Controlled source of entangled photonic qubits

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(Received 14 December 1998; published 17 May 2000)

We consider a general proposal for generating a train of entangled single-photon wavepackets. The photons are created inside a resonator via an interaction with an active medium. In the course of the generation process photons are transferred to the continuum outside the resonator through cavity loss. We show that wave packets generated in this way can be regarded as independent logical qubits. This and the possibility of producing strong entanglement between the qubits suggests many applications in quantum communication. We give a specific example in the context of cavity QED and show that undesired decoherence effects can be efficiently reduced in the considered scheme.

PACS number(s): 03.67.Hk

I. INTRODUCTION

A controlled source of entangled qubits is necessary for most quantum communication and computation procedures [1,2]. For communication purposes the choice of photons as qubits is especially appropriate, since they propagate reliably over long distances [3]. In this sense, an important experimental step has been taken in quantum communication with the recent successful teleportation of a qubit between two spatially distant locations [4,5]. The quintessential ingredient to a successful implementation of this quantum communication process was the availability of a source of entangled qubits. The source presently in use is parametric downconversion in a crystal, which is a reliable source of entangled twin-photons. However, the generation process of entangled photons is random and largely untailorable, and the efficiency of the process is quite low. Although there is an intrinsic limitation on the number of entangled qubits that can be produced directly by downconversion, novel schemes have been found to extract a GHZ state from two pairs of twin-photons by measuring the state of one qubit [6]. Yet the efficiency of this process is low and it is foreseeable that for instance teleporting simultaneously several qubits will require a novel source of entangled qubits. In principle, the obvious choice for such a source appears to be a higher order nonlinear process. However, the efficiency of such processes is extremely small which renders their use unpractical.

In this paper we propose a scheme for the generation of a train of entangled single-photon wave packets which are well resolved in time [7]. This feature permits us to regard them as individual qubits. Our scheme is based on well-established techniques from cavity quantum electrodynamics (CQED) [8–10]. We consider a nonclassical medium (for instance, an atom or ion) inside a high quality optical resonator. With the use of external pump fields, the coupling of the active medium to the resonator allows a single photon transfer to the resonator and therefrom via cavity decay to the continuum outside. The process may be regarded as a *quantum data bus* where the logical state is encoded in the active medium and then mapped onto a photonic qubit generated in the continuum outside the resonator. Before the process is repeated the medium is coherently recycled to its initial state, which provides the basis for a build-up of entanglement between subsequently generated photonic qubits. An encoding of quantum information in the one-photon wave packets could either take place by identifying two orthogonal polarization states of the single photon with logical ''0'' and ''1,'' or by regarding the absence of a photon as logical ''0'' while its presence would correspond to logical ''1.''

The advantage of our scheme over other conventional sources of single-photon wave packets such as down-conversion is twofold. First of all, our scheme offers excellent control over the instances in time when a qubit is created as well as over the spectral composition of the wave packet. The qubits may thus be generated with a well defined *tact frequency*. Second, the coherent recycling of the state of the active medium after the generation of a qubit gives rise to entanglement between subsequent qubits. In particular n-qubit maximally entangled states (MES), e.g., GHZ-states [11] for $n=3$, can be conveniently generated in this way.

The qubits produced by our scheme can be utilized in various quantum cryptography [12] and teleportation [13] schemes, as well as in tests of non-locality and multiparticle interference [14]. Because the production of MES of n qubits is straightforward, our scheme is well suited for quantum communication between three or more parties. Several recent proposals for n-users quantum cryptography key distribution [15], quantum secret sharing [16], cryptographic conferencing, and multiparticle generalizations of superdense coding [17] assume the parties involved to share n-qubit maximally entangled states. In a future quantum network our scheme could be used by a trusted provider to generate entangled states on demand and distribute them to clients through a network with star-shaped topology. Subsequent communication between the recipients who now share an entangled state only needs to take place classically thus freeing them from the need to maintain separate quantum channels between each other.

While in [7] we introduced our proposal by considering a specific type of active medium, namely, a trapped and externally driven atom or ion inside an optical resonator, we here give a model independent formulation. This is important insofar as other implementations based on quantum dots [18] or molecules [19,20] may emerge in the course of time. In Sec. II we give a general description of the requirements for both the source and type of interaction necessary to warrant entangled photon wave-packet generation. We also show how the physical creation process may be represented in terms of abstract quantum circuit diagrams. The general formalism is presented in Sec. III. In Sec. IV we demonstrate that if there is no significant temporal overlap between successively generated photons we may regard the multiphoton state as a product state of independent entities. This is essential for an interpretation of the individual wave packets as independent logical quantum bits. In a further section we underpin the validity of several assumptions made in the derivation by comparing the theoretical predictions with the corresponding numerical solutions. In Sec. V we present a cavity QED based soluble example to illustrate the general argument. We briefly recapitulate the explicit model considered in [7] to illustrate the more general results obtained in the preceding sections. Finally, we discuss the effect of decoherence in this system and show that up to ten maximally entangled qubits can be generated with a fidelity of at least 90%. In Sec. VI we summarize our findings and give a brief discussion of possible applications of multiqubit entangled states in quantum communication.

II. MOTIVATION

As already pointed out in the preceding introductory part one-photon wave packets are presently perceived as the most amenable carriers of quantum information. There is thus a clear need for a tailorable source of one-photon wave packets. Any such source should be able to produce a well resolved train of one-photon wave packets with an adjustable spectral envelope. The most promising proposal so far has been the *photon gun* proposed by Law and Kimble [21,22]. This proposal differs from previously made ones [18–20] in so far as it offers perfect control over the pulse shape as well as the time window T during which the wave packet is created, cf. Fig. 1(a). Several quantum information protocols [3] which so far have to rely on the use of attenuated coherent fields would benefit from such a cleaner source. For other purposes in future quantum networks the source proposed in [21] may not be adequate since it lacks one key ingredient that is *entanglement* between subsequently created photons, cf. Fig. 1(b).

With multiparty quantum communication in mind the need for a source of *entangled* photonic qubits becomes immediately obvious. The scheme we propose here employs conditional wave-packet creation. This means that the polarization of the created wave packet depends on the internal state of the medium, see Fig. 1(b). The encoding of quantum information in the photonic qubit is accomplished most conveniently using two orthogonal polarization states of the field. Thus we have to consider a medium which reflects this fact: in our proposal it has two possible initial states each giving rise to a photon with a different polarization state. If the medium is initially in a superposition of these two states we may create entangled medium-field states. Figure 5 shows a schematic representation of the proposal.

An algorithm for the creation of entangled states would thus read:

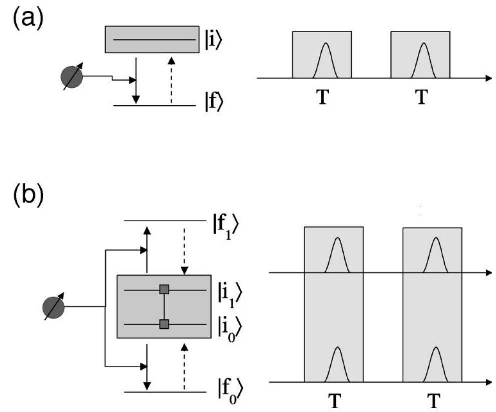

FIG. 1. This figure illustrates the main differences between the photon gun (a) and a scheme capable of conditional wave-packet generation (b). In Fig. 1(a) a system initially in a unique state $|i\rangle$ is transferred to an ancilla state $|f\rangle$ by applying an externally controllable stimulus. In the course of this a single photon is created. Upon recycling the medium back to its initial state (dashed arrow) the generation sequence may be reinitiated. This produces a stream of tailorable but disentangled wave packets. Fig. 1(b) uses a similar principle but the medium is initially in a superposition state of $|i_0\rangle$ and $|i_1\rangle$. Corresponding to each of these two states a wave-packet component with a different (orthogonal) polarization state is generated. The system winds up in an entangled state. Recycling of the ancilla states $|f_{0/1}\rangle$ back to the initial state and reinitiating the sequence allows one to create higher order entangled states.

(1) Prepare initially the medium in a state $|\Psi\rangle_m = c_0|i_0\rangle + c_1|i_1\rangle$, here $|i_0\rangle$ and $|i_1\rangle$ span the medium qubit.

(2) Find an interaction which maps each medium state $|i_\alpha\rangle$ to a unique ancilla state $|f_\alpha\rangle$ thereby creating a single photon with *logical* state $|\alpha\rangle_p$, i.e., after some time T: $|i_\alpha\rangle|\text{vacuum}\rangle \rightarrow |f_\alpha\rangle|\alpha\rangle_p$. Here p denotes photon.

(3) Next recycle the medium: $|f_\alpha\rangle \rightarrow |i_\alpha\rangle$. We have created an entangled state $|\Psi'\rangle = c_0|i_0\rangle|0\rangle_p + c_1|i_1\rangle|1\rangle_p$ in the process.

(4) Alter the state of the medium as desired by a one-qubit rotation of the states $|i_\alpha\rangle$ and repeat steps (2)–(4). This way we are able to create higher order entangled medium-field states.

In the next section we will give a detailed description of the physics involved in this procedure. For now we would like to stress a more general schematic point of view. We may employ a quantum circuit diagram as in Fig. 2 to illustrate the algorithm. The creation of a photon wavepacket involves three distinct steps (1)–(3) as listed above which are reiterated for each qubit to be generated (cf. the numbers shown on top of the gate operation symbols in Fig. 2). In step (1) photon creation takes place, in step (2) the medium is recycled back to its initial state, finally in step (3) the medium is prepared for the next generation sequence. In Fig. 2 the medium is decribed by a qubit initially in state say $|i_0\rangle$. In phase I we perform a one-qubit operation on the medium to create the desired initial medium state. Note that the photonic qubits initially do not actually exist since they are only created at a later stage in the course of the interaction. They still can be incorporated in the diagram assuming each of them initially in logical state $|0\rangle_j$, where j labels the instance

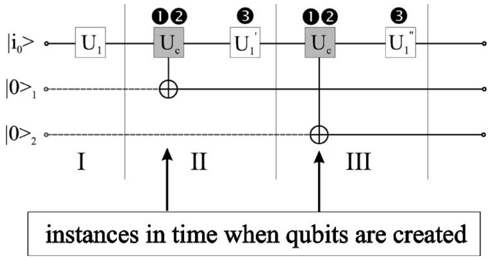

FIG. 2. A quantum network diagram of the algorithm for preparing entangled photonic qubits. In phase I the medium qubit is initialized with a one-qubit gate operation (U_1), in phases II and III the first and second qubit states are prepared by applying two (U_c) and one-qubit gate operations (U_1', U_1'', etc.). The numbers shown above the gate operations refer to the steps described in the text.

in time when they will later come into existence. In phase II the first photon is created by a two qubit operation U_c which has the effect of a controlled-NOT (CNOT) gate,

$$|i_0\rangle|0\rangle_1 \rightarrow |i_0\rangle|0\rangle_1, \quad \text{and} \quad |i_1\rangle|0\rangle_1 \rightarrow |i_1\rangle|1\rangle_1 .$$

This operation concatenates thus steps (1) and (2) mentioned above. A further single qubit rotation (U_1') concludes phase II. In phase III another photon is created. Upon completion we have created a three particle entangled state comprising two photonic qubits and a medium qubit.

III. MODEL

In this section we will develop a theory to describe the wavepacket generation in a rather general context. The emphasis is on the exact treatment of the cavity-mode decay into the continuum of modes outside the cavity, rather than on a detailed investigation of the medium-cavity dynamics as in [8]. Right now we are not concerned with any specific model for the active medium, e.g., a single trapped atom or ion inside the resonator, and we first consider interactions of the medium with a single (polarization) mode only. These simplifications will be abandoned in Sec. IV C when we establish the interpretation of 1-photon wavepackets as qubits. The Hamiltonian of our model reads as follows (in the standard dipole and rotating wave approximations):

$$H/\hbar = H_0 + V,$$

$$H_0 = \omega_c a^\dagger a + \int d\omega\, \omega b^\dagger(\omega) b(\omega) + H_m ,$$

$$V = i \int d\omega k(\omega)[a b^\dagger(\omega) - b(\omega) a^\dagger] + \sum_j \theta(t-t_j)\theta(t_j+T-t) V_{\mathrm{mc}}(t), \tag{3.1}$$

where ω_c denotes the cavity frequency; $a^\dagger$, $b^\dagger(\omega)$ [a, $b(\omega)$] are the creation (annihilation) operators of the cavity and reservoir modes, respectively. The reservoir modes satisfy standard bosonic commutation relations: $[b(\omega), b^\dagger(\nu)] = \delta(\omega-\nu)$. The coupling between the cavity and the reservoir modes is described by $k(\omega)$. Assuming the coupling to be flat around the cavity resonance frequency (i.e., adopting the first Markov approximation) we may set $k(\omega) \approx \sqrt{\kappa_c/\pi}$ [23]. The term H_0 contains the free dynamics with H_{m} denoting the free Hamiltonian of the active medium. Finally, V_{mc} describes the interaction between the cavity mode and the active medium. We assume that the (repeated) creation of a single photon inside the cavity takes place within a time window of duration T starting at times t_j, with the condition that $t_{j+1} - t_j > T$. The Heaviside step functions employed in Eq. (3.1) serve the purpose of expressing in a convenient fashion the limited duration T during which the interaction in each wave-packet generation sequence is switched on; note that we assume that V_{mc} vanishes smoothly at the end points of each time window. In an interaction picture with respect to the free dynamics given by H_0 we have

$$V_I = i \sqrt{\frac{\kappa_c}{\pi}} \int d\omega [a b^\dagger(\bar{\omega}) e^{i\omega t} - b(\bar{\omega}) e^{-i\omega t} a^\dagger] + V^I_{\mathrm{mc}}(t), \tag{3.2}$$

where $\bar{\omega} \equiv \omega + \omega_c$ and

$$V^I_{\mathrm{mc}}(t) = \sum_j \theta(t-t_j)\theta(t_j+T-t) e^{iH_0 t} V_{\mathrm{mc}}(t) e^{-iH_0 t}. \tag{3.3}$$

The evolution operator corresponding to the medium-cavity interaction is defined as

$$U_{\mathrm{mc}}(t) = \mathcal{T} e^{-i\int_0^t dt' V^I_{\mathrm{mc}}(t')},$$

where $\mathcal{T}$ denotes temporal ordering. The family of interactions that we wish to consider may be characterized by two requirements:

(i) the interaction transforms a given initial (metastable) state of the medium into a known final (metastable) state: $|i\rangle_m \rightarrow |f\rangle_m$.

(ii) in the course of the interaction *at most* one photon is deposited in the cavity mode. The interaction is slow on the timescale given by the cavity lifetime κ_c^{-1}, i.e., $T \gg \kappa_c^{-1}$.

We assume both the cavity and the reservoir to be initially in the vacuum state. Then, endowed with the above mentioned premisses the evolution for $t < t_1$ is restricted to a subspace with only one photonic excitation present. Thus the following ansatz is appropriate for the wave function evolution:

$$|\psi(t)\rangle = |\varphi(t)\rangle_{\mathrm{mc}} |0\rangle_r + \int d\omega |\varphi_{\bar{\omega}}(t)\rangle_{\mathrm{mc}} |1_{\bar{\omega}}\rangle_r . \tag{3.4}$$

Here $|\varphi_{\bar{\omega}}(t)\rangle_{\mathrm{mc}}$ and $|\varphi(t)\rangle_{\mathrm{mc}}$ denote medium-cavity states with and without the transfer of a photon having taken place into the reservoir mode with frequency $\bar{\omega}$, respectively. Their behavior is governed by the following equations:

$$|\dot{\varphi}(t)\rangle_{\rm mc} = -iV^I_{\rm mc}(t)|\varphi(t)\rangle_{\rm mc} - \sqrt{\frac{\kappa_c}{\pi}}\int d\omega e^{-i\omega t}a^\dagger|\varphi_{\vec{\omega}}(t)\rangle_{\rm mc},$$

$$|\dot{\varphi}_{\vec{\omega}}(t)\rangle_{\rm mc} = -iV^I_{\rm mc}(t)|\varphi_{\vec{\omega}}(t)\rangle_{\rm mc} + \sqrt{\frac{\kappa_c}{\pi}}e^{i\omega t}a|\varphi(t)\rangle_{\rm mc},$$

with $|\varphi(0)\rangle_{\rm mc} = |i\rangle_{\rm m}|0\rangle_{\rm c}$ Formal integration of the evolution equation for $|\varphi_{\vec{\omega}}(t)\rangle_{\rm mc}$ yields

$$|\varphi_{\vec{\omega}}(t)\rangle_{\rm mc} = \sqrt{\frac{\kappa_c}{\pi}}\int_0^t dt' U_{\rm mc}(t,t')e^{i\omega t'}a|\varphi(t')\rangle_{\rm mc}, \tag{3.5}$$

where $U_{\rm mc}(t,t') = U_{\rm mc}(t)U^\dagger_{\rm mc}(t')$. Insertion of Eq. (3.5) into the evolution equation of $|\varphi(t)\rangle_{\rm mc}$ yields, within the validity of the Markov approximation [24],

$$|\varphi(t)\rangle_{\rm mc} = U^{\rm eff}_{\rm mcr}(t)|\varphi(0)\rangle_{\rm mc}, \tag{3.6}$$

where

$$U^{\rm eff}_{\rm mcr} = \mathcal{T}\exp\left(-i\int_0^t dt'[V^I_{\rm mc}(t') - i\kappa_c a^\dagger a]\right) \tag{3.7}$$

is the effective evolution operator generated by a non-Hermitian Hamiltonian describing both the medium-cavity interaction and the decay into the reservoir. Hence we obtain

$$|\varphi_{\vec{\omega}}(t)\rangle_{\rm mc} = \sqrt{\frac{\kappa_c}{\pi}}\int_0^t dt' e^{i\omega t'}U_{\rm mc}(t,t')aU^{\rm eff}_{\rm mcr}(t')|\varphi(0)\rangle_{\rm mc}. \tag{3.8}$$

We are interested in the state of the reservoir after the photon has left the resonator with near certainty. We therefore only consider times t exceeding T by a few cavity lifetimes. The contribution to the integral from times larger than T vanishes since there $U^{\rm eff}_{\rm mcr}(t)$ represents merely an exponential decay, i.e.,

$$|\varphi(t)\rangle_{\rm mc} = e^{-\kappa_c a^\dagger a(t-T)}|\varphi(T)\rangle_{\rm mc} \tag{3.9}$$

Any one-photon component of $|\varphi(t)\rangle_{\rm mc}$ will thus be strongly suppressed. The assumption of efficient transfer (i) guarantees that the projection of $|\varphi(T)\rangle_{\rm mc}$ on the vacuum component is insignificant. We may thus assume that the contribution of state $|\varphi(t)\rangle_{\rm mc}|0\rangle_r$ to the total state $|\psi(t)\rangle$ vanishes. We may therefore write

$$|\psi(t)\rangle = \sqrt{\frac{\kappa_c}{\pi}}\int\int_0^T d\omega dt' \times e^{i\omega t'}U_{\rm mc}(t,t')aU^{\rm eff}_{\rm mcr}(t')|\varphi(0)\rangle_{\rm mc}|1_{\vec{\omega}}\rangle_r. \tag{3.10}$$

According to the requirement (i) the interaction with the medium must guarantee an efficient transfer of a single photon to the resonator within its duration T, i.e., it will connect the initial state $|i\rangle_{\rm m}|0\rangle_c$ only to the state $|f\rangle_m|1\rangle_c$. Consequently, the action of the term $U_{\rm mc}(t,t')$ in the integral kernel on the right hand side of Eq. (3.10) will be twofold. First it will erase the state component proportional to $|i\rangle_{\rm m}|0\rangle_c$ and transform $|f\rangle_m|1\rangle_c$ into $|f\rangle_m|0\rangle_c$. The latter is not affected by the interaction thus $U_{\rm mc}(t,t')$ leaves it unaltered. In formal terms this can be rephrased as: $V_{\rm mc}(t)|f\rangle_{\rm m}|0\rangle_c = 0$. Physically, this means that the generation of photons stops after the first generated photon has left the cavity. We may then recast Eq. (3.10) in the following simpler form

$$|\psi(t)\rangle = \int d\omega aG(\omega,T)|\varphi(0)\rangle_{\rm mc}|1_{\vec{\omega}}\rangle_r, \tag{3.11}$$

where

$$G(\omega,T) \equiv g(\omega,T)a^\dagger\sigma_{fi} = \sqrt{\frac{\kappa_c}{\pi}}\int_0^T dt e^{i\omega t}U^{\rm eff}_{\rm mcr}(t). \tag{3.12}$$

The operator σ_{fi} maps the state $|i\rangle_{\rm m}$ to the state $|f\rangle_m$. The spectral envelope of the created wave packet is given by $g(\omega,T)$ (a mere c-number function). Conservation of the norm of $|\psi\rangle$ implies $\int d\omega|g(\omega,T)|^2 = 1$. Note that we have already shown in a previous publication [22] that a time-dependent decay rate $\kappa_c(t)$ and central frequency $\omega_c(t)$ can be employed to generate a one-photon wavepacket of an arbitrary spectral composition. Such an approach may be regarded as a description of the presence of a medium in terms of an effective model.

We may now recycle the medium back to its initial state $|i\rangle_m$ and repeat the generation anew starting at some time $t_1 > T$. For times larger than $t_1 + T$ we have then created a two-photon wavepacket with no significant temporal overlap of the individual photons. After a few lines of algebra one obtains the following approximate result for the state for times $t > t_1 + T$

$$|\psi(t)\rangle = \int d\omega\, d\nu\, aG(\nu,T) \times e^{i\nu t_1}\sigma_{if}aG(\omega,T)|\varphi(0)\rangle_{\rm mc}|1_{\vec{\omega}}1_{\vec{\nu}}\rangle_r. \tag{3.13}$$

It can be quite instructive to illustrate the theory developed by considering a simple example which is exactly soluble within the Markov approximation. Let us imagine an *instantaneous* creation of a photon by the medium-cavity interaction at the time $t_0 = 0$, i.e., we start from the initial state $a^\dagger\sigma_{fi}|\varphi(0)\rangle_{\rm mc}$. With this initial condition the effective evolution operator defined by Eq. (3.7) becomes $U^{\rm eff}_{\rm mcr}(t) = e^{-\kappa_c a^\dagger a t}a^\dagger\sigma_{fi}$, and $G(\omega,T)$ can be easily calculated from Eq. (3.12). We obtain

$$G(\omega,T) = \sqrt{\frac{\kappa_c}{\pi}}\frac{1}{\kappa_c a^\dagger a - i\omega}a^\dagger\sigma_{fi}. \tag{3.14}$$

With regard to the chosen initial condition, i.e., the cavity being in the vacuum state at $t = t_0$ we may replace $a^\dagger a$ in the denominator of the rhs of Eq. (3.14) by unity. By direct comparison of the preceding equation with Eq. (3.12) we may identify the spectral envelope $g(\omega,T)$ of the generated wavepacket. Thus the corresponding wave function after the instantaneous injection of a second photon inside the resonator explicitly reads as

$$|\psi(t)\rangle=\frac{\kappa_c}{\pi}\int d\omega\frac{1}{\kappa_c-i\omega}b^\dagger(\overline{\omega})\times\int d\nu\frac{1}{\kappa_c-i\nu}e^{i\nu t_1}b^\dagger(\overline{\nu})|f\rangle_m|0\rangle_c|0\rangle_r. \tag{3.15}$$

The theory just presented is independent of the type of medium used. Before considering an explicit example from CQED in Sec. V we discuss the interpretation of the created wavepackets as independent entities—as logical quantum bits.

IV. PHOTON WAVE PACKETS AS QUBITS

Any physical realization of logical qubits is contingent upon the availability of distinguishable and independently controllable (two-state) quantum systems which interact with each other only in a controlled manner. In other words, the annihilation and creation operators corresponding to different qubits must commute. We will now show that within the framework of the present theory we may isolate such approximately commuting creation and annihilation operators. They can be identified with the multimode operators that create (annihilate) the individual wave packets. This thus allows us to regard the wave packets as independent quantum entities. The physical prerequisite for this to hold true is a vanishing temporal overlap between successively generated wave packets.

A. Factorized photon wave packets

Let us assume that the time between subsequent photon generation processes is so long that the previous photon has leaked out of the cavity with almost certainty before the next photon is produced. Then we can generalize the result for the state of two photonic qubits as given by Eq. (3.13) to the case of n qubits, and formally express the state as a product of multimode creation operators applied to the vacuum of the continuum. The n-photon wave packet state can be written as

$$|\psi\rangle_{\rm r}=\prod_{j=0}^{n}B^\dagger(t_j,T)|0\rangle_r \tag{4.1}$$

with $t_0=0$ and

$$B^\dagger(t_j,T)=\int d\omega\, e^{i\omega t_j}g(T,\omega)b^\dagger(\overline{\omega}). \tag{4.2}$$

In order to arrive at a description of the n-photon wave packet in terms of a product of n photonic qubits we have to show that

$$[B(t_j,T),B^\dagger(t_k,T)]=\delta_{jk} \tag{4.3}$$

holds to a very good approximation. Inserting Eq. (4.2) yields

$$[B(t_j,T),B^\dagger(t_k,T)]=\int d\omega e^{i\omega(t_k-t_j)}|g(\omega,T)|^2.$$

It is clear that the shape of $g(\omega,T)$ will depend on the type of interaction used to implement the scheme. It is thus not possible to give a general result for the commutator. In fact it would need to be checked for each chosen implementation. We may however adopt the following general line of argument to show why Eq. (4.3) should nevertheless hold whenever successively generated wave packets do not temporally overlap. It is plausible to assume that the mode distribution function $|g(\omega)|^2$ is a reasonably well behaved smooth function which may be assigned something like a half width at half maximum. Let us refer to this quantity as $\delta\omega<\kappa_c$. Its inverse may by associated with the pulse duration $\tau=1/\delta\omega$. The Fourier transform of $|g(\omega,T)|^2$ will thus be approximately given by an expression of the following type:

$$\int d\omega e^{i\omega(t_j-t_k)}|g(\omega,T)|^2\approx\{\exp[-(|t_j-t_k|/\tau)^\rho]\}^\xi, \tag{4.4}$$

with $\xi>0$ and $\rho\geqslant1$ (the latter only holds provided the spectral density drops off asymptotically at least as fast as ω^{-2} which may be assumed since the distribution is limited in width by the Lorentzian line shape of the resonator). We realize that the rhs of Eq. (4.4) becomes extremely small if $|t_j-t_k|\gg\tau$, that is if there is no overlap between successively generated wave packets.

For instance, in the example of instantaneous photon injection discussed above

$$[B(t_j,T),B^\dagger(t_k,T)]=e^{-\kappa_c|t_j-t_k|}, \tag{4.5}$$

which approaches δ_{jk} provided $|t_j-t_k|\sim T\gg\kappa_c^{-1}$ for $j\neq k$. This thus permits us to interpret each of the $B(t_j,T)$ as acting on its own vacuum. Physically, this corresponds to the fact that the individual wavepacket can be thought of as being localized each within a box of length cT, all of which do not overlap and travel at the speed of light, cf. Fig. 1. So we can rewrite Eq. (4.1) as

$$|\psi\rangle_{\rm r}=\prod_{j=0}^{n}|\psi\rangle_j,$$

$$|\psi\rangle_j=B^\dagger(t_j,T)|0\rangle_{r_j}. \tag{4.6}$$

This paves the way for an interpretation of the wave packets as distinguishable quantum entities.

B. Numerical simulation of the wave-packet generation and factorization

We have numerically studied the validity of both the Markov approximation and the interpretation of the wave packets as independent entities in the case of instantaneous photon injection into the cavity. Of course, this can be extended to the case of an atom playing the role of the medium. Numerical simulations are restricted to the case of few photons due to the exponential growth of the whole system's Hilbert space with increasing excitation number. A numerical treatment brings with it the need to model the reservoir by a set of M discrete modes. Morover we assume that the distribution of reservoir frequencies is flat and centered around the cavity frequency. Thus the coupling between the cavity and reservoir modes in a rotating frame with respect to the cavity center frequency (discrete version of Eq. (3.2) for the case of instantaneous photon injection) is given by

$$V_I = i\sum_{j=0}^{M-1} k'(\omega_j)(ab_j^\dagger e^{i(\omega_j-\omega_c)t} - b_j a^\dagger e^{-i(\omega_j-\omega_c)t}),$$

with

$$k'(\omega_j) \equiv \sqrt{\kappa_c \delta\omega/\pi}$$

and $\omega_j = \omega_0 + j\delta\omega$, $\delta\omega = \Delta\omega/(M-1)$, $\omega_0 = \omega_c - \Delta\omega/2$. $\Delta\omega$ is the considered full range of reservoir frequencies. Here we approximate the reservoir by a large number M of *quasicontinuum* modes. The scaled interaction parameter in the expression of $k'(\omega_j)$ was chosen to give the same decay constant as for the continuum limit (i.e., κ_c). This approach remains valid for times smaller than the spurious revival time, $t_R = 2\pi/\delta\omega$, which emerges as a consequence of our using a finite number of reservoir modes with equidistant spacing.

We will assume that the cavity was initially prepared in a one photon state and the reservoir modes are in their vacuum states. The system evolves on a subspace of dimension $M+1$ and the discrete version of Eq. (3.4) holds as

$$|\Psi(t)\rangle = \alpha(t)|1\rangle_c|0\rangle_r + \sum_{j=0}^{M-1} \beta_j(t) b_j^\dagger |0\rangle_c |0\rangle_r,$$

where $|0\rangle_r$ denotes the vacuum state of reservoir modes. After a time $T \gg \kappa_c^{-1}$ a new photon is injected into the cavity thereby enlargening the accessible portion of the Hilbert space. The system will then evolve on a subspace of dimension M^2+M+1, which is numerically still manageable for M up to around 1000. The probability amplitude $\alpha(t)$ for the photon still being inside the resonator is the solution to an integral equation

$$\dot{\alpha}(t) \simeq -\frac{2\kappa_c}{\pi}\int_0^{\Delta\omega t/2} \frac{\sin(z)}{z}\,\alpha\!\left(t - \frac{2z}{\Delta\omega}\right)dz, \quad \alpha(0)=1. \tag{4.7}$$

If the size of the frequency window and the number of reservoir modes go to infinity, i.e., $\Delta\omega$, $M\to\infty$, we recover the continuum limit. This becomes obvious from the following approximate solution to Eq. (4.7) which is valid for $t \ll t_R$:

$$\alpha(t) \simeq e^{-\kappa_c t\,\mathrm{sinc}(2\kappa_c t/\pi\chi)(1+\chi) + \chi[1-\cos(2\kappa_c t/\pi\chi)]},$$

$$\beta_j(t) = \sqrt{\frac{\kappa_c \delta\omega}{\pi}} \int_0^t \alpha(t') e^{i(\omega_j-\omega_c)t'} dt', \tag{4.8}$$

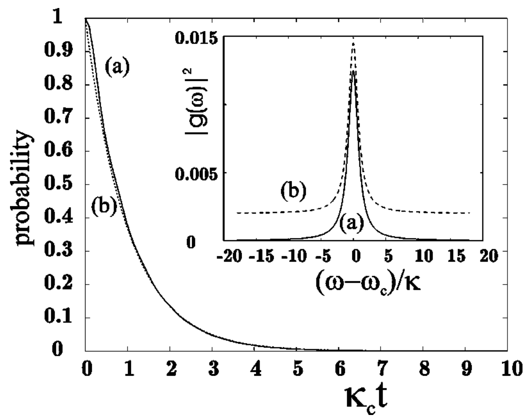

FIG. 3. Plot of the probability for the initial photon to remain inside the cavity as predicted by the discrete model $|\alpha(t)|^2$ [curve (a)] and the corresponding continuum limit result [curve (b)]. The inset depicts the spectral density of the wave packet $|g(\omega)|^2$ generated by our discrete model for $\kappa_c t = 12$ [curve (a)]. In curve (b) we plot the Lorentzian distribution predicted by the continuum model. Note that the frequency is given in units of κ_c. Since both distributions completely overlap we have added an offset of 0.002 to the continuum result. The system was initially prepared in state $|1\rangle_c|0\rangle_r$.

where $\chi = 4\kappa_c/\pi\Delta\omega$ is a small perturbation parameter, and the sinc function

$$\mathrm{sinc}(x) = \frac{2}{\pi}\int_0^x \frac{\sin z}{z}dz, \tag{4.9}$$

with $\mathrm{sinc}(\infty)=1$. From Eq. (4.8) we note that the length of the frequency window $\Delta\omega$ gives rise to a time dependent cavity decay rate, which approaches the continuum limit κ_c for $t \simeq \kappa_c^{-1}$.

Our primary goal here is to demonstrate the validity of the factorisation approximation used in Eq. (3.13). To this end we first have to convince ourselves that the chosen numerical model closely reproduces the continuum limit. In Fig. 3 we plot the probability to find the photon still inside the cavity as a function of time. Curve (a) corresponds to $|\alpha(t)|^2$ while curve (b) is the corresponding exponential decay $\exp(-2\kappa_c t)$ valid for the continuum limit. For short times we observe a slight deviation from the continuum result which is due to the finite size of the frequency window $\Delta\omega$. The inset depicts the mode densities of the wavepacket generated for both models. Note the excellent agreement which made it necessary to introduce an offset for the continuum result in order for both of them to be distinguishable. In our numerical treatment we have model the continuum by $M=1024$ discrete modes embedded in a frequency window $\Delta\omega$ of width $40\kappa_c$ centered around the cavity resonance frequency ω_c. We are now in the position to test the factorisation assumption. We assume that at time t_1 the cavity is again prepared in a one-photon Fock state. The new initial condition reads

$$|\Psi(t_1)\rangle = \sum_{j=0}^{M-1} \beta_j(t_1) b_j^\dagger |1\rangle_c |0\rangle_r.$$

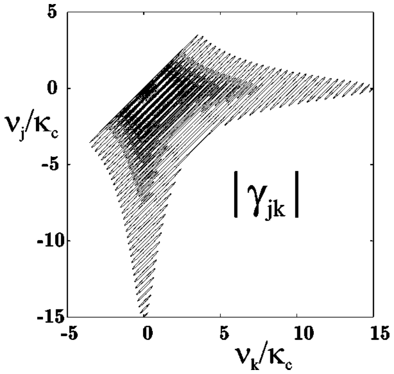

FIG. 4. Modulus of the two-photon spectral density distribution $|\gamma_{jk}|$ for the case of instantaneous photon injection.

We make the following ansatz for the wave function valid for times t greater than t_1

$$|\Psi(t)\rangle=\alpha(t)|2\rangle_c|0\rangle_r+\sum_{j=0}^{M-1}\beta_j(t)b_j^\dagger|1\rangle_c|0\rangle_r+\sum_{j\leq k=0}^{M-1}\gamma_{jk}(t)b_j^\dagger b_k^\dagger|0\rangle_c|0\rangle_r. \tag{4.10}$$

We again let the system evolve up to a time T satisfying $T-t_1=12/\kappa_c$. As the temporal width of the wave packet is roughly given by $2\kappa_c^{-1}$ we should find the following relation to be fulfilled to good approximation:

$$\gamma_{jk}(T)\approx\beta_j(t_1)\beta_k(t_1)\times\begin{cases}e^{i\nu_j\Delta t}+e^{i\nu_k\Delta t}, & \text{for } j<k\\ \sqrt{2}e^{i\nu_j\Delta t}, & \text{for } j=k\text{'}\end{cases} \tag{4.11}$$

where $\Delta t=T-t_1$, and $\nu_j=\omega_j-\omega_c$. The rhs of Eq. (4.11) is a discrete version of Eq. (3.15). A contour plot of $|\gamma_{jk}(T)|$ is given in Fig. 4. It displays both the bifurcation and interference fringes predicted by the rhs of Eq. (4.11). The fringes and bifurcation are due to the same origin. The modulus of the phasefactors on the rhs of Eq. (4.11) is a constant for $\nu_k=\nu_j\pm 2\pi n/\Delta t$, for integer n. Along these rays a bifurcation into two local maxima occurs because of the product of the β coefficients of which one is now centered around $\omega_c\pm 2\pi n/\Delta t$. Let us refer to the rhs of Eq. (4.11) as γ_{jk}^{fac}. We have found that γ_{jk}^{fac} closely reproduces γ_{jk}. We thus refrain from plotting portions of both distributions side by side. We rather try to characterize their closeness by introducing the following quantity:

$$d=\frac{\sum_{j\leq k=0}^{M-1}|\gamma_{jk}-\gamma_{jk}^{\text{fac}}|}{\sum_{j\leq k=0}^{M-1}(|\gamma_{jk}|+|\gamma_{jk}^{\text{fac}}|)}. \tag{4.12}$$

For the chosen scenario we find that d is of the order of $10^{-4}-10^{-3}$ which thus proves that the distribution function of the generated two-photon wave packet factorizes to good approximation in the limit of temporally nonoverlaping single photon pulses. Evidently this will also hold for all subsequently generated wave packets.

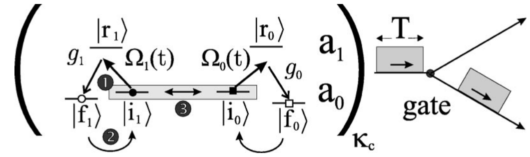

FIG. 5. A single atom with six internal states interacts with two cavity modes of orthogonal polarization a_0, a_1. In a Raman process (step 1) an initial superposition state of levels $|i_0\rangle$ and $|i_1\rangle$ is transformed into an entangled cavity-atom state. Due to cavity leakage the photon will leave the cavity and produce a photon wave packet in the continuum modes outside the resonator. In step 2 the atom is recycled back to $|i_0\rangle$ and $|i_1\rangle$. Inbetween two photon generation sequences levels $|i_0\rangle$ and $|i_1\rangle$ can be coupled (step 3) to tailor the outgoing state.

C. Logical qubit states with polarized photons

Up to now, for reasons of notational simplicity we have restricted our condiderations to a resonator supporting only one polarization mode of the field. Let us now make use of both polarization states (e.g., σ_+, σ_-) of a single longitudinal mode of an optical resonator. Formally, we will thus have to consider two independent resonator modes a_α, $\alpha=0,1$. In order to warrant clean conditional wave-packet generation as shown in Fig. 1(b) the medium-cavity interaction must not contain any polarization mixing terms. As indicated in the motivational section the medium must be prepared in a superposition state. Let us denote by $|i_\alpha\rangle_m$, $|f_\alpha\rangle_m$ ($\alpha=0,1$) the states involved in the creation of a σ_+ (0) and σ_- (1) polarized photon, respectively. In the absence of polarization mixing terms the subsystems corresponding to different values of α are completely decoupled and we may expect the scheme to implement the following mapping:

$$|i_\alpha\rangle_{\text{m}}|0\rangle_c\rightarrow a_\alpha^\dagger|f_\alpha\rangle_{\text{m}}|0\rangle_c, \quad \alpha=0,1.$$

The generalization of the developed formalism to the case of two polarization degrees of freedom is straightforward. If we assume the medium to be initially prepared in a superposition state $|\varphi(0)\rangle_m=c_0|i_0\rangle+c_1|i_1\rangle$ then the final state of the total system can be directly obtained from the superposition principle by issuing each partial solution with the appropriate probability amplitude c_α. We can define wavepacket creation operators like in Eq. (4.6) for both polarizations, $B_0^\dagger(t_j,T)$ and $B_1^\dagger(t_j,T)$, which are composed of the single frequency creation operators $b_0^\dagger(\overline{\omega})$ and $b_1^\dagger(\overline{\omega})$. We are thus able to identify the one-photon wave packets as qubits with logical states

$$|\alpha\rangle_j\equiv B_\alpha^\dagger(t_j,T)|0\rangle_r, \tag{4.13}$$

where $\alpha=0,1$ refers to the logical (polarization) state.

The process described above can be realized by making use of two Λ-type energy-level configurations, cf. Fig. 5 which we will discuss in more detail in Sec. V. We have thus shown that it is permissible to regard the n-photon pulse generated by our scheme as an array of n qubits in an entangled state with the logical state being assoociated with the polarization state of each one-photon wave packet the pulse is made up of.

D. Quantum state engineering

Logical states of the multiqubit system are created by preparing the active medium initially in a superposition state:

$$|\psi(0)\rangle=(\cos\phi_1|i_0\rangle_m+\sin\phi_1 e^{i\varphi_1}|i_1\rangle_m)|0\rangle_c|0\rangle_r .$$

This is accomplished by connecting levels $|i_0\rangle$ and $|i_1\rangle$ through an appropriate interaction mechanism, e.g., in the case of the medium being an atom, a Raman laser pulse or a microwave field of appropriate duration and detuning is applied to the transition. Such a one-quibt rotation is realized a unitary transformation parametrized by the angles ϕ_1 and φ_1. After the generation of the first photon the state results in

$$(\cos\phi_1|f_0\rangle_m|0\rangle_1+\sin\phi_1 e^{i\varphi_1}|f_1\rangle_m|1\rangle_1)|0\rangle_c .$$

Note that after each generation process the medium and the polarization state of the generated wave packet will be in an entangled state. We can now recycle the states $|f_\alpha\rangle_m$ back to $|i_\alpha\rangle_m$ and repeat the procedure, with angles ϕ_2 and φ_2:

$$|i_0\rangle_m\rightarrow\cos\phi_2|i_0\rangle_m+e^{i\varphi_2}\sin\phi_2|i_1\rangle_m ,$$

$$|i_1\rangle_m\rightarrow\cos\phi_2|i_1\rangle_m-e^{-i\varphi_2}\sin\phi_2|i_0\rangle_m .$$

After a successfully completed second generation sequence (including the recycling step) we obtain the state

$$(\cos\phi_1\cos\phi_2|i_0\rangle_m|0\rangle_1|0\rangle_2+\cos\phi_1\sin\phi_2 e^{i\varphi_2}|i_1\rangle_m|0\rangle_1|1\rangle_2+\cos\phi_2\sin\phi_1 e^{i\varphi_1}|i_1\rangle_m|1\rangle_1|1\rangle_2$$
$$-\sin\phi_1\sin\phi_2 e^{i(\varphi_1-\varphi_2)}|i_0\rangle_m|1\rangle_1|0\rangle_2)|0\rangle_c ,$$

and similarly for all subsequently generated photons.

In this way we are able to generate various logical states of the n-qubit system. Not all possible ones, though: a general state in a basis spanned by n qubits is defined by $N=2^{n+1}-2$ independent coefficients, which correspond to 2^n complex coefficients minus a normalization and a global phase factor. In the above described state engineering procedure we have only $2n$ independent coefficients at hand, thus the size of the class of states that can be prepared is limited by this fact. This limitation is due to the structure of the active medium which is solely a two-state system as far a conditional wave packet generation is concerned. We have chosen the active medium to be well suited for the engineering of various kinds of strongly entangled states.

Eventually the residual entanglement with the medium can be removed by measuring the medium state in an appropriate basis. For example, in the three-photon case, if we choose $\phi_1=\pi/4$, $\phi_2=\phi_3=\varphi_1=\varphi_2=\varphi_3=0$, disregarding a normalization constant the final state is $(|f_0\rangle_m|0\rangle_1|0\rangle_2|0\rangle_3-|f_1\rangle_m|1\rangle_1|1\rangle_2|1\rangle_3)|0\rangle_c$. By measuring the atom in the basis $|f_0\rangle_m\pm|f_1\rangle_m$ we can remove the entanglement with the medium and obtain one of the two Greenberger-Horne-Zeilinger states $|000\rangle\mp|111\rangle$; the preparation of a higher dimentional MES

$$|\Psi\rangle=1/\sqrt{2}(|s_1,s_2,\ldots,s_n\rangle+|1-s_1,1-s_2,\ldots,1-s_n\rangle),$$

$$s_j=0,1,$$

is analogous.

V. REALIZATION WITH THREE-LEVEL Λ ATOMS

In this part we will briefly recapitulate the results of Ref. [7] the intention behind this being our wish to illustrate how the general formalism carries over to an explicit model. We consider a single atom or ion trapped inside an optical cavity [9,10] as the implementation closest to present day experimental reality. For the atom we assume a double three-level Λ structure in the large detuning limit as depicted in Fig. 5. The levels $|i_\alpha\rangle$ ($\alpha=0,1$) are coupled to the upper levels $|r_\alpha\rangle$ via classical fields $\Omega_\alpha(t)e^{-i[\omega_\alpha t+\phi_\alpha(t)]}$, where ω_α are the field center frequencies and the subscript refers to the two polarization states. The external control parameters are the real amplitudes $\Omega_\alpha(t)$ and the phases $\phi_\alpha(t)$. The levels $|f_\alpha\rangle$ are coupled to the upper levels by the cavity modes a_α, with a common frequency ω_c but with orthogonal polarization, with coupling constants g_α. The large detuning (δ) assumption allows us to adiabatically eliminate the upper atomic levels. We are left with two effective two-level systems describable by generalized spin operators $\sigma_{i_\alpha j_\alpha}=|i_\alpha\rangle\langle j_\alpha|$. The center frequencies of the external laser pulses fulfil the Raman resonance condition. Note that any offsets can still be accommodated in the phases $\phi_\alpha(t)$. The field outside the resonator is described as outlined in Sec. III. During the process of photon wave-packet generation (step 1 in Fig. 5) there is no interaction connecting the two Λ-systems. Thus, as pointed out in Sec. IV C, we may independently work with each system corresponding to a single index α, and drop this index for the time being. In this example the total system consists of three building blocks: the continuum outside the resonator, the cavity modes and the internal degrees of freedom of the atom inside the resonator. We switch to an interaction picture with respect to the free dynamics of the compound system, cf. Eq. (3.3). This eliminates the fast optical timescales from the dynamics and leaves us with a simpler effective interaction term:

$$V^I_{\mathrm{mc}}(t)=\frac{r^2(t)}{\Delta_s}|i\rangle\langle i|+\Delta_s a^\dagger a|f\rangle\langle f|+ir(t)$$
$$\times(e^{-i\phi(t)}a^\dagger|f\rangle\langle i|-e^{i\phi(t)}|i\rangle\langle f|a),$$

where $r(t)=g\Omega(t)/2\delta$ (Rabi frequency of Raman transition) and $\Delta_s=g^2/\delta$ (cavity induced Stark shift). For simplicity, we have assumed a common detuning parameter between the pump fields and the cavity modes and real coupling constants. Any additional frequency shifts can be included in the phase of the classical field. The time and intensity dependent terms correspond to dynamical shifts arising from the adiabatic elimination of the upper atomic level. We assume that the atom-cavity system is initially in the state $|\varphi(0)\rangle_{\mathrm{mc}}=|i\rangle|0\rangle_{\mathrm{c}}$. Therefore, the action of the effective evolution operator $U^{\mathrm{eff}}_{\mathrm{mcr}}(t)$ [cf. Eq. (3.7)], on the chosen initial state is completely characterized by the following expression:

$$U^{\mathrm{eff}}_{\mathrm{mcr}}(t)=C_i(t)e^{-i\theta(t)}\mathbb{1}+C_f(t)e^{-i\Delta_s t}a^\dagger\sigma_{fi}, \tag{5.1}$$

where $\theta(t)=\int_0^t dt'|\Omega|^2(t')/4\delta$. The slowly varying functions evolve according to

$$\dot{C}_i(t)=-r(t)C_f(t)e^{i\theta_c(t)},$$
$$C_f(t)=\int_0^t dt' r(t')C_i(t')e^{-i\theta_c(t')-\kappa_c(t-t')}, \tag{5.2}$$

where $\theta_c(t)=\theta(t)+\phi(t)-\Delta_s t$. In the bad cavity limit the main contribution to the integral will stem from time differences $t-t'$ of the order of the intracavity photon lifetime κ_c^{-1}. If both coefficients vary slowly on this time scale we may approximate the solution of the previous equation by

$$C_f(t)\simeq\frac{r(t)}{\kappa_c}e^{-\mu(t)-i\theta_c(t)},$$
$$C_i(t)=e^{-\mu(t)},$$

where $\mu(t)$ is an effective decay constant defined by

$$\mu(t)\equiv\int_0^t dt'\frac{r^2(t')}{\kappa_c}.$$

So far we have found an expression for the state $|\varphi(t)\rangle_{\mathrm{mc}}$. In order to warrant reliable wave packet creation during a time window T we have to require that $\mu(T)\gg 1$. Note that this sets a lower bound on the size of the pulse area of the externally applied laser fields. Assuming this to be the case we simply have to work out the expression given by Eq. (3.10) which now fully characterizes the state. To this end we have to apply the mode destruction operator a to the state $U^{\mathrm{eff}}_{\mathrm{mcr}}(t)|\varphi(0)\rangle_{\mathrm{mc}}$. As we have already argued in Sec. III doing so will yield a state that is not affected by the interaction Hamiltonian $V^I_{\mathrm{mc}}(t)$; in the present case this state is proportional to $|f\rangle|0\rangle_c$. One may easily convince oneself that it does not evolve under the influence of the interaction Hamiltonian, i.e., $U_{\mathrm{mc}}(t)|f\rangle|0\rangle_c=|f\rangle|0\rangle_c$. Finally, the spectral envelope as introduced in Eq. (3.12) can be written as

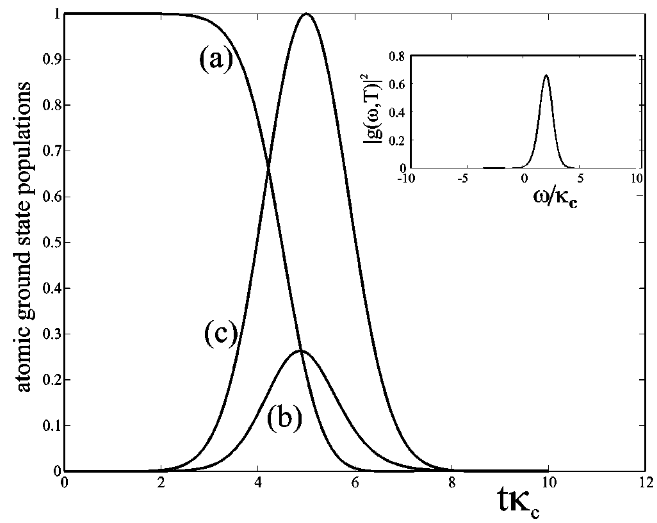

FIG. 6. Temporal evolution of the atomic populations (levels $|i\rangle$ (a) and $|f\rangle$ (b)) of one the of atomic Λ configurations and the time dependence of the applied external pump field (normalized) (c). The parameters are as listed in Sec. V B. Since the cavity decay constant exceeds all other rates we are in the overdamped regime, which explains why we do not observe Rabi oscillations. The inset shows the asymptotic spectral envelope of the created photon wave packet, i.e., $\kappa_c t=10$.

$$g(\omega,T)=\sqrt{\frac{\kappa_c}{\pi}}\int_0^T dt'\frac{r(t')}{\kappa_c}e^{-[\mu(t')-i(\omega-\Delta_s)t'+i\theta_c(t')]}.$$

We may now also check whether $|g(\omega)|^2$ is a normalized distribution as it should be in the case of reliable wave-packet generation. We find

$$\int d\omega|g(\omega)|^2=-\int_0^T dt'\dot{\mu}(t')e^{-\mu(t')}=1-e^{-\mu(T)}. \tag{5.3}$$

If $e^{-\mu(T)}$ is small, which is the underlying assumption, we have created a wave packet with near certainty. Subsequent wave packets can be generated by recycling the Λ-system back to state $|i\rangle$ through auxiliary atomic levels immediately before switching on the external light pulse again. Logical qubit states are generated by using a two-mode cavity and two Λ-systems, as explained in Sec. IV C and Fig. 5. We simply obtain two wavepackets each weighed with the appropriate probabilty amplitude and spectral envelopes g_α where $\alpha=0,1$ now accounts for the fact that both Λ systems need not be identical in terms of coupling strengths and detunings.

Figure 6 shows the evolution of the atomic populations for an external driving field with Gaussian pulse shape. The phase of the pump field was chosen in order to compensate the time dependent Stark shift introduced by the external field, i.e., $\theta_c(t)\equiv 0$. The inset displays the spectral envelope of the created one-photon wave packet. We observe that the wavepacket has a Gaussian distribution centered around $\omega_c+\Delta_s$. This shift is due to the Stark shift introduced by the cavity field which has not been compensated for. The condition $\mu(t) \gg 1$ ensures a full transfer of the atomic state from level $|i\rangle$ to the ancilla level $|f\rangle$ with concomitant photon wave-packet generation in the continuum outside the cavity by way of photon leakage from the cavity.

A. Wave-packet tailoring

The present proposal offers great control over the shape of the generated wave packet since we may manipulate both the amplitudes and phases of the external fields used in the generation process. This provides two external knobs which may be used to independently tailor each outgoing logical wave-packet component.

An interesting question arising in this context is that of *wave-packet reverse engineering*. Suppose we wish to create a wavepacket with a certain pulse shape. It is then crucial to find a way to determine the temporal behavior of amplitude and phase of the external field whose action will eventually produce the desired wave packet.

Let us refer to the desired temporal pulse shape of the wave packet to be generated as $\tilde{g}(t)$:

$$\tilde{g}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \;\; g(\omega, T) e^{-i\omega t}. \tag{5.4}$$

As shown previously, the spectral envelope of the wave packet is determined by the Fourier transform of the coefficient $C_f(t)$. We thus find

$$\begin{aligned} \tilde{g}(t) &= \sqrt{\frac{\kappa_c}{2\pi^2}} \int_0^T dt' C_f(t') e^{-i\Delta_s t'} \int d\nu \;\; e^{-i\nu(t-t')} \\ &= \sqrt{2\kappa_c} C_f(t) e^{-i\Delta_s t}, \end{aligned} \tag{5.5}$$

with $\tilde{g}(t)$ being nonzero only for $t \in [0,T]$. In the general case the time dependence of the Rabi frequency and phase of the laser pulse can in principle be determined from Eqs. (5.2) although this might represent quite a challenge.

A simple relation may, however, be established in the bad cavity limit, for which we obtain the following relations:

$$\frac{r^2(t)}{\kappa_c} = \frac{|\tilde{g}(t)|^2/2}{1 - \int_0^t |\tilde{g}(t')|^2 dt'}, \tag{5.6}$$

$$\dot{\theta}_c(t) + \Delta_s = i \frac{\left(\frac{d\tilde{g}(t)}{dt} \tilde{g}^*(t) - \frac{d\tilde{g}^*(t)}{dt} \tilde{g}(t) \right)}{2|\tilde{g}(t)|^2}. \tag{5.7}$$

We would like to note that in the the limit of an overdamped atom-cavity system the presence of the atom can be accounted for by an effective model which comprises an empty cavity (prepared initially in a one-photon Fock state) with time dependent cavity decay constant and center frequency, cf. [22].

B. Decoherence

Up to now, we have considered an idealized model with no sources of errors present during the generation of the entangled multiphoton wave packets. A rough estimate of the qubit generation rate our scheme would seem capable of is given by the inverse of the duration of a generation/preparation sequence. For a classical pump field $\Omega(t)$ with a peak frequency $\Omega_0 = 55$ Mhz and a Gaussian pulse shape, $\delta = 1.5$ GHz, $g = 55$ Mhz, and $\kappa_c = 50$ Mhz one-photon pulse durations of around 10 cavity lifetimes are feasible, as can be seen from Fig. 6. Taking into account recycling and reinitialization of the medium a conservative estimate would yield a generation rate of around 1 MHz. However, the maximum number n of entangled photon wave packets (qubits) that our scheme can generate with a certain fidelity is limited by technical imperfections, atomic spontaneous emission, and random loss of photons inside the output coupler (cavity mirror). Below we will analyze the impact of each of these perturbations on the reliability with which a certain state can be generated from the point of view of the model just considered.

1. Laser phase and amplitude fluctuations

In principle, stabilization of laser phase fluctuations below 1 kHz represents a technical challenge. In the present scheme, the state produced after each cycle only depends on the phase difference between the laser beams driving both transitions in Fig. 5. Note that the logical encoding takes place in the polarization state of the photon wave packets. Therefore both driving fields can be derived from the same source (there is no need to distinguish logical components by frequency). Thus fluctuations in the phase difference of driving fields are effectively suppressed. Amplitude fluctuations cause a distortion of the pulse form and lead to incomplete population transfer. An estimate gives that $n \ll I/\Delta I \sim 10^4$, where $\Delta I/I$ are the relative intensity fluctuations.

2. Spontaneous emission during the atomic transfer

Spontaneous emission from the auxiliary levels $|r\rangle_\alpha$ at rate Γ is quenched by choosing a large detuning $|\delta|$ which leads to an effective decay rate $\Gamma_{\text{eff}} = \Gamma\Omega^2/4\delta^2$ with $\Omega = \max(\Omega_\alpha, g_\alpha)$. For the above mentioned parameters and $\Gamma = 5$ MHz the probability of spontaneous emission per cycle is $< 10^{-3}$. So even in the case of a 100 entangled qubits to be generated the probability for a single spontaneous emission event remains small.

3. Absorption in the cavity mirrors

Photon absorption in the cavity mirrors is an essential effect for high-Q optical cavities. In general, it leads to two types of errors: Photon absorption and concomitant destruction of the entanglement. The first type of errors are evaded by using postselection schemes, which correspond here to discarding sequences with a number of detected photons smaller than n. State distortion [25] can occur even in the absence of loss of a photon according to

$$|\Psi\rangle=\sum_{x\in\{0,1\}^n} q_x|x\rangle \to \sum_{x\in\{0,1\}^n} q_x e^{-\kappa_1 n_1(x)T-\kappa_0 n_0(x)T}|x\rangle$$
$$=e^{-\kappa_0 nT}\sum_{x\in\{0,1\}^n} q_x e^{-(\kappa_1-\kappa_0)Tn_1(x)}|x\rangle, \qquad (5.8)$$

where the x are binary representations of different photon states, $n_1(x)$ and $n_0(x)$ are the numbers of ones and zeros contained in x $[n_1(x)+n_0(x)=n]$, respectively, and $\kappa_{0,1}$ are the loss rates for modes 0 and 1, respectively. Errors as in Eq. (5.8) vanish for a cavity with equal absorption rates for both polarizations, i.e., $\kappa_0 \simeq \kappa_1$.

4. Atomic motion

The optimal way to suppress fluctuations induced by the motion of the atom is to place the atom at an antinode of both the cavity modes and the laser beam (which therefore have to be standing waves). In these positions the effect of spatial variations is minimum. The system must operate in the Lamb-Dicke regime, where the size of the atomic wave packet L is smaller than the optical wavelength $\eta=L/\lambda \ll 1$ and the population left behind in the wrong level is $\eta^4/4$, which puts a limit on the number of photon pulses $n \ll 4/\eta^4$.

5. Imperfections in the recycling of states

Rather unexpectedly, the most prominent source of decoherence is due to imperfections in the recycling between the ground states $|f_\alpha\rangle \to |i_\alpha\rangle$. The required π pulses could be ill timed or their effect influenced by the motion of the atom. Moreover, there could be a dephasing between the probability amplitudes of the states $|f_\alpha\rangle$ of the two Λ systems due to laser fluctuations in the Raman transitions. To assess their importance we assume the following imperfect mapping in each of the generation sequences:

$$|f_\alpha\rangle \to A_\alpha|i_\alpha\rangle + B_\alpha|f_\alpha\rangle,$$

where $A_\alpha=(1-\epsilon_\alpha)\exp(i\delta_\alpha)$. When preparing an n-qubit MES the resulting state is, for fixed A_α^j, of the form

$$|\Psi\rangle=(A_0^1A_0^2,\ldots,A_0^n|00,\ldots,0\rangle+A_1^1A_1^2,\ldots,A_1^n|11,\ldots,1\rangle + \text{others})/\sqrt{2},$$

where ''others'' refers to states containing less than n photons (qubits). We assume that the magnitude of ϵ_α is evenly distributed over a range of $[0,\epsilon_m]$. The dephasing angle is evenly distributed over $[-\delta_m,\delta_m]\pi$. The final density matrix is given as an average over different realizations, $\rho=\text{avg}[|\Psi\rangle\langle\Psi|]$. In Fig. 7 we plot the fidelity $\mathcal{F}(n)$ of an n-qubit MES produced by our model source. The fidelity is defined as

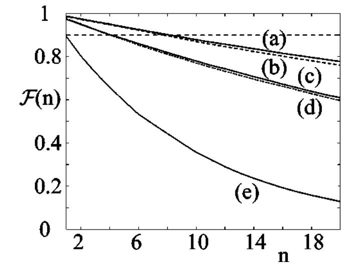

FIG. 7. Ensemble averaged fidelity as a function of the number of qubits n. Curves (a), (c), and (e) assume $\delta_m=0$ and $\epsilon_m=0.0125,0.025,0.1$, respectively. Curves (b) and (d) are the same as (a) and (c) with $\delta_m=0.05$.

$$\begin{aligned}\mathcal{F}(n)&=\tfrac{1}{2}(\langle 00,\ldots,0|+\langle 11,\ldots,1|)\\&\quad\times\rho(|00,\ldots,0\rangle|+|11,\ldots,1\rangle)\\&=\tfrac{1}{4}\text{Avg}[|A_0^1A_0^2,\ldots,A_0^n|^2+|A_1^1A_1^2,\ldots,A_1^n|^2\\&\quad+2\,\text{Re}(A_0^1A_0^2,\ldots,A_0^n(A_1^1A_1^2,\ldots,A_1^n)^*)].\end{aligned} \qquad (5.9)$$

and represents the average overlap between the desired n-photon wave packet and the one resulting from an imperfect sequence of recycling steps. Figure 7 shows that the process is rather robust against global dephasing (δ_α) but that the correct timing of the π-pulses is critical. From curve (a) we gather that for errors in the 2% range approximately 10 qubits can be created with a fidelity of 90%.

The fidelity $\mathcal{F}(n)$ drops off exponentially with the number of qubits, as can be seen from a simple analytical estimate in the case of Gaussian-distributed error. Let us consider for simplicity the case $\delta_\alpha^j=0$. We assume that the width σ of the error distribution is so small that we can extend the upper limit $\epsilon_\alpha^j=1$ in the integrations into infinity. The fidelity turns out to be

$$\begin{aligned}\mathcal{F}(n)&=\frac{1}{4}\int_0^\infty \sum_{\alpha=0,1}\prod_{j=1}^n \frac{2}{\sigma\sqrt{\pi}} d\epsilon_\alpha^j e^{-\epsilon_\alpha^{j\,2}/\sigma^2}\left[\prod_{j=1}^n (1-\epsilon_0^j)^2\right.\\&\quad\left.+\prod_{j=1}^n (1-\epsilon_1^j)^2+2\prod_{j=1}^n (1-\epsilon_0^j)(1-\epsilon_1^j)\right]\\&=\frac{1}{2}\left[\frac{2^n}{\sigma^n\sqrt{\pi}^n}\left[\int_0^\infty d\epsilon e^{-\epsilon^2/\sigma^2}(1-\epsilon)^2\right]^n\right.\\&\quad\left.+\left[\int_0^\infty d\epsilon e^{-\epsilon^2/\sigma^2}(1-\epsilon)\right]^{2n}\right].\end{aligned} \qquad (5.10)$$

For $\sigma \ll 1$ this yields $\mathcal{F}(n)\simeq[1-2\sigma/\sqrt{\pi}]^n$. For large n we obtain the exponential decay

$$\mathcal{F}(n)\sim e^{-2\sigma/\sqrt{\pi}\,n}. \qquad (5.11)$$

In accordance with the simulation, $\sigma=0.0125$ gives $\mathcal{F}(10) \sim 0.87$.

VI. SUMMARY

We have presented a general proposal for the controlled generation of entangled n-qubit states, where nonoverlapping one-photon wave packets are identified with the logical qubits. The theory developed is general although a CQED based implementation seems most promising from the present point of view. However, it can be easily adapted to other implementations, for instance quantum dots or single atoms embedded in a host material [18], which may become available in the future as quantum systems with long coherence times. We have also given an explicit example of a CQED based implementation, where the medium is an atom with two Λ-type level configurations. The logical qubit states are identified with two orthogonal transverse polarization states (e.g., $\sigma_{+/-}$ of the cavity and reservoir modes. Our investigation of decoherence in this model indicates that such a proposal is in principle experimentally doable with state-of-the-art equipment. It could thus form the experimental basis for multiparty communication in future quantum networks. Of particular interest are several recent proposals [16,17] which have suggested to make use of many-qubit entangled states in order to share quantum information between several parties. An example is quasi-simultaneous transfer of information to n clients using an n-qubit MES. In this way a key to be used later in an RSA encryption scheme is transferred to n parties in one fell swoop. As has been shown by Molotkov and Nazin [15] such a procedure not only implements key distribution but also allows one to check efficiently for the presence of an eavesdropper. Another interesting application is a controlled quantum communciation network, which consists of a single ''provider'' (P) who is connected to the clients (A–D) via photonic channels [8]. The provider creates entangled photonic qubits on request and transmits them to the clients, e.g., A and B in Fig. 8. Upon transmission all three parties involved share an entangled three-qubit state. In order for A or B to teleport the state of an additional qubit between each other the central station P has to measure the state of its qubit (which would be the internal state of the medium) and disclose the result to A and B. The advantage of such a scheme is its relatively simple topology which eliminates the need for photonic channels between the individual nodes A-D. Moreover, no successful transfer of information is possible without the consent of the provider. We have thus a network in which the transfer of information between two parties is always controlled by a third one. This may be of practical importance if the time at which information transfer takes place should be determined by the third party. Another scenario in which GHZ states and their higher order analogues may be useful is quantum secret sharing [16]. The provider may use his qubit to teleport the state of another qubit he holds to clients A and B. These two now share a Bell state but they are only able to extract the teleported state if they are willing to cooperate. This could be useful in situations when neither of them can be completely trusted on his own or if at the time of teleportation it is not yet clear who of the two is supposed to eventually receive the information.

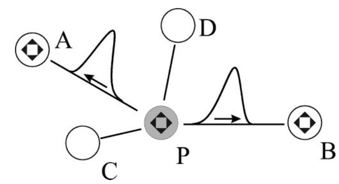

FIG. 8. Topology of a quantum communication network which consists of a single provider of entanglement P and several clients A–D.

ACKNOWLEDGMENTS

This work was supported by the Austrian Research Foundation under grant no. S06514-TEC, the European TMR network ERB-FMRX-CT96-0087. C.S. thanks Fundacion Andes and FONDECYT 1970995 (Chile) for support, and P.T. the TMR Program of the European Commission (ERB-FMBI-CT98-3061).

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