Parametric instability of a coupled atom-cavity system driven by a strong resonant pulse

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We investigate the behavior of a coupled atom-cavity system (a collection of two-level atoms inside a high-Q cavity) driven by a strong external resonant pulse. A different effect is predicted: parametric instability of a coupled atom-cavity system appears if the atomic number density in a cavity exceeds the threshold value that is proportional to the strong driving field. The increment of this instability is proportional to the atom-cavity coupling constant.

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The properties of atoms inside a cavity have been investigated in recent years by many authors [1,2]. Atoms inside a cavity exhibit many different radiative properties from those in free space, such as cavity-modified spontaneous emission (the spontaneous emission rate into the particular cavity mode is enhanced [3]; otherwise, it is inhibited [4]), and vacuum Rabi splitting. The phenomena of enhanced and inhibited spontaneous emission, resulting in a broadening or narrowing emission, have been observed experimentally in the microwave regime [5], the infrared regime [6], and the visible regime [7,8]. An oscillatory exchange of energy between the atom and the cavity field leads to a splitting of the spontaneous emission spectrum. Experiments exhibiting the phenomena of vacuum Rabi splitting in the strong coupling regime were performed in the optical regime [9,10]. Most experiments in strong-coupling cavity quantum electrodynamics with few atoms have been performed in the linear regime. Recently, a nonlinear spectroscopic investigation [11] of a strongly coupled atom-cavity system was also presented. Atomic radiative properties inside a cavity can also be modified by imposing a coherent driving field on the atoms. The steady-state response of a two-level atom in a cavity driven by an external field was investigated in [12,13].

The driven two-level atom in a cavity has been investigated, usually in steady state. It is interesting to consider the driving of a two-level atom in a cavity by a short pulse. In this paper we investigate the behavior of a coupled atom-cavity system strongly driven by a resonant pulse collection of two-level atoms inside a high-Q cavity. We reveal a different effect: the instability of a coupled atom-cavity system driven by a strong resonant pulse appears if the atom-cavity coupling constant exceeds a critical value. The coupling constant is proportional to the atomic number density in a cavity; therefore, there is the threshold value of the atomic density in a cavity for the appearance of the instability. The increment of this instability is proportional to the atom-cavity coupling constant.

We consider a strongly driven collection of N two-level atoms inside a single-mode cavity, where $N_0=N/V$ is the atomic number density and V is the effective cavity-mode volume. A two-level atom with the atomic transition frequency ω_{21} is coupled to a single-mode cavity field Ω, and the cavity mode is tuned to resonance with the frequency ω_{21}; the atom is driven directly by an external field χ of frequency ω; the Rabi frequency of the total field is $\Omega+\chi$.

We assume that the external field is in the form of a short pulse of the square-wave form [$\chi(t)=\chi=\text{const}$ for $0\leqslant t\leqslant\tau$ and $\chi(t)=0$ for other times], whose length τ obeys the condition $\tau<T_1$, T_2, where T_1 is the population decay time and T_2 is the dipole dephasing time, and our consideration is related to times $t\leqslant\tau$. Note that the consideration for the external pulse of the non-square-wave form is given in the Appendix. We assume that the external field is strong $\chi\tau\gg 1$. The Maxwell-Bloch equations for the atom-cavity system driven by an external pulse are given by

$$\begin{aligned}
\dot{v}&=-\frac{v}{T_2}+(\Omega+\chi)w+\Delta u,\\
\dot{u}&=-\frac{u}{T_2}-\Delta v,\\
\dot{w}&=-\frac{w-w_0}{T_1}-(\Omega+\chi)v,\\
\dot{\Omega}+\gamma\Omega&=Av,
\end{aligned}\tag{1}$$

where the slowly varying part of the off-diagonal density matrix element $\sigma_{12}(t)$ is expressed in terms of two real quantities u and v as $2\sigma_{12}=u_{12}-iv_{12}$; $w=\sigma_{22}-\sigma_{11}$ is the population difference, $\Delta=\omega_{21}-\omega$ is the detuning, γ is the cavity damping rate, the coefficient

$$A=\frac{2\pi\omega N_0|d|^2}{h}$$

determines the coupling rate between the intracavity field and the collection of two-level atoms in a cavity, and d is the transition dipole moment.

The off-diagonal density matrix element $v(t)$ is also determined by the intracavity field $\Omega(t)$, and the intracavity field also depends on $v(t)$. As a result, there is the self-consistent behavior of the off-diagonal matrix element $v(t)$. This behavior is described by the equation of motion, which we derive from Eqs. (1) for a collection of N two-level atoms

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inside a high-Q cavity driven by a strong resonant $\Delta=0$ driving field $\chi \gg \Omega, \gamma, T_i$, where

$$\ddot{v}(t)+\frac{\dot{v}(t)}{T_2}+\chi^2 v(t)\left(1-\frac{A}{\chi^2}w(t)\right)=0. \tag{2}$$

This equation of motion contains the term with the time-dependent coefficient caused by the coupling. For a driving field $\chi \gg \Omega, \sqrt{A}$ the population difference $w(t)$ oscillates with the Rabi frequency χ:

$$w(t)=-\cos \chi t, \tag{3}$$

and we have the well-known differential Mathieu equation

$$\ddot{v}(t)+\frac{\dot{v}(t)}{T_2}+\chi^2 v(t)(1+h\cos \chi t)=0, \tag{4}$$

where $h \equiv (A/\chi^2) \ll 1$.

Without the atom-cavity coupling $h=0$ we have the damping oscillations (the damping Rabi nutations). However, due to the atom-cavity coupling the oscillator has the frequency with the small $h \ll 1$ oscillating part. The equation of the Mathieu type is known [14] to display parametric resonance close to the frequencies $\omega = 2\omega_0/n$, where ω is the frequency of the small oscillating part, ω_0 is the natural frequency of the oscillator, and order n is a positive integer [for $n=1$ there is first-order (main) parametric resonance]. In our case $\omega \equiv \chi$, $n=2$, and for second-order parametric instability of order $n=2$ we obtain

$$v(t)=\exp(st)\sin \chi t, \tag{5}$$

where

$$s=\frac{1}{2}\left[\left(\frac{A}{A_{\rm th}}\right)^2-1\right]T_2^{-1}, \quad A_{\rm th}^2 \equiv \frac{2\chi^3}{T_2}. \tag{6}$$

Because $\chi T_2 \gg 1$, there is

$$\frac{A_{\rm th}}{\chi^2}=\sqrt{\frac{2}{\chi T_2}} \ll 1.$$

For $A_{\rm th} < A \ll \chi^2$ we have $s \ll \chi$.

From Eq. (1) we derive the equation of motion of the intracavity field for an atom-cavity system driven by a strong resonant driving pulse $\chi \gg \Omega, \sqrt{A}$, where

$$\dot{\Omega}(t)-\gamma^2\Omega(t)=-A[\chi \cos \chi t+v(t)(\gamma+T_2^{-1})]. \tag{7}$$

Using the solution (5) and the initial condition for $\Omega(t)$, we obtain, for a high-Q cavity $\chi \gg \gamma$,

$$\Omega(t)=\frac{A(\gamma+T_2^{-1})}{\chi^2}\exp(st)\sin \chi t+\frac{A}{\chi}(\cos \chi t-1) + \Omega(0)\exp(-\gamma t). \tag{8}$$

The exponentially growing field $\Omega(t) \sim \exp(st)$ is proportional to the atom-cavity coupling A. The intracavity field $\Omega(t)$ for $t > \tau$ is

$$\Omega(t)=\Omega(\tau)\exp[-\gamma(t-\tau)]. \tag{9}$$

At the stability edge $A=A_{\rm th}$ for the instability, the parametric gain due to the atom-cavity coupling exactly balances the losses. As a result, we obtain that if the parameter A exceeds a critical value

$$A > A_{\rm th} \equiv \frac{2\chi^{3/2}}{\sqrt{T_2}}, \tag{10a}$$

and corresponding

$$N_0 > N_{0\rm th} \equiv \frac{h\chi^{3/2}}{\pi\sqrt{T_2}\omega_{21}|d|^2}, \tag{10b}$$

there is second-order parametric instability with the increment $s>0$. This instability appears if the atomic number density N_0 of two-level atoms in a cavity exceeds the threshold value $N_{0\rm th}$, which is proportional to the strong driving field. The positive net gain $s>0$ corresponds to the region of instability, where the parametric gain exceeds the losses. This parametric instability leads to the amplification of the intracavity field with the gain $s\tau$ under the action of pulse τ. The parametric amplification and the appearance of the exponentially growing field is possible if the atom-cavity coupling exceeds the threshold value. The gain $s\tau > 1$ under the action of pulse τ if

$$\frac{A}{A_{\rm th}} > \sqrt{2\frac{T_2}{\tau}+1}. \tag{11}$$

Note that including the atomic motion also requires the consideration of the spatial dependence of the field inside the cavity. We restricted our study to atomic motion along one axis (z axis) so that it would be necessary to consider only the z dependence of the field. This is in accordance with respect to the present cavity QED experiments. The atomic motion can be incorporated as [15] $f(z) \to f(vt)$, where v denotes the atomic velocity and $f(z)$ is the shape function of the field inside the cavity. The atomic motion in a cavity is characterized [16] by the parameter $b(t) \equiv \pi p v t/L$, where p stands for the number of half wavelengths of the field inside a cavity of length L in the z direction. If this parameter $b(\tau) \ll 1$, then a tipping angle χt, which determines the population difference $w(t)$ [Eq. (3)], has the small addition $\Delta\chi(t)t$

$$\frac{\Delta\chi(t)}{\chi} \simeq b(t) \sim \frac{v}{c}\omega t, \tag{12}$$

where c is the velocity of light. In the case $b(t) \ll 1$, the critical parameter $A_{\rm th}$ also has the small addition

$$\frac{\Delta A_{\rm th}}{A_{\rm th}} \sim \frac{3}{2}b(\tau). \tag{13}$$

Because this addition is positive, the instability has the greater threshold due to the atomic motion and the spatial dependence of the field inside the cavity.

Finally, we repeat the requirements to an external driving field and to a cavity: (i) a field is in the form of a pulse, $\tau < T_1, T_2$; (ii) a strong driving field, $\chi\tau \gg 1$, $\chi \gg T_i^{-1}$; (iii) a high-Q cavity $\gamma \ll \chi$, or $Q \gg \omega_{21}/\chi$, where Q is the cavity-mode quality factor; (iv) an atomic number density of two-level atoms $N_0 > N_{0\mathrm{th}}$ and corresponding atom cavity coupling constant $A_{\mathrm{th}} < A \ll \chi^2$.

For example, we will make rough numerical estimates. For the resonance line (i.e., the $3s \rightarrow 3p$ transition) of atomic sodium [17] $T_1 = 1.6\times10^{-8}$ s, $T_2 = 2T_1$ we assume a strong external driving optical pulse, with duration $\tau \sim 5\times10^{-9}$ s, amplitude $\chi \sim 3\times10^{10}$ CGSE (the power density $I \sim 3$ kW/cm^2), and a good [18] cavity with the cavity-mode quality factor $Q \sim 10^6$; then we obtain the threshold value of the atomic density $N_{0\mathrm{th}} \sim 3.5\times10^9$ cm^3 (we have $A_{\mathrm{th}} \sim 0.05\chi^2$) and the instability with the gain $s\tau \sim 2$ for $N_0 \sim 1.4\times10^{10}$ cm^3. Note that the Doppler width [8] is 12 MHz, and the inequality $b(\tau) \ll 1$ is fulfilled for $\tau \sim 5\times10^{-9}$ s very well.

In conclusion, we have shown that there is a different effect; parametric instability of a coupled atom-field system driven by a strong resonant pulse occurs if the atomic number density in a cavity exceeds the threshold value.

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APPENDIX

Let us consider the external pulse of length τ of the non-square-wave form

$$\chi(t) = \chi_0 \frac{t}{\tau_0} \qquad \text{for} \quad t < \tau_0,$$

$$\chi(t) = \chi_0 \qquad \text{for} \quad \tau_0 \leqslant t \leqslant -\tau_0,$$

$$\chi(t) = \chi_0 \frac{\tau - t}{\tau_0} \qquad \text{for} \quad \tau - \tau_0 < t \leqslant \tau.$$

We suppose that the edges of a pulse satisfy the condition $\tau_0 \ll \tau$ and $\chi_0 \tau \gg 1$. The consideration of the paper concerns the time-independent external pulse and in this case times $\tau_0 \leqslant t \leqslant \tau - \tau_0$. For the time-dependent external field $\chi(t)$ on the edges of a pulse there is no damping because length τ_0 of the short edges obeys the condition $\tau_0 \ll T_i$. In this case, using the well-known solution [19] of the optical Bloch equations for resonant $\Delta = 0$ field $\varepsilon(t)$, yields

$$v(t) = \sin \int^t \varepsilon(t')\,dt'; \tag{A1}$$

for the edge $t < \tau_0$ we have

$$v(t) = \sin \int^t \left[\Omega(t') + \frac{\chi_0 {t'}^2}{2\tau_0} \right] dt'. \tag{A2}$$

Suppose $\chi_0 \tau_0 \ll 1$ and for $\chi_0 \gg \Omega$, $\chi_0 \tau_0 \gg \int^{\tau_0} \Omega(t')\,dt'$ we have

$$v(\tau_0) = \frac{\chi_0 \tau_0}{2} \tag{A3}$$

Passing to the limit $\tau_0 \rightarrow 0$ (the square-wave form) the contribution of the edge of a pulse is also going to zero $v(\tau_0) \rightarrow 0$.

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