 \dots 1} = 0$ (since the corresponding eigenvalue $\lambda_2 \lambda_1^{n-1}$ is positive at $\epsilon=0$), and we have $x_1 = y_1$. Similarly, other a 's, whose subscripts are obtained by permuting $\{21 \dots 1\}$, may be constrained to zero giving $\mathbf{x} = \mathbf{y}$. However this implies that the vector is of Schmidt rank 1 if it is to satisfy these null space constraints. Thus no Schmidt rank-2 vector of the form $\psi^{ee \dots e}$ belongs to the null space of $\rho^{PT}(c, \epsilon=0)$.

Next we consider the case of Schmidt rank-1 vectors, where without loss of generality we may assume $|\beta|=0$. Then, the null space constraint $a_{11 \dots 1} = 0$ implies that $|\alpha|=0$, thus proving the result. ■

Now we are ready for the main result:

Theorem 3. The null space of $(\rho^{PT}(c, \epsilon=0))^{\otimes n}$ for $d \geq 3$ and $n \geq 1$ does not contain any vector of Schmidt rank less than 3.

Proof. For $n=1$ the result is obvious, because the null space consists of the span of the vector $|\Phi_0\rangle$ which has Schmidt rank $d \geq 3$. For purpose of illustrating the proof technique, we next prove the result for two copies, i.e., $n=2$. Then we will show how the proof generalizes to n copies.

Recalling Eq. (3.11) and the fact that the eigenvectors form a basis for the one-copy Hilbert space of $d \otimes d$, a general vector $|\psi\rangle$ in the Hilbert space of two copies can be written as

$$|\psi\rangle = |\psi^{ee}\rangle + |\psi^{ep}\rangle + |\psi^{pe}\rangle + |\psi^{pp}\rangle, \quad (4.9)$$

with $|\psi^{ee}\rangle = \sum_{i^e, j^e} \alpha_{i^e, j^e}^{ee} |i^e\rangle \otimes |j^e\rangle$, $|\psi^{ep}\rangle = \sum_{i^e, j^p} \alpha_{i^e, j^p}^{ep} |i^e\rangle \otimes |j^p\rangle$, $|\psi^{pe}\rangle = \sum_{i^p, j^e} \alpha_{i^p, j^e}^{pe} |i^p\rangle \otimes |j^e\rangle$, and $|\psi^{pp}\rangle = \sum_{i^p, j^p} \alpha_{i^p, j^p}^{pp} |i^p\rangle \otimes |j^p\rangle$. Here the α 's are complex coefficients for the vectors. The ψ^{pp} term must be zero if the vector is to belong to the null space, because the corresponding eigenvalue λ_2^2 is positive at $\epsilon=0$. Now assuming ψ has Schmidt rank less than 3, we will show that the coefficients α^{ep} 's and α^{pe} 's are zero. To show this we repeatedly use the fact that local projections cannot increase the Schmidt rank of a vector. Alice and Bob can project locally on the vector $|k^p\rangle$, for any k^p of the first copy, which results in a vector proportional to $|k^p\rangle \otimes \sum_{j^e} \alpha_{k^p, j^e}^{pe} |j^e\rangle$. By lemma 5 this vector has a Schmidt rank greater than 2 unless it is zero. Thus all the α^{pe} 's are zero. Similarly applying this argument to the ψ^{ep} term, with the projection now done on a product vector $|k^p\rangle$ of the second copy, we see that the α^{ep} 's are zero. The only term left now is the ψ^{ee} term, for which lemma 5 applies and gives us the result.

We write the general proof for n copies along the lines of the two-copy proof, albeit with considerable notational complications. Generalizing the notation of Eq. (4.9), we define \mathcal{P}_k to be the set of all distinct permutations of k p 's and $(n-k)$ e 's. We also denote the strings representing permutations in \mathcal{P}_k by bold font, e.g., $\mathbf{s} = s_1 s_2 \dots s_k$, where the s_j are the characters in the permutation string, e.g., for $\mathbf{s} = pep \in \mathcal{P}_2$, then $s_1 = p$, $s_2 = e$, and $s_3 = p$.

A general state in the n -copy Hilbert space can be written in the form

$$|\psi\rangle = \sum_{k=0}^n \sum_{\mathbf{s} \in \mathcal{P}_k} |\psi^{\mathbf{s}}\rangle, \quad (4.10)$$

with

$$\begin{aligned} |\psi^{\mathbf{s}}\rangle &= |\psi^{s_1 s_2 \dots s_n}\rangle \\ &= \sum_{i_1^{s_1}, i_2^{s_2}, \dots, i_n^{s_n}} \alpha_{i_1^{s_1}, i_2^{s_2}, \dots, i_n^{s_n}} |i_1^{s_1}\rangle \otimes |i_2^{s_2}\rangle \otimes \dots \otimes |i_n^{s_n}\rangle. \end{aligned} \quad (4.11)$$

Again, the $\psi^{pp \dots p}$ term is zero if the vector is to be in the null space, because the corresponding eigenvalue λ_2^n is positive at $\epsilon=0$. Define ψ_m by

$$|\psi_m\rangle = \sum_{i=0}^m \sum_{\mathbf{s} \in \mathcal{P}_i} |\psi^{\mathbf{s}}\rangle. \quad (4.12)$$

Then to prove the result we show that there is no vector with Schmidt rank less than 3 of the form $|\psi_i\rangle$ for all $m \leq n-1$. This we show by induction on m . For $m=0$, the result immediately follows from lemma 5. For the induction step, we write

$$|\psi_m\rangle = |\psi_{m-1}\rangle + \sum_{\mathbf{s} \in \mathcal{P}_m} |\psi^{\mathbf{s}}\rangle. \quad (4.13)$$

Now if Alice and Bob locally project $|\psi_m\rangle$ onto $|r_i^p\rangle$ of the i th copy, for $i=1 \dots m$, the result is

$$\begin{aligned} &|r_1^p\rangle \otimes |r_2^p\rangle \otimes \dots \otimes |r_m^p\rangle \\ &\otimes \sum_{k_{m+1}^e \dots k_n^e} \alpha_{r_1^p r_2^p \dots r_m^p k_{m+1}^e \dots k_n^e} |k_{m+1}^e\rangle \otimes \dots \otimes |k_n^e\rangle. \end{aligned} \quad (4.14)$$

Since local projection cannot increase the Schmidt rank, by lemma 5 the vector inside the sum above must be zero. Doing this for all the different values of the r_i^p 's we see that $\psi^{pp \dots p e e \dots e} = 0$, where the superscript contains m p 's and $(n-m)$ e 's. Similarly we can prove that $|\psi^{\mathbf{s}}\rangle$ is zero for any permutation string $\mathbf{s} \in \mathcal{P}_m$. This shows that $|\psi_m\rangle$ has to be of the form $|\psi_{m-1}\rangle$, for which the result is true by the induction hypothesis. ■

V. DISTILLABILITY AND 2-POSITIVE LINEAR MAPS

In this section we find a formulation of the problem of distillability of an arbitrary bipartite density matrix ρ . This formulation uses the notion of 2-positive linear maps. We will explicitly show how the problem of distillability of the density matrices ρ_{bc} that were discussed in the preceding sections can be cast in the language of positive linear maps.

Let us first recall the definition of a k -positive linear map [17]. Let $B(\mathcal{H}_n)$ denote the matrix algebra of operators on an n -dimensional Hilbert space, and let $B(\mathcal{H}_n)^+$ denote the set of positive semidefinite matrices. A linear map $\Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ is called positive when $\Lambda : B(\mathcal{H}_n)^+ \rightarrow B(\mathcal{H}_m)^+$, that is, the map preserves the set of positive semidefinite matrices. A linear map $\Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ is called k -positive when the map $\mathbf{1}_k \otimes \Lambda : B(\mathcal{H}_k \otimes \mathcal{H}_n) \rightarrow B(\mathcal{H}_k \otimes \mathcal{H}_m)$ is positive. Note that 1-positivity is equivalent to positivity. It is not hard to show that when a map $\Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ is n -positive, it is completely positive.

We will now give an alternative characterization of k -positivity. The next lemma says that to test a linear map for k -positivity we only need to apply it to pure states of at most Schmidt rank k .

Lemma 6. A positive linear map $\Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ is k -positive if and only if

$$(\mathbf{1}_n \otimes \Lambda)(|\psi\rangle\langle\psi|) \geq 0, \quad (5.1)$$

for all vectors $|\psi\rangle \in \mathcal{H}_n \otimes \mathcal{H}_n$ which have a Schmidt rank of at most k .

Proof. If Eq. (5.1) holds for all states $|\psi\rangle$ of a Schmidt rank of at most k , then it follows that $(\mathbf{1}_k \otimes \Lambda)(|\psi\rangle\langle\psi|) \geq 0$ for all vectors $|\psi\rangle \in \mathcal{H}_k \otimes \mathcal{H}_n$. Therefore $(\mathbf{1}_k \otimes \Lambda)(\rho) \geq 0$ for all $\rho \in B(\mathcal{H}_k \otimes \mathcal{H}_n)^+$, and thus Λ is k -positive. On the other hand, if there exists a vector $|\psi\rangle$ of at most Schmidt rank k , for which $(\mathbf{1}_n \otimes \Lambda)(|\psi\rangle\langle\psi|) \not\geq 0$, then Λ cannot be k -positive. ■

We would like to make an additional simplification in characterizing 2-positive maps. The next lemma says that in order to test a linear map for 2-positivity we only need to apply it to maximally entangled pure states of Schmidt rank 2.

Lemma 7. A linear positive map $\Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ is 2-positive if and only if, for all $|\Psi^\beta\rangle = |0, \beta_0\rangle + |1, \beta_1\rangle$ with $\langle\beta_0|\beta_1\rangle = 0$, $\langle\beta_0|\beta_0\rangle = \langle\beta_1|\beta_1\rangle = 1$,

$$(\mathbf{1}_2 \otimes \Lambda)(|\Psi^\beta\rangle\langle\Psi^\beta|) \geq 0. \quad (5.2)$$

The proof of this lemma is given in the Appendix. It is possible to formulate a similar lemma for k -positive maps, in which k -positivity or the lack thereof can be deduced from applying the map on all maximally entangled vectors of Schmidt rank k .

With a Hermitian operator $H \in B(\mathcal{H}_d \otimes \mathcal{H}_d)$ we can always associate a hermiticity-preserving linear map Λ in the following way:

$$H = (\mathbf{1}_d \otimes \Lambda)(|\Phi^+\rangle\langle\Phi^+|), \quad (5.3)$$

where

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle. \quad (5.4)$$

In the Appendix of Ref. [6] it was proved that the operator H is positive semidefinite if and only if the linear map Λ is completely positive. From this we conclude that any bipartite density matrix ρ on $d \otimes d$ can always be written as

$$\rho = (\mathbf{1}_d \otimes \mathcal{S})(|i\rangle\langle ii|), \quad (5.5)$$

where $\mathcal{S} : B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d)$ is a completely positive map. Note that \mathcal{S} need not be trace preserving.

As an example, we derive the completely positive map \mathcal{S}_{bc} associated with the density matrices ρ_{bc} given in Eq. (2.1). We can specify \mathcal{S}_{bc} on the input states:

$$\begin{aligned} \mathcal{S}_{bc}(|i\rangle\langle i|) &= a|i\rangle\langle i| + \frac{b+c}{2} \sum_{j \neq i} |j\rangle\langle j|, \\ \mathcal{S}_{bc}(|i\rangle\langle j|) &= \frac{c-b}{2} |j\rangle\langle i|, \quad i \neq j. \end{aligned} \quad (5.6)$$

The following main theorem expresses the connection between 2-positivity and distillability of a density matrix ρ .

Theorem 4. Let ρ be a bipartite density matrix on $d \otimes d$. Let $\mathcal{S} : B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d)$ be a completely positive map which is uniquely determined by

$$\rho = (\mathbf{1}_d \otimes \mathcal{S})(|\Phi^+\rangle\langle\Phi^+|). \quad (5.7)$$

Let $\Lambda : B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d)$ be a linear positive map defined as

$$\Lambda = T \circ \mathcal{S}, \quad (5.8)$$

where T is matrix transposition in the basis $\{|i\rangle\}_{i=0}^{d-1}$. There exists no projections $P_A : \mathcal{H}_d^A \rightarrow \mathcal{H}_2$ and $P_B : \mathcal{H}_d^B \rightarrow \mathcal{H}_2$ such that $(P_A \otimes P_B) \rho (P_A^\dagger \otimes P_B^\dagger)$ is entangled if and only if the map Λ is 2-positive. Let

$$\Lambda^{\otimes n} = \underbrace{\Lambda \otimes \dots \otimes \Lambda}_n. \quad (5.9)$$

The density matrix ρ is not distillable if and only if for all $n = 1, 2, \dots$ the map $\Lambda^{\otimes n}$ is 2-positive.

Proof. We will prove the theorem in two parts. First we will prove the relation between 2-positivity of Λ and the nonexistence of a $2 \otimes 2$ subspace on which ρ is entangled. Then we prove the result relating undistillability to 2-positivity of $\Lambda^{\otimes n}$.

Let us assume that there does not exist a $2 \otimes 2$ subspace on which the density matrix ρ is entangled. We can write any projector $P_A : \mathcal{H}_d \rightarrow \mathcal{H}_2$ as

$$P_A = |0\rangle\langle\alpha_0| + |1\rangle\langle\alpha_1|, \quad (5.10)$$

where $\langle\alpha_0|\alpha_1\rangle = 0$. Lemma 2 implies that

$$(\mathbf{1}_2 \otimes T)[(P_A \otimes \mathbf{1}_d) \rho (P_A^\dagger \otimes \mathbf{1}_d)] \geq 0 \quad (5.11)$$

for all projectors P_A . This expression, using Eqs. (5.7) and (5.8), is equal to

$$(\mathbf{1}_2 \otimes \Lambda)(|\Psi^{\alpha^*}\rangle\langle\Psi^{\alpha^*}|) \geq 0, \quad (5.12)$$

with $|\Psi^{\alpha^*}\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_d$ defined as

$$|\Psi^{\alpha^*}\rangle = \frac{1}{\sqrt{2}}(|0, \alpha_0^*\rangle + |1, \alpha_1^*\rangle). \quad (5.13)$$

The vectors $|\alpha_{0,1}^*\rangle$ are defined as $|\alpha_{0,1}^*\rangle = \sum_{i=0}^{d-1} \langle\alpha_{0,1}|i\rangle |i\rangle$. Note that $\langle\alpha_0^*|\alpha_1^*\rangle = 0$. We now invoke the property of a 2-positive map as given in lemma 7; if Eq. (5.12) holds for all $|\alpha_0^*\rangle, |\alpha_1^*\rangle \in \mathcal{H}_n$ with $\langle\alpha_0^*|\alpha_1^*\rangle = 0$, then Λ is a 2-positive map. Conversely, invoking lemma 7, if Λ is a 2-positive linear map, then Eq. (5.12) holds for all states $|\Psi^{\alpha^*}\rangle$. This implies that Eq. (5.11) holds for all projectors P_A , and thus there does not exist a $2 \otimes 2$ subspace on which ρ is entangled.

Now we turn to the second part of the proof. The necessary and sufficient condition for distillability of a density matrix was given in lemma 1. Let $\rho^{\otimes n} = \rho \otimes \rho \otimes \dots \otimes \rho$ on $d^n \otimes d^n$. The density matrix ρ is undistillable if and only if

there exists no projections $P_A : \mathcal{H}_{d^n}^A \rightarrow \mathcal{H}_2$ and $P_B : \mathcal{H}_{d^n}^B \rightarrow \mathcal{H}_2$, such that $(P_A \otimes P_B) \rho^{\otimes n} (P_A^\dagger \otimes P_B^\dagger)$ is entangled. Thus if a density matrix is undistillable, we have, similar to Eq. (5.11),

$$(\mathbf{1}_2 \otimes T)[(P_A \otimes \mathbf{1}_{d^n}) \rho^{\otimes n} (P_A^\dagger \otimes \mathbf{1}_{d^n})] \geq 0, \quad (5.14)$$

for all projectors $P_A : \mathcal{H}_{d^n}^A \rightarrow \mathcal{H}_2$ and all $n = 1, 2, \dots$. We use the fact that $T : \mathcal{B}(\mathcal{H}_{d^n}^n) \rightarrow \mathcal{B}(\mathcal{H}_{d^n}^n)$ is equivalent (up to a unitary transformation) to $T_d^{\otimes n}$, where T_d is a matrix transposition in \mathcal{H}_d . Then Eq. (5.14) can be rewritten as

$$(\mathbf{1}_2 \otimes \Lambda^{\otimes n})(|\Psi\rangle\langle\Psi|) \geq 0 \quad (5.15)$$

for all maximally entangled states $|\Psi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_{d^n}$ for all $n = 1, 2, \dots$. This implies, with lemma 7, that $\Lambda^{\otimes n}$ is 2-positive for all $n = 1, 2, \dots$. Conversely, when $\Lambda^{\otimes n}$ is not 2-positive for some n , there will exist a $2 \otimes 2$ subspace on which $\rho^{\otimes n}$ is entangled. ■

Remarks. Note that the theorem also holds for entangled density matrices ρ that have the PPT property or density matrices which are separable. In this case, however, the positive map Λ is completely positive, and therefore the map $\Lambda^{\otimes n}$ for all $n = 1, 2, \dots$ is 2-positive trivially.

We note that theorem 4 can also be made to apply to a situation in which one is given a large number of copies of, say, two different density matrices ρ_1 and ρ_2 . With each of these density matrices we associate a positive linear map Λ_1 and Λ_2 . Distillability of ρ_1 and ρ_2 together can be formulated as the problem of determining whether $\Lambda_1^{\otimes n_1} \otimes \Lambda_2^{\otimes n_2}$ is 2-positive. This provides a method for searching for nonadditivity in the property of distillability [12]. We could encounter a situation in which both ρ_1 and ρ_2 are undistillable, but ρ_1 and ρ_2 taken together are distillable.

In general, given two 2-positive maps Λ_1 and Λ_2 , the tensor product $\Lambda_1 \otimes \Lambda_2$ is not necessarily 2-positive. As an example we take Λ_1 to be the identity map $\mathbf{1}_d$ and Λ_2 a 2-positive map which is not $2d$ -positive. Then by definition, $\mathbf{1}_2 \otimes \mathbf{1}_d \otimes \Lambda_2$ is not positive. In the cases that we consider here however, the maps are of a special form, namely $\Lambda = T \circ \mathcal{S}$, where \mathcal{S} is completely positive. For this special form, it is possible that the composed maps are always 2-positive.

The positive map Λ_{bc} of Eq. (5.9) corresponding to the example Eq. (5.6) is

$$\begin{aligned} \Lambda_{bc}(|i\rangle\langle i|) &= a|i\rangle\langle i| + \frac{b+c}{2} \sum_{j \neq i} |j\rangle\langle j|, \\ \Lambda_{bc}(|i\rangle\langle j|) &= \frac{c-b}{2} |i\rangle\langle j|, \quad i \neq j. \end{aligned} \quad (5.16)$$

For the states on the line FH this corresponds to the positive map τ_W which acts as

$$\tau_W(X) = d\lambda \mathbf{1} \text{Tr} X - (\lambda + 1)X, \quad (5.17)$$

where λ is the parameter in Eq. (3.17).

It has been shown [18] that this map τ_W is 2-positive in the region $\lambda \geq 2/(d-2)$. This thus establishes an alternative proof of theorem 2 in Sec. III A.

VI. CONCLUSION

Our alternative formulation of the problem of distillability in terms of the 2-positivity property of linear maps has not yet led to a solution of the problem of NPT density matrices which are (likely to be) undistillable (see the conjecture at the end of Sec. IV). We present the formulation here, as it points to a new connection between the structure of positive linear maps and the classification of bipartite mixed state entanglement. We expect that fruitful results will flow from understanding in more detail the classification schemes for these NPT states that are based directly on their 2-positivity properties.

In conclusion, we have shown that most of the distillability properties of NPT mixed states can be restricted to the study of the canonical set ρ_{bc} . Many of the questions about one-copy and few-copy distillability of these states are completely answered by our analysis. A final, general proof of the full undistillability of these states eludes us, but we have shown that if they are distillable, it involves a much more difficult protocol than any which has been needed up until now.

Note added. Recently, we became aware of closely related work by Dür *et al.* [19]. This paper studies the states along the line HGF in Fig. 2; for these states it provides an alternative proof to the one discussed here in Sec. IV that, approaching point H , the states are pseudo- n -copy undistillable for any n . Reference [19] also obtained the same theorem as here (our theorem 2) about the pseudo-one-copy distillability of these states, as well as obtaining additional numerical results indicating that the region of pseudo-two- and -three-copy undistillability is the same as that for one-copy undistillability. All the results of Ref. [19] and the present work are consistent.

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APPENDIX: PROOF OF LEMMA 7

The proof is similar in structure to the proof of the lemma in the Appendix of Ref. [6]. By definition a linear positive map $\Lambda : \mathcal{B}(\mathcal{H}_n) \rightarrow \mathcal{B}(\mathcal{H}_m)$ is 2-positive if and only if, for all $|\psi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_n$,

$$(\mathbf{1}_2 \otimes \Lambda)(|\psi\rangle\langle\psi|) \geq 0. \quad (\text{A1})$$

We will show that we only need to consider states $|\psi\rangle$ that are maximally entangled. Note that any (unnormalized) maximally entangled state can be written as $|\Psi^\beta\rangle = |0, \beta_0\rangle + |1, \beta_1\rangle$ with $\langle\beta_0|\beta_1\rangle = 0$, $\langle\beta_0|\beta_0\rangle = \langle\beta_1|\beta_1\rangle = 1$. We start with the following observation: When we apply the map Λ on some maximally entangled state in $\mathcal{H}_2 \otimes \mathcal{H}_n$,

$$D = (\mathbf{1}_2 \otimes \Lambda)(|\Psi^\beta\rangle\langle\Psi^\beta|), \quad (\text{A2})$$

the matrix D uniquely determines the action of the map Λ on any input matrix that has support on the two-dimensional space spanned by the vectors $|\beta_0\rangle$ and $|\beta_1\rangle$.

For the first part of the lemma, let $D \geq 0$ in Eq. (A2). Since D is Hermitian, we can write it in its eigendecomposition,

$$D = \sum_i \mu_i |\phi_i\rangle\langle\phi_i|, \quad (\text{A3})$$

with the eigenvalues $\mu_i \geq 0$ and the eigenvectors $|\phi_i\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_n$. Each eigenstate $|\phi_i\rangle$ can be written in a Schmidt decomposition as $|\phi_i\rangle = \sqrt{\lambda_{0,i}} |\alpha_{0,i}, \beta_{0,i}\rangle + \sqrt{\lambda_{1,i}} |\alpha_{1,i}, \beta_{1,i}\rangle$, with $\langle\beta_{0,i}|\beta_{1,i}\rangle = \langle\alpha_{0,i}|\alpha_{1,i}\rangle = 0$, and all vectors normalized. Note that the states $|\beta_{0,i}\rangle$ and $|\beta_{1,i}\rangle$ can span a different two-dimensional subspace of \mathcal{H}_n for each i . There exists a local filter W_i^β [6] from which we can obtain the state $|\phi_i\rangle$ from the maximally entangled state $|\Psi^\beta\rangle$:

$$|\phi_i\rangle\langle\phi_i| = (\mathbf{1}_2 \otimes W_i^\beta)(|\Psi^\beta\rangle\langle\Psi^\beta|)(\mathbf{1}_2 \otimes W_i^{\beta\dagger}). \quad (\text{A4})$$

W_i^β includes (1) a unitary transformation from the basis $\beta'_{(0,1),i}$ to $\beta_{(0,1),i}$, where β' are the Schmidt vectors of Ψ^β when it is written in the form $|\Psi^\beta\rangle = |\alpha_{0,i}, \beta'_{0,i}\rangle + |\alpha_{1,i}, \beta'_{1,i}\rangle$ (taking advantage of the degeneracy of the Schmidt decomposition of the maximally entangled state); and (2) a diagonal filter which reduces the Schmidt coefficients to $\lambda_{(0,1),i}$.

Thus we may write D as

$$D = \sum_i \mu_i (\mathbf{1}_2 \otimes W_i^\beta) |\Psi^\beta\rangle\langle\Psi^\beta| (\mathbf{1}_2 \otimes W_i^{\beta\dagger}). \quad (\text{A5})$$

We see that since $D \geq 0$ by assumption, we are able to write the action of the map Λ on the input $|\Psi^\beta\rangle$ in a ‘‘completely positive form’’ with operation elements $\sqrt{\mu_i} W_i^\beta$ that depend on β . We observed above that this input determines the action of the map uniquely on the subspace spanned by the vectors $|\beta_0\rangle$ and $|\beta_1\rangle$. Therefore, the map acts as a completely positive map on any input that has support on a two-dimensional space. This implies that Eq. (A1) holds for any state $|\psi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_n$. Conversely, if Λ is 2-positive then Eq. (5.2) holds for any maximally entangled state $|\Psi^\beta\rangle$. ■

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