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Ideal gases in time-dependent traps

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We investigate, theoretically, the properties of an ideal trapped gas in a time-dependent harmonic potential. Using a scaling formalism, we are able to present exact analytical results for two important classes of experiments: free expansion of the gas upon release of the trap and the response of the gas to a harmonic modulation of the trapping potential. We present specific results relevant to current experiments on trapped fermions.

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Recently, the onset of Fermi degeneracy has been observed for a trapped ultracold spin-polarized gas of ^{40}K atoms [1]. Such a weakly interacting, quantum-degenerate Fermi-Dirac gas provides a new platform for exploring fundamental quantum many-body physics, and several theoretical studies of the equilibrium properties of these gases have been published [2].

One class of experiments that is compatible with standard atom trapping protocols, and which has gleaned valuable information on the dynamics of dilute Bose-Einstein condensates (BECs), involves monitoring the response of the gas to a change in trapping potential. For example, transient modulation of the trapping potential induces free ringing of the gas, which in the case of BECs has led to a direct determination of low-lying regions of the quasiparticle spectrum [3]. Complete release of the trapping potential enables one to view the free expansion of the gas; early on, this established the essential validity of the time-dependent Gross-Pitaevski equation for describing the dynamics of the present generation of gaseous BECs [4], a matter that has subsequently been put to stringent quantitative tests [5].

A trapped, single-component gas of ultracold fermionic atoms can, for experiments of current interest, be considered ideal (noninteracting), since binary atomic collisions can only occur in partial waves of angular momentum $l>0$, which are strongly suppressed by the centrifugal barrier [6]. This Rapid Communication presents a theoretical study of the dynamics of such an ideal gas in a time-dependent harmonic trap. Using a scaling formalism similar to that which has been successfully applied to trapped BECs [7], we derive exact results for the above-mentioned excitation spectroscopy and release experiments. In the cases studied here, the only effect of quantum statistics is to establish the initial equilibrium distribution of particles in the trap; the subsequent time evolution of this distribution, under changes of the trapping potential that preserve its harmonicity, is rigorously equivalent to that of an ensemble of noninteracting particles, independent of statistics. We present a simple formula that describes the free expansion of such an ideal gas, and our results suggest an alternative approach to the problem of quantitative thermometry in the nanokelvin regime. Due to their weak pair interactions, single-component Fermi-Dirac gases are attractive candidates for nanokelvin thermometry; with an improved theoretical understanding of finite-temperature properties of BECs [8], which are much more robust candidates for experiment at present, we can envisage a direct comparison of temperatures of ultracold Bose-Einstein and Fermi-Dirac gases. We also examine in detail the nonlinear response of the gas to a harmonic oscillation of the trapping frequency. This reveals a domain of driving frequencies and amplitudes that generates a resonant response of the gas. Since interactions in general play a smaller role for Fermi-Dirac vs Bose-Einstein systems, the present approach may provide a useful starting point for consideration of the dynamics of multiple-component Fermi gases.

We start with a derivation of the equations describing the scaling properties of an ideal gas trapped in a time-dependent harmonic potential. Consider a classical particle of mass m trapped in a potential $V(\mathbf{r},t)=m\Sigma_j\omega_j(t)^2r_j^2/2$ with $j=x,y,z$ denoting the three spatial dimensions. We set $\omega_j(t)=\omega_{0j}$ for $t\leqslant 0$, i.e., the trap potential is constant prior to $t=0$. Newton's law is then expressed by $\dot{k}_j=-m\omega_j(t)^2r_j$, with $k_j=m\dot{x}_j$. By invoking a scaling transformation, $q_j=r_j/\gamma_j(t)$ and $p_j=\gamma_j(t)k_j-\dot{\gamma}_j(t)mr_j$, we obtain $\partial_{\tau_j}p_j=-m\omega_{0j}^2q_j$, with $\tau_j(t)=\int^t dt'\gamma_j(t')^{-2}$, where the scaling parameters $\gamma_j(t)$ satisfy the equations

$$\ddot{\gamma}_j(t)=\frac{\omega_{0j}^2}{\gamma_j(t)^3}-\omega_j(t)^2\gamma_j(t). \tag{1}$$

In quantum mechanics, the Heisenberg equation for a noninteracting gas in a time-dependent harmonic potential takes the form $i\hbar\partial_t\hat{\psi}(\mathbf{r},t)=[-(\hbar^2/2m)\nabla^2+V(\mathbf{r},t)]\hat{\psi}(\mathbf{r},t)$, where $\hat{\psi}(\mathbf{r})$ is the field operator for an atom at position $\mathbf{r}$, which obeys the usual Fermi-Dirac anticommutation relations. The quantum-mechanical analog of the classical rescaling described above is obtained by writing [7]

$$\hat{\psi}(\mathbf{r},t)=\frac{\hat{\Phi}(\mathbf{q}(t))}{\sqrt{\gamma_x\gamma_y\gamma_z}}\exp\left(im\sum_j r_j^2\dot{\gamma}_j/2\hbar\gamma_j\right), \tag{2}$$

with $q_j(t)=r_j/\gamma_j(t)$ as defined previously. If each γ_j satisfies Eq. (1), by writing $\hat{\Phi}(\mathbf{q}(t))=\Pi_j\hat{\phi}(q_j(t))$, we obtain

$$i\hbar\partial_{\tau_j}\hat{\phi}(q_j(t))=\left[-\frac{\hbar^2}{2m}\partial_{q_j}^2+\frac{1}{2}m\omega_{0j}^2q_j^2\right]\hat{\phi}(q_j(t)). \tag{3}$$

Thus the time-dependent problem has been reduced to the trivial case of evolution in a time-independent harmonic trap in the rescaled variables (τ_j, q_j). Determination of these variables requires only the solution of the three ordinary uncoupled differential equations expressed by Eq. (1). From Eq. (2), it follows that the density $\rho(\mathbf{r},t)$ of the gas for a given time t is given by

$$\rho(\mathbf{r},t) \equiv \langle \hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\rangle = \frac{1}{\gamma_x(t)\gamma_y(t)\gamma_z(t)}\rho_0(\mathbf{q}(t)), \tag{4}$$

where $\rho_0(\mathbf{r})$ is the particle density for $t=0$. Thus we can calculate $\rho(\mathbf{r},t)$ for any time $t>0$ for modulated frequencies $\omega_j(t)$ by solving Eq. (1), subject to the boundary conditions $\gamma_j(0)=1$ and $\dot{\gamma}_j=0$. It has been shown previously [7] that Eq. (1) also describes a two-dimensional, nonideal BEC subject to isotropic variations of the trap potential, with Eq. (4) also being applicable if ρ_0 is obtained by a solution of the Gross-Pitaevski equation.

We now analyze the solution of Eq. (1) for two cases of experimental relevance: free expansion of the gas upon release of the trapping potential and response of the gas to harmonic modulation of the trapping potential.

To model a free expansion experiment, we take $\omega_j(t)=0$ for $t>0$. Solving Eq. (1) with the requirement that the gas be in equilibrium for $t\leqslant 0$, we obtain $\gamma_j(t)=1$ for $t\leqslant 0$ and

$$\gamma_j(t) = \sqrt{1+\omega_{0j}^2 t^2} \tag{5}$$

for $t>0$. The time-dependent width of the cloud after the trap has been dropped is given by $[\langle \hat{r}_j^2\rangle(t)]^{1/2} = \gamma_j(t)[\langle \hat{r}_j^2\rangle(0)]^{1/2}$. To describe the aspect ratio $\alpha(t)$ of a cylindrically symmetric cloud, we find

$$\alpha(t) \equiv \sqrt{\frac{\langle x^2\rangle(t)}{\langle z^2\rangle(t)}} = \alpha(0)\sqrt{\frac{1+\omega_{0x}^2 t^2}{1+\omega_{0z}^2 t^2}}. \tag{6}$$

We now apply these results to an ideal gas in two limiting cases.

We first treat the case of $T=0$, with the chemical potential $\mu_F(T=0)/\hbar\omega_{0j}\gg 1$ for $j=x,y,z$. This corresponds to the semiclassical limit appropriate to current experiments in which the number of atoms N is greater than a few hundred. In this limit, the initial density profile is well described by the Thomas-Fermi (TF) approximation [2]. This gives the integrated density $\rho(x,z,t) \equiv \int dy\, \rho(\mathbf{r},t)$,

$$\rho(x,z,t) = \frac{m\mu_F}{4\pi\hbar^3\omega_{0y}\gamma_x\gamma_z}\left[1-\frac{x^2/\gamma_x^2+\lambda_z^2 z^2/\gamma_z^2}{R_F^2}\right]^2, \tag{7}$$

with $\lambda_z \equiv \omega_{0z}/\omega_{0x}$ and $R_F=\sqrt{2\mu_F/m\omega_{0x}^2}$. We see that the cloud becomes isotropic for $\omega_{0j}t\gg 1$. This is consistent with the initial isotropic momentum distribution implicit in the TF approximation. We have $\alpha(0)=\lambda_z$ and Eq. (6) yields $\alpha(t)\to 1$ as $t\to\infty$. In Fig. 1, we plot $\alpha(t)$ as given in Eq. (6) for $T=0$, $\mu_F(T=0)=20\hbar\omega_{0x}$, and $t=0$ (a) and $t=20/\omega_{0x}$ (b).

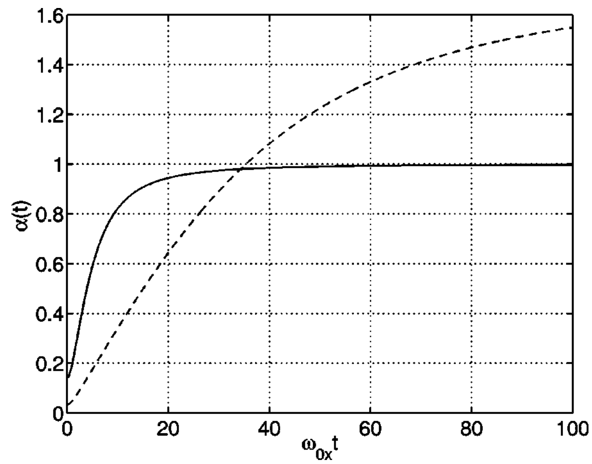

FIG. 1. The aspect ratio for a freely expanding gas. The solid line is for $\lambda_z=19.5/137$, $\lambda_y=1$ and $\mu_F=20\hbar\omega_{0x}$. The dashed line is for $\lambda_z=\lambda_y=1/50$ and $\mu_F=\hbar\omega_{0x}$.

We have taken $\lambda_z=19.5/137$ and $\lambda_y\equiv\omega_{0y}/\omega_{0x}=1$ corresponding to current experiments on trapped ^{40}K atoms [1].

Note that $\alpha(0)=\lambda_z$ for any initial density distribution of the form $\rho_0(\mathbf{r})=f(\Sigma_j\omega_{0j}^2 r_j^2)$. Such initial distributions will become isotropic in a free expansion experiment. For instance, for $T>T_F\equiv\mu_F(T=0)/k_B$, the density is well described by a classical Gaussian profile, i.e., $\rho_0(\mathbf{r})\propto\exp[-\beta(\mu_F-m\Sigma_j\omega_{0j}^2 r_j^2/2)]$, and Fig. 1 thus describes a trapped gas of fermions for any T within the TF approximation. This means that one cannot detect the onset of Fermi degeneracy by looking at the aspect ratio of the expanding gas.

As an example where $\rho_0(\mathbf{r})\neq f(\Sigma_j\omega_{0j}^2 r_j^2)$, we now consider a case with $\mu_F/\hbar\omega_{0x}<3/2$ such that only one level is occupied in the x direction and $\mu_F/\hbar\omega_{0j}\gg 1$ for $j=y,z$. The gas is initially strongly confined in the x direction. The integrated density profile is then

$$\rho(x,z,t)\propto e^{-x^2/l_\hbar^2\gamma_x^2}\left(1-\frac{m\omega_{0z}^2 z^2/\gamma_z^2}{2\mu_F-\hbar\omega_{0x}}\right)^{3/2} \tag{8}$$

for $t\geqslant 0$, yielding $\alpha(0)=\sqrt{3\hbar\omega_{0x}}\lambda_z/\sqrt{2\mu_F-\hbar\omega_{0x}}$. In Fig. 1, we plot the aspect ratio for a free expansion for $\mu_F=\hbar\omega_{0x}$ and $\lambda_y=\lambda_z=1/50$ using Eq. (6). We have $\alpha(0)=\sqrt{3}/50$ and $\alpha(t)\to\sqrt{3}$ for $t\to\infty$. The gas, which is initially strongly confined in the x direction will for $\omega_{0j}t\gg 1$ become most confined in the y, and z directions. This is, of course, a direct reflection of the uncertainty relation giving higher average momentum in the x direction. However, observation of such a quantum effect requires a highly anisotropic trap. For the case of 10^4 trapped atoms we would require $\lambda_y=\lambda_z\simeq 1/250$. The anisotropy of the expanded cloud is due to the Heisenberg uncertainty principle, whose effect for Fermi-Dirac particles diminishes as N increases and numerous quantum states become populated. This contrasts with the Bose-Einstein case, where there is a macroscopic population of a single initial quantum state and the anisotropy is preserved with increasing N.

From a measurement of the density at any time t under free expansion, it is straightforward to determine the initial density distribution using Eqs. (4) and (5). This suggests that a trapped, single-component Fermi-Dirac gas could serve as a low–temperature thermometer. Assuming the gas is in thermodynamic equilibrium for $t \leqslant 0$, one could infer the temperature of the gas from a knowledge of its density distribution, the calculation of which is a trivial problem of summing over fractionally occupied trap levels [2]. Hence a determination of T is simply a matter of fitting a calculated density to the measured $\rho(\mathbf{r})$.

We now theoretically examine the nonlinear response of a trapped gas to a harmonic modulation of the trapping potential. The noninteracting case treated here is in some sense opposite to the hydrodynamic limit we have treated elsewhere [9]. To model a typical experiment, we assume that the trapping frequencies take the form $\omega_j(t)^2 = \omega_{0j}^2[1 - 2\eta\cos(\omega_D t)]$ for $t>0$, with ω_D being the driving frequency and η the driving amplitude. Instead of solving Eq. (1) with this form for $\omega_j(t)$, it turns out to be easier to go back to the original classical equation of motion $\ddot{r}_j = -\omega_j(t)^2 r_j$ by using $r_j(t) = q_j(t)\gamma_j(t)$ and $q_j(t) = \exp[\pm i\omega_{0j}\tau(t)]$. Thus by writing $\xi_j(t) = \gamma_j(t)\exp[i\omega_{0j}\tau(t)]$ and $\chi = \omega_D t/2$ we obtain

$$\partial_\chi^2 \xi_j + [a - 2q\cos(2\chi)]\xi_j = 0 \tag{9}$$

with $a = 4/\tilde{\omega}^2$, $q = 4\eta/\tilde{\omega}^2$, and $\tilde{\omega} = \omega_D/\omega_{0j}$. Equation (9) is a variant of Mathieu's equation, whose properties have been extensively studied [10]. We consider the case when the trap frequency is modulated during a finite interval $0 < t \leqslant t_D$, and then returned to its original value. The subsequent motion of the cloud is described by a solution of the time-independent problem $\omega_j(t) = \omega_{0j}$ with arbitrary initial conditions.

$$\gamma_j(t) = \sqrt{\sqrt{E^2-1}\sin(2\omega_{0j}t + c) + E}, \tag{10}$$

where $E = (\omega_{0j}^{-2}\dot{\gamma}_j^2 + \gamma_j^2 + \gamma_j^{-2})/2 \geqslant 1$ is a conserved quantity for $t > t_D$, $c = \arcsin[(\gamma_{0j}^2 - E)/\sqrt{E^2-1}]$, and $\gamma_{0j} = \gamma_j(t_D)$ is the value of $\gamma_j(t)$ immediately after the modulation ceases. Equation (10) has a discrete frequency spectrum (frequencies of $2n\omega_{0j}$ with $n = 0,1,2\ldots$), as expected for a noninteracting gas in a harmonic trap. The linear response limit is recovered by letting $E \to 1_+$ in Eq. (10) yield $\gamma_j(t) = 1 + \delta\sin(2\omega_{0j}t)$.

In essence, the problem of predicting the response of the gas to a harmonic driving with frequency ω_D and amplitude η is reduced to an analysis of the well-known solutions to Mathieu's equation. In the parameter space (a,q), there are regions where the solutions of Eq. (9) are unstable, i.e., their amplitude increases exponentially with time. Also, there are stable regions where the solutions remain bounded. The solutions on the boundaries between these regions are the Mathieu functions [10]. Using $|\gamma_j(t)| = |\xi_j(t)|$, $a = 4/\tilde{\omega}^2$, and $q = 4\eta/\tilde{\omega}^2$, this means that for certain regions in the (ω_D, η) space the response of the gas diverges as the driving time t_D increases, i.e., there is a resonant response, whereas in other regions the response of the gas remains finite for any value of t_D. For $a = n^2$ with $n = 0,1,2\ldots$, the solutions to Eq. (9) diverge in time for an arbitrarily small q [10]. Thus for the fractions

$$\omega_D = 2\omega_{0j}/n, \quad n = 1,2,3\ldots, \tag{11}$$

the response of the gas is resonant for an arbitrarily small driving amplitude and the amplitude of its oscillations will diverge with the driving time. In terms of Eq. (10), if the gas remains trapped in the time-independent potential $\omega_j(t) = \omega_{0j}$ after the driving $(t > t_D)$, we have $E \gg 1$, and the oscillations of the gas will be large and contain many harmonics. The resonance for $\omega_D = 2\omega_{0j}$ is, of course, the usual excitation frequency for an even-parity perturbation for a noninteracting gas. However, the resonances for $\omega_D = \omega_{0j}, 2\omega_{0j}/3, \omega_{0j}/2$, etc., do not correspond to new modes. They simply reflect the fact that for these driving frequencies, the trapping potential is doing resonant work on the gas.

We now examine the width of the unstable (resonant) regions around $\omega_D = 2\omega_{0j}/n$ for $\eta \to 0$. For the $\omega_D = 2\omega_{0j}$ resonance, the unstable region of the Mathieu equation is bounded by $1 - q < a < 1 + q$ for $q \to 0$ [10]. This implies that the response of the gas is divergent for driving frequencies and amplitudes that satisfy

$$2 - \eta < \tilde{\omega} < 2 + \eta, \quad \eta \ll 1. \tag{12}$$

Hence the resonance region of the gas for finite but small driving amplitude η has a reasonable width and should be relatively easy to access experimentally. Likewise, for the $\omega_D = \omega_{0j}$ resonance, the unstable region is bounded by

$$1 - 5\eta^2/6 < \tilde{\omega} < 1 + \eta^2/6, \quad \eta \ll 1, \tag{13}$$

and it should be experimentally accessible. For the lower-frequency resonances $(\omega_D = 2\omega_{0j}/n,\ n \geqslant 3)$, it turns out that the resonance regions are very narrow for $\eta \to 0$ as the boundary lines between the stable and unstable solutions of Eq. (9) only differ by terms of order q^3 or higher [10]. These resonances are therefore less attractive experimental candidates for small-amplitude driving. One could instead increase the driving amplitude η for a given frequency ω_D; for η sufficiently large, an unstable region is reached and the response of the gas diverges. The above results are illustrated in Fig. 2, where we plot the response of the gas as a function of driving frequency ω_D and amplitude η. The lines in the plot separate regions where the response of the gas remains finite from regions where the oscillations of the gas diverge with driving time. For example, if one modulates the external potential with an off-resonance frequency of $\tilde{\omega} = 5/3$, the response of the gas remains finite for small driving amplitudes, whereas it diverges for $\eta \gtrsim 0.33$. The plot is generated using the well-known properties of the Mathieu equation and the mapping $(a,q) \to (4/\tilde{\omega}^2, 4\eta/\tilde{\omega}^2)$. Note the unstable (resonance) regions for $\eta \to 0$ for $\omega_D = 2\omega_{0j}/n$. The regions have zero width for $\eta \to 0$ for $n \geqslant 3$ as predicted above. For increasing driving amplitude η, the unstable regions grow as expected. For $\omega_D \to 0$, the response of the gas for $\eta \lesssim 0.5$ is finite, whereas it diverges for $\eta \gtrsim 0.5$. Physically, this corre-

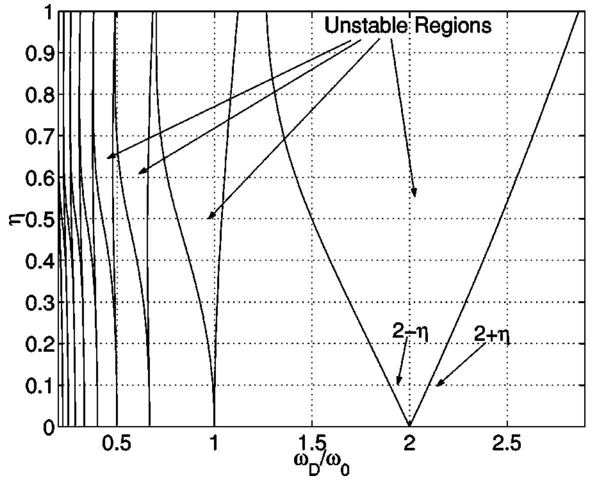

FIG. 2. The regions of stability/unstability for the response of a noninteracting gas to a modulation of the trapping potential with frequency ω_D and amplitude η.

sponds to the fact that for $\omega_D/\omega_{0i} \ll 1$ and $\eta < 0.5$, the gas returns adiabatically to its initial state after one period of trap modulation, and there is no net work done. Contrary to this, for $\eta > 0.5$ the trapping potential becomes inverted for certain times, resulting in a divergent response of the gas. Mathematically, the transition region for $\eta \simeq 0.5$ comes from the fact that for $a \sim 2q$ the even and odd periodic solutions (Mathieu functions) of Eq. (9) start to differ in ''energy'' (the Mathieu characteristic value a [10]), as the tunneling between successive minima of the potential $\cos(2\chi)$ becomes significant.

It should be noted that the above results are valid for any ideal gas (Bose or Fermi) at any T. They are in that sense universal: The effects of quantum statistics and finite temperature enter the problem only through the initial distribution of the gas. It is worth noting that, if one chooses to describe the results of this section in the language of perturbation theory, our scaling formalism provides results summed to infinite order in powers of the characteristic parameter of the perturbation (the driving amplitude). It is valid even in such highly nonlinear cases as when the trapping potential is inverted for certain times ($\eta > 0.5$). Equation (11) clearly indicates that the analysis contains nonlinear dynamics: The resonance at $\omega_D = 2\omega_{0j}/n$ for $n=1$ can easily be explained within linear response theory. However, the resonances for $n \geqslant 2$ are due to divergencies in increasing orders in perturbation theory and they are nontrivial. Hence we believe that it would be of interest to experimentally verify the analysis presented here as an example of exact results concerning nonlinear dynamics.

In conclusion, using a scaling formalism we have been able to derive analytical results for the dynamics of ideal gases trapped in time-dependent harmonic traps. We have concentrated on two important classes of experiments. For free expansion, we showed how the initial density profile of the gas can easily be determined from a measurement of the density profile at any time t after the trap has been dropped. We have proposed a low-temperature thermometer based on these results. Also, we showed how the problem of determining the nonlinear response of the gas to a harmonic modulation of the trapping frequency can be mapped to an analysis of the well-known properties of the solutions to Mathieu's equation. We identified regions in (ω_D, η) space where the response of the gas was divergent with the modulation time. In particular, we were able to predict nontrivial unstable regions for $\eta \to 0$, reflecting the fact that the trap is doing resonant work on the gas. Since ultracold spin-polarized fermions are noninteracting to a very good approximation, our results should be directly relevant for current experiments in this very active field of research.

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