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Hydrodynamic quantization approach to Bose-Einstein condensation

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In this paper we employ the hydrodynamic quantization approach to studying the macroscopic state occupation of a bosonic system. We propose that the condensed state is a coherent state of the phase field component associated with the quantized field operator. By making Ansätze on functional states, we transform the problem of finding the phase and density associated with the condensate into a variational problem. We also discuss some properties of the condensed state, like superfluidity. In close analogy with Bose-Einstein condensation in the $\vec{p}=0$ state, we propose some possible wave functions for the excited states of a condensed system.

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I. INTRODUCTION

The experimental realization of Bose-Einstein condensates [1] have opened up new frontiers, and brought back fundamental questions related to many-body systems. Until recently, ^{4}He was the only experimental system to offer possibilities for comparison between theory and experiment. Nonetheless, the produced quantum liquid in this system is far from being considered a dilute gas, where only a two-body theory permits a reasonable treatment. The recent realization of degenerated quantum gases using laser-cooled alkalis opened a new perspective to real possibilities for comparison with theory.

Experiments involving interference between condensates have been realized [2]. In this context questions related to the nature of the condensate become relevant, i.e., whether it represents a coherent state or any other type of state, and the question related to the phase [3] of a condensate.

In this paper, we employed a hydrodynamic quantization approach, as originally used by Chan and Valantin [4] and recently by Recami and Salesi [5], to demonstrate that the condensate is a coherent state of the phase field component associated with the quantized field operator. We discuss the properties of the condensate phase and propose possibilities for excited state wave functions.

In field theory, a description of scalar bosons is achieved by associating a complex field $\psi(x)$ to a nonrelativistic Bose particle. The quantization is done through typical equal-time commutation relations between the field and its canonically conjugated variable ψ^*, followed by a typical quantization scheme implemented with the introduction of the Fock space. This method is especially useful when we deal with systems for which the number of particles is well defined, for example in the description of scattering processes.

Here we propose an alternative procedure: a so-called hydrodynamic quantization of the field. In this scheme we use a new set of variables $\rho(x)$ and $\varphi(x)$, which are more convenient in the description of properties related to Bose-Einstein condensation. These new variables are defined as

$$\begin{aligned}\rho(x)&=\psi^*(x)\psi(x),\\ \psi(x)&=\sqrt{\rho(x)}e^{i\varphi(x)},\end{aligned} \tag{1}$$

denominated density and phase variables, respectively. They are canonically conjugated to each other, as we shall see below. For this reason we impose the following commutation relations among the density operator and the phase operator:

$$[\hat{\rho}(x),\hat{\varphi}(x')]=i\delta(x-x'). \tag{2}$$

The quantization method, based on commutation relations among density and phase operators, is called a hydrodynamic quantization, and its application in superfluidity, where Bose-Einstein condensation, is known to occur [6].

The commutation relation in Eq. (2), in conjunction with an explicit representation of these operators, requires a departure from typical quantization approaches. Within the density representation, in which the number density is a c-number operator, the states are represented as functionals of the number density. These functionals, as well as other physically relevant quantities, are written as integrals over space-time ''densities.'' In terms of the field ψ, the classical Lagrangian density $\mathcal{L}$, for the nonrelativistic scalar particles, is

$$\mathcal{L}=\frac{i}{2}\{\psi^*(x)[\partial_t\psi(x)]-[\partial_t\psi^*(x)]\psi(x)\} -\frac{\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x)}{2m}-\mathcal{H}^I[\psi^*(x)\psi(x)], \tag{3}$$

where $\mathcal{H}^I$ is the interaction Hamiltonian density. We define the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H}(x)\equiv\frac{\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x)}{2m}+\mathcal{H}^I[\psi^*(x)\psi(x)]. \tag{4}$$

Placing Eq. (1) into Eq. (3), one can see that the classical Lagrangian density can be written in the general form

$$\mathcal{L}=-\rho(x)\frac{\partial\varphi(x)}{\partial t}-\mathcal{H}(x). \tag{5}$$

The use of hydrodynamical quantization introduces a rather nonconventional description of a quantum bosonic system. This approach might be useful in describing physical systems in which the number of particles in a given state is very large, and therefore followed by great uncertainty. In this case, as we have learned in the example of quantum coherence in optics [2], another set of states, like coherent states, are more appropriate. This is the main reason to look again into this question in the context of Bose-Einstein condensation. We shall see that by using variational techniques, hydrodynamic quantization provides a framework for studying Bose-Einstein condensation.

We have divided this paper as follows. In Sec. II we present a hydrodynamic quantization of a quantum complex Bose field, presenting the Hamiltonian in terms of the density and phase operators. In order to make clear the transition to the first quantization method, in Sec. III we discuss the vacuum state and free-particle states. These states will be relevant in a description of excited states of the system. In Sec. IV we deal with a very special set of states: coherent states. These states are eigenstates of the field operator

$$\hat{\psi}_\rho(x)\equiv e^{i\hat{\varphi}(x)}, \tag{6}$$

which is the pure phase component of the boson complex field. In Sec. V, we apply the functional approach to a study of the ground state. The basis of the method is an ansatz functional that we extremize in order to obtain the energy of the ground state and the respective wave function. From the variational approach we obtain a general method for condensate states, which is done in Sec. VI. In the remaining sections, we present examples. These examples cover well-known cases of condensates, and we have focused our main interest on superfluid ^{4}He. Finally, we present explicit examples of possible wave functionals (and wave functions) for excited states of those systems. These examples are very much like Feynman's wave functional to superfluid ^{4}He [7]. This, once again, shows the unifying picture for Bose-Einstein condensation provided by the hydrodynamic approach.

II. HYDRODYNAMIC QUANTIZATION APPROACH

In dealing with the hydrodynamic quantization, we shall use the so-called density representation [4], in which the density operator is a classical *c* number and the phase is represented by the operator

$$\hat{\varphi}(x)\equiv -i\frac{\delta}{\delta\rho(x)}, \tag{7}$$

in this representation the commutation relation (2) is automatically guaranteed.

In the density representation, the state vectors are functionals of the density only, that is

$$\psi=\psi[\rho]. \tag{8}$$

The physical reasoning for this dependence is that Lagrangian (3) is U(1) symmetric, carrying that to the eigenstates. The discrete state ψ_0, for which

$$H\psi_0 \tag{9}$$

is a minimum, is the ground state of the system. We shall assume that the ground state is a functional of the density under the form

$$\psi_0[\rho]=e^{-W[\rho]} \tag{10}$$

and a specific form for $W[\rho]$ will be discussed in Sec. V.

All other states of the system will be generated by applying some functional $\mathcal{F}\rho(x)$ to the ground state

$$\psi[\rho]=\mathcal{F}[\rho]\psi_0[\rho]. \tag{11}$$

The local momentum $\vec{P}(x)$ is defined as

$$\hat{\vec{P}}(x)\equiv +\vec{\nabla}\hat{\varphi}(x)=-i\vec{\nabla}\left(\frac{\delta}{\delta\rho(x)}\right). \tag{12}$$

In terms of local momentum, the Hamiltonian density operator is written as

$$\hat{\mathcal{H}}(x)=\frac{\hat{\vec{P}}^{+}(x)\rho(x)\cdot\hat{\vec{P}}(x)}{2m}+\mathcal{H}^{I}[\rho(x)]. \tag{13}$$

For the interaction with electromagnetic field, we just use the minimal coupling substitution in Eq. (13):

$$\vec{\nabla}\rightarrow\vec{\nabla}-ie\vec{A}. \tag{14}$$

Finally the field operator $\hat{\psi}$ is represented by

$$\hat{\psi}(x)=\sqrt{\rho(x)}e^{i\hat{\varphi}}. \tag{15}$$

The time evolution of an operator $\hat{O}(x)$ is given by the Heisenberg equation of motion,

$$\frac{\partial\hat{O}(x)}{\partial t}=i[\hat{H},\hat{O}(x)]. \tag{16}$$

Within the density representation the equations of motion of the density and phase operators can be written, formally, as

$$\frac{\partial\hat{\varphi}(x)}{\partial t}=-\frac{\delta\hat{H}}{\delta\rho(x)}, \tag{17}$$

$$\frac{\partial\rho(x)}{\partial t}=\frac{\widehat{\delta H}}{\delta\varphi(x)}. \tag{18}$$

From the above equation it follows that ρ and φ are canonically conjugated variables.

The momentum equation of motion is given by

$$\frac{\partial\hat{\vec{P}}(x)}{\partial t}=\hat{\vec{F}}(x), \tag{19}$$

where the local force operator $\hat{\vec{F}}(x)$ is, formally, written as

$$\hat{\vec{F}}(x) = -\vec{\nabla}\left(\frac{\partial \hat{H}}{\partial \rho(x)}\right). \tag{20}$$

The Hamiltonian operator in Eqs. (16)–(20) is the integral of the Hamiltonian density

$$\hat{H} = \int dx \hat{\mathcal{H}}[\hat{\varphi}, \rho]. \tag{21}$$

We shall assume that the interaction Hamiltonian density has the general form

$$\begin{aligned} \mathcal{H}^I[\rho(x)] &= -\mu\rho(x) + U(x)\rho(x) + \rho(x)e(x,\rho) \\ &= [-\mu + U(x)]\rho + \mathcal{H}^{\mathrm{int}}[\rho(x)] \end{aligned} \tag{22}$$

where μ in Eq. (19) is the chemical potential, $U(x)$ is an external potential, and $e(x,\rho)$ is the internal energy per particle. The last term $\mathcal{H}^{\mathrm{int}}[\rho(x)]$ describes internal forces among the particles. If one admits only two-body interactions, we write

$$\mathcal{H}^{\mathrm{int}}[\rho] = \tfrac{1}{2}\int\int V(x-x')\rho(x')\rho(x)dx'dx. \tag{23}$$

One of the interesting features of the hydrodynamic quantization is that the quantum equation of motion resembles that a classical fluid (from this fact derives the hydrodynamic name). In fact, from Eqs. (2), (13), (17) and (19) the formal time evolution equations

$$-\frac{\partial \hat{\varphi}(x)}{\partial t} = \frac{\hat{\vec{P}}^2(x)}{2m} - \mu + U(x) + h(x), \tag{24}$$

$$\frac{\partial \hat{\rho}(x)}{\partial t} = -\vec{\nabla}\cdot\frac{\rho(x)\hat{\vec{P}}(x)}{m}, \tag{25}$$

$$\frac{\partial \vec{P}}{\partial t} = -\vec{\nabla}\left(\frac{\hat{\vec{P}}^2}{2m}\right) + \vec{F}_{\mathrm{ext}}(x) + \vec{\nabla}h(x) \tag{26}$$

follows where $h(x)$ in Eq. (24) and (26) is the entropy per particle (as we shall see later)

$$h(x) \equiv \frac{\delta\mathcal{H}^{\mathrm{int}}}{\delta\rho(x)}. \tag{27}$$

The term $\vec{F}_{\mathrm{ext}}(x)$ in Eq. (26) is the force exerted on each of the charged particles of the system as a result of the external field $U(x)$,

$$\vec{F}_{\mathrm{ext}}(x) = -\vec{\nabla}U(x); \tag{28}$$

defining the velocity operator $\vec{V}$ by

$$\vec{V}(x) = \frac{\vec{P}(x)}{m}, \tag{29}$$

the current density becomes

$$\hat{\vec{j}}(x) = \rho(x)\hat{\vec{V}}(x). \tag{30}$$

Equation (25) is the continuity equation

$$\frac{\partial\rho(x)}{\partial t} + \vec{\nabla}\cdot\hat{\vec{j}}(x) = 0. \tag{31}$$

III. VACUUM AND FOCK STATES

To simplify the interpretation, we shall go from the functional formalism to the wave function interpretation. Simply, we take for ρ, the density associated with pointlike particles. Within the time-independent method we write

$$\begin{aligned} \rho(\vec{x}) &= \rho_p(\vec{x}) \equiv \sum_{i=1}^{n} \delta(\vec{x}-\vec{x}_i), \\ \vec{P}(x) &= -i\vec{\nabla}, \end{aligned} \tag{32}$$

so that, formally, the kinetic term becomes

$$\hat{k} = \int d^3x \vec{P}(x)\rho\cdot\vec{P}(x) \rightarrow +\sum \vec{\nabla}_i\cdot\vec{\nabla}_i. \tag{33}$$

Using Eq. (32), for the two-body, time-independent interaction, we obtain the expression

$$H = \sum_n \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{N}[-\mu + U(x_i)] + \sum_i\sum_j \frac{1}{2}V(\vec{x}_i - \vec{x}_j) \tag{34}$$

The equation for the wave functional,

$$\hat{H}\psi[\rho] = E\psi[\rho], \tag{35}$$

becomes the equation for the wave function by means of the identification

$$\psi(\vec{x}_1, \ldots, \vec{x}_n) = \psi(\rho)\big|_{\rho=\sum_{i=1}^{n}\delta(\vec{x}-\vec{x}_i)}, \tag{36}$$

whereas for the time-dependent approach one replaces, in all integrals over x', the density by

$$\rho(x') = \delta(t'-t)\rho_p(\vec{x}'), \tag{37}$$

so that the wave function is obtained from $\psi[\rho]$ by

$$\psi(\vec{x}_1, \ldots, \vec{x}, t) = \psi[\rho]\big|_{\rho(x')=\delta(t'-t)\rho_p(\vec{x}')}. \tag{38}$$

The state for which there are no particles in the system is the vacuum state. We shall assume that the vacuum state is characterized by the property

$$\hat{\varphi}(x)\psi_0 = 0. \tag{39}$$

This ensures that the vacuum has zero momentum,

$$\vec{p}\,\psi_0 = \vec{0}. \tag{40}$$

In view of Eqs. (40) and (7), we shall associate with the vacuum state a constant field

$$\psi_0[\rho] = \psi_0 . \tag{41}$$

A generic state will be built by successive products of the density to the vacuum. Let us illustrate this point by constructing the wave functionals associated with particles with well-defined momenta. The simplest states we can build are states describing free particles possessing well-defined momentum $\vec{k}$.

The wave functional associated with just one particle with momentum $\vec{k}$ and energy $\vec{k}^2/2m$, is

$$\psi_k[\rho] = \int e^{ikx}\rho(x)\,dx\,\psi_0 , \tag{42}$$

with $k = (\vec{k}^2/2m, \vec{k})$, whereas for two particles carrying momentum k_1 and k_2 we write

$$\psi_{k_1k_2}[\rho] = \int e^{ik_1x_1}e^{ik_2x_2}\rho_2(x_1,x_2)\,\psi_0\,dx_1\,dx_2 , \tag{43}$$

where

$$\rho_2(x_1,x_2) = \rho(x_1)[\rho(x_2) - \delta(x_1 - x_2)]. \tag{44}$$

For l free particles the associated wave functional can be generalized as

$$\psi_{k_1 \ldots k_l[\rho]} = \int dx_1 \cdots dx_l\, e^{i[\sum_{i=1}^{l} k_ix_i]}\rho_n(x_1, \ldots ,x_l) \tag{45}$$

where

$$\begin{aligned}\rho_n(x_1, \ldots ,x_l) = \rho(x_1)&[\rho(x_2) - \delta(x_2 - x_1)]\\ &\times[\rho(x_3) - \delta(x_3 - x_2) - \delta(x_3 - x_1)]\\ &\times[\rho(x_l) - \delta(x_l - x_{l-1}) - \cdots\\ &- \delta(x_l - 1)].\end{aligned} \tag{46}$$

It should be stressed then, and in view of the above examples, that the analog of Fock states are obtained by applying products of densities to the vacuum states.

For one free particle and ρ given by Eq. (32), we obtain

$$\psi_k(x_1, \ldots ,x_n) = \sum_{i=1}^{n} e^{ikx_j}\psi_0 , \tag{47}$$

$$\psi_{k_1k_2}(x_1, \ldots ,x_n) = \sum_{l \neq j = 1}^{n} e^{ik_1x_l}e^{ik_2x_j}\psi_0 , \tag{48}$$

$$\psi_{k_1, \ldots ,k_l}(x_1 \ldots x_n) = \sum_{l \neq m \neq \cdots \neq p = 1} e^{ik_lx_l}e^{ik_mx_m}\cdots e^{ik_px_p}\psi_0 . \tag{49}$$

IV. COHERENT STATES

In this section we introduce a set of states which, as we shall see later, are associated with a macroscopic occupation of the ground state. We start by looking for phase eigenstates, and we consider the eigenvalue problem

$$\hat{\varphi},(x)\,\psi[\rho] = -i\,\frac{\delta}{\delta\rho(x)}\,\psi[\rho] = \varphi(x)\,\psi[\rho]. \tag{50}$$

Within the density representation this is equivalent to finding eigenstates for the part of the field ψ related to the phase. Defining the field operator as a product of two operators

$$\hat{\psi}(x) = \rho^{1/2}(x)\,\hat{\psi}_p(x), \tag{51}$$

it follows from Eq. (1) that a pure phase component of the field operator can be defined. This pure phase component will be written as

$$\hat{\psi}_p(x) \equiv e^{i\hat{\varphi}(x)}. \tag{52}$$

It should be noted that the phase eigenstates are eigenstates of the phase component of the field ψ. Therefore,

$$\hat{\psi}_p(x)\,\psi[\rho] = e^{i\varphi(x)}\psi[\rho]. \tag{53}$$

Since they are eigenstates of the field $\hat{\varphi}$, phase eigenstates are then coherent states, for which the following property holds true:

$$\psi^*[\rho]e^{i\hat{\varphi}(x_n)}\cdots e^{i\hat{\varphi}(x_1)}\psi[\rho] = e^{i\varphi_0(x_n)}\cdots e^{i\varphi_0(x)}\psi^*[\rho]\psi[\rho]. \tag{54}$$

The solution of Eq. (50) is straightforward, and results in

$$\psi[\rho] = e^{+i\int\varphi_0(x)\rho(x)dx}. \tag{55}$$

We notice that, due to the commutation relation, we cannot find eigenvalues of the field itself. However, we can further improve our coherent states in such a way that they look like eigenstates of the field. Let us consider the eigenstate

$$\psi[\rho] = e^{\int \ln \psi_0(x)\rho(x)} = \exp\int dx[i\varphi_0(x) + \ln\rho_0^{1/2}(x)]\rho(x) \tag{56}$$

which are eigenstates such that the following holds:

$$e^{i\hat{\varphi}(x)}\psi[\rho] = \psi_0(x)\,\psi[\rho] = \sqrt{\rho_0(x)}e^{i\varphi_0(x)}\psi[\rho]. \tag{57}$$

As a consequence, these states are functional eigenstates of the pure phase of the field

$$\hat{\psi}_p(x)\,\psi[\rho] = \psi_0(x)\,\psi[\rho], \tag{58}$$

and the following factorization property holds true:

$$\psi^*[\rho]e^{i\hat{\varphi}(x_n)}\cdots e^{i\hat{\varphi}(x_1)}\psi[\rho]=\prod_{i=1}^{n}\psi_0(x_i)\psi^*[\rho]\psi[\rho]. \tag{59}$$

We notice that, from Eq. (59), the expectation value

$$\frac{\psi^*[\rho]\hat{\psi}_p(x_1)\cdots\hat{\psi}_p(x_n)}{\psi^*[\rho]\psi[\rho]}\psi[\rho]=\prod_{i=1}^{n}\psi_0(x_i) \tag{60}$$

exhibits the property of being a product of fields, and this property is similar to the propositions of Anderson [8] in the context of superfluidity.

The introduction of these states, and the possibility of their physical existence, is one of the main motivations for working with the hydrodynamic quantization approach. The interesting property of these coherent states, which can be explored in the condensation phenomena, is that they are eigenstates of the kinetic energy per particle defined as

$$K(x)=\frac{\vec{P}^*(x)\vec{P}(x)}{2m}, \tag{61}$$

as, alternatively,

$$\hat{K}(x)\psi[\rho]=\frac{K(x)}{2m}\psi[\rho], \tag{62}$$

where the eigenvalue $K(x)$ is

$$K(x)=\frac{[\vec{\nabla}\varphi_0(x)]^2}{2m}+\frac{[\vec{\nabla}\ln\sqrt{\rho_0(x)}]^2}{2m}. \tag{63}$$

Further consideration of expression (63) will be provided in Sec. V.

V. FUNCTIONAL APPROACH TO THE GROUND STATE

Although the method employed here makes a closer connection with hydrodynamics, the only way of obtaining information on the wave functionals associated with the ground state, as well with excited states, is to guess a trial wave functional, followed by the use of a variational method.

From the trial wave functional one can reduce the problem to a minimization of the functional given by

$$\langle\mathcal{H}[\rho,\varphi]\rangle=\psi^*[\rho]\hat{\mathcal{H}}[\rho,\hat{\varphi}]\psi[\rho]\frac{1}{\psi^*[\rho]\psi[\rho]}, \tag{64}$$

which corresponds to the same functional used by Feynman [7] in his approach to superfluid in ^{4}He.

Several *Ansätze* for the wave functional can be used, and they are divided into two categories. In the first category the *n* particle systems have their positions correlated. In the second the particles are not correlated, but behave as collective modes of the system. For the first category an example of the two-particle correlation wave functional is

$$\psi[\rho]=e^{-\int dx\int dx'\rho(x)g(x-x')\rho(x')}, \tag{65}$$

whose wave function is, from Eq. (32),

$$\psi(\vec{x}_1,\ldots,\vec{x}_n)=\prod\phi(\vec{x}_i-\vec{x}_j), \tag{66}$$

where

$$\phi(\vec{x}_i-\vec{x}_j)=e^{-g(\vec{x}_i-\vec{x}_j)}. \tag{67}$$

We shall start with a simple *Ansatz*, exhibiting no correlations for the trial wave functional. Let us start with a wave functional of the form (56), keeping in mind that we are dealing with a coherent states of the pure phase component. We propose

$$\psi_0[\rho,\varphi_0]=e^{+\int dx[\ln\psi_0(\rho,x)]\rho(x)}, \tag{68}$$

where

$$\psi_0(\rho,x)=\sqrt{\rho(x)}e^{i\varphi_0(x)}, \tag{69}$$

and the problem now consists of finding the extremum of the functional

$$\mathcal{H}[\rho,\varphi_0]=\frac{\rho[\vec{\nabla}\varphi_0(x)]^2}{2m}+\frac{[\vec{\nabla}\sqrt{\rho(x)}]^2}{2m}+\mathcal{H}^I[\rho]. \tag{70}$$

In the presence of an electromagnetic field, the functional to be minimized is modified to

$$\mathcal{H}[\rho,\varphi_0]=\frac{\rho(x)}{2m}[\vec{\nabla}\varphi_0(x)-\vec{A}]^2+\frac{[\vec{\nabla}\sqrt{\rho_0(x)}]^2}{2m}+\mathcal{H}^I[\rho]. \tag{71}$$

The condensate state is defined as the field configuration for which $\mathcal{H}[\rho,\varphi_0]$ is minimum. Therefore,

$$\left.\frac{\delta\mathcal{H}}{\delta\rho}\right|_{\rho=\rho_c}=0, \tag{72}$$

$$\left.\frac{\delta\mathcal{H}}{\delta\rho}\right|_{\varphi_0=\varphi_c}=0. \tag{73}$$

The subindex *c* denotes the condensate state. Details about the solution of Eqs. (72) and (73) will be presented in Sec. VI. For the time being let us consider the problem of finding the energy of the ground state for simple condensed states. The simplest condensate that one can think of is the one for which all particles are in the zero momentum state.

In the absence of external electromagnetic field, the minimum of Eq. (72) will occur for

$$\varphi_0(x)=\varphi_c=0, \tag{74}$$

and for a uniform ρ, that is

$$\rho(x)=\rho_b. \tag{75}$$

This type of condensate is normally referred to as a Bogoliubov condensate [9]. As we shall see later, they describe the condensation of particles in the zero-momentum state, and in this case the Hamiltonian density of the condensed state is given by

$$\mathcal{H}=\mathcal{H}^{\rm int}(\rho_b), \tag{76}$$

and the energy of the condensate is

$$H_{\rm min}=\int dx\mathcal{H}^{\rm int}[\rho_b]. \tag{77}$$

In the presence of electromagnetic fields, the structure of the ground state is more complex.

VI. CONDENSED STATES

We shall define condensed states as those corresponding to the trial functional (68) for which the Lagrangian density is the lowest. Therefore,

$$\delta\mathcal{L}|_{\varphi=\varphi_c}=0, \tag{78}$$

$$\delta\mathcal{L}|_{\rho=\rho_c}=0, \tag{79}$$

where the Lagrangian density, from Eq. (5), is

$$-\mathcal{L}=\rho\dot{\varphi}+\mathcal{H}[\rho,\varphi]. \tag{80}$$

The equations that emerge from the above conditions are

$$\frac{\partial\rho_c(x)}{\partial t}=+\left.\frac{\delta\mathcal{H}}{\delta\varphi(x)}\right|_{\varphi=\varphi_c}, \tag{81}$$

$$\frac{\partial\varphi_c(x)}{\partial t}=-\left.\frac{\delta\mathcal{H}}{\delta\rho(x)}\right|_{\rho=\rho_c}. \tag{82}$$

Considering the Hamiltonian given by Eq. (71) the corresponding equations become

$$\begin{aligned}-\frac{\partial\varphi_c(x)}{\partial t}=&\frac{[\vec{\nabla}\varphi_c(x)-e\vec{A}]^2}{2m}\\ &+\frac{1}{4m}\left[\frac{1}{2}\left(\frac{\vec{\nabla}\rho_c(x)}{\rho_c(x)}\right)^2-\frac{\Delta\rho_c(x)}{\rho_c(x)}\right],\\ &-\mu+U(x)+h(x),\end{aligned} \tag{83}$$

and

$$\frac{\partial\rho_c}{\partial t}=-\vec{\nabla}\cdot\vec{J}_c, \tag{84}$$

where $h(x)$ is the per particle enthalpy, defined in Eq. (27). Using Eq. (22),

$$h(x)=e_{\rm int}(x,\rho_c)+\rho(x)\frac{\delta e_{\rm int}(x_\rho)}{\delta\rho(x)} \tag{85}$$

$$\equiv e_{\rm int}(x,\rho_c)+\frac{P(x)}{\rho(x)}. \tag{86}$$

Hamiltonian (71) describes a quantum system of particles under the action of external electric and magnetic fields whose associated potentials are $U(x)$ and $\vec{A}(x)$.

The momentum of the condensate is

$$\vec{P}_c(x)=-\vec{\nabla}\varphi_c(x) \tag{87}$$

whereas the velocity is

$$\vec{V}_c(x)=\frac{\vec{P}_c(x)-e\vec{A}(x)}{m}, \tag{88}$$

and the current

$$\vec{J}=\rho_c\vec{V}_c=\frac{\rho_c}{m}(\vec{P}_c-e\vec{A}). \tag{89}$$

Time-independent condensates are associated with the microscopic occupation of a given state if the number of particle n in Eq. (32) corresponds to a large fraction of the total number of particles. Notice that the wave functional

$$\psi_c[\rho,\psi_c(x)]=e^{\int\ln\psi_c(x)\rho(x)dx}\psi_0 \tag{90}$$

is endowed with the following essential properties which simplify the solution of the problem.

(1) For $\rho(x)=\rho_c(x)$, $\psi_c[\rho,\psi_c]$ is an eigenstate of the Hamiltonian density

$$\begin{aligned}&[\hat{H}(\rho_c(x),\hat{\varphi}(x))]\psi_c[\rho,\psi_c(x)]\\ &\quad=[\mathcal{H}(\rho_c(x),\varphi_c(x))]\psi_c[\rho,\psi_c(x)].\end{aligned} \tag{91}$$

The eigenvalue is the Hamiltonian density of the condensate.

(2) The quantum time evolution equations are satisfied in the following senses:

$$\left(\frac{\partial\hat{\varphi}(x)}{\partial t}\right)\psi_c(\rho,\varphi)\bigg|_{\substack{\rho=\rho_c\\ \varphi=\varphi_c}}=\frac{\partial\hat{\mathcal{H}}}{\partial\rho}\psi_c(\rho,\varphi)\bigg|_{\substack{\rho=\rho_c\\ \varphi=\varphi_c}}, \tag{92}$$

$$\frac{\partial\rho(x)}{\partial t}\psi(\rho,\varphi)\bigg|_{\substack{\rho=\rho_c\\ \varphi=\varphi_c}}=-\frac{\partial\hat{\mathcal{H}}}{\partial\varphi}\psi_c(\rho,\varphi)\bigg|_{\substack{\rho=\rho_c\\ \varphi=\varphi_c}}. \tag{93}$$

The quantum wave functional (90) is the one for which the action is the least.

(3) Functional (90) describes a state in which $U(1)$ symmetry is spontaneously broken, because this state depends explicitly on the phase $\varphi_c(x)$. As a consequence,

$$\frac{\psi_c^*[\rho,\psi_c]e^{i\hat{\varphi}(x)}\psi_c[\rho,\psi_c]}{\psi_c^*[\rho,\psi_c]\psi_c[\rho,\psi_c]}=\psi_c(x). \tag{94}$$

(4) The Green's functions of the theory

$$G(x_1,\ldots,x_n)\equiv\frac{\psi^*[\rho]\hat{\psi}(x_n)\cdots\hat{\psi}(x_1)\psi[\rho]}{\psi^*[\rho]\psi[\rho]}\tag{95}$$

assumes, for the wave functional (90), the following form:

$$G(x_1,\ldots,x_n)=\psi_c(x_1)\cdots\psi_c(x_n)\rho(x_1)\cdots\rho(x_n).\tag{96}$$

Thus, for ρ close to ρ_b (the Bogoliubov condensate), one can write

$$G(x_1,\ldots,x_n)=\prod_{i=1}^{n}\ \psi_c(x_n).\tag{97}$$

The above factorization property of the Green's functions is known as off-diagonal long-range order, and it has been shown to be relevant in the context of superfluidity [8,10].

(5) The most important property of functional (90) is that for *n* particles in some definite positions $\vec{x}_1,\ldots,\vec{x}_n$ at time *t*, the associated wave function is

$$\begin{aligned}\psi_0(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_n;t)&\equiv\psi_0[\rho]|_{\rho(x')=\delta(t'-t)\delta_p(\vec{x})}\\&=\prod_{i=1}^{n}e^{i\varphi_c(x)}\sqrt{\rho_c(x)}\psi_0,\end{aligned}\tag{98}$$

corresponding to a state in which all particles occupying the condensate.

We can now identify $\varphi_c[\rho]$ with wave functional of the condensed, therefore for the condensate wave function one can write

$$\psi_c(x)=e^{i\varphi_c(x)}\sqrt{\rho_c(x)}.\tag{99}$$

With a knowledge of the condensate wave function, all the relevant properties can be determined.

VII. SOME APPLICATIONS

In this section we shall analyze some applications of the functional variational method that we have proposed here, applying it to cases of interest.

A. Single particle

The first obvious application of our approach, and the simplest one, is the study of a single particle interacting with an external potential. In this case we are obviously not talking about Bose-Einstein condensation presented, but we would like to check how reliable the approach is presented here. The wave functional is again Eq. (90), and since there is only a single particle the wave function can be easily written as

$$\begin{aligned}\psi(x)&=\int e^{\ln\psi_c(x')\rho(x')}|_{\rho(x')=\delta(x-x')\delta(t'-t)}\\&=\psi[\psi_c,\rho]|_{\rho=\delta(t'-t)\delta(x-x')}\\&=e^{\ln\psi_c(x)}\\&=\sqrt{\rho_c(x)}e^{i\varphi_c(x)},\end{aligned}\tag{100}$$

showing that, for a single particle, the condensate wave function coincides with the particle wave function.

From Eqs. (83) and (84), it follows that the density and phase fields satisfy the equations

$$\begin{aligned}-\frac{\partial\varphi_c(x)}{\partial t}=&\frac{[\vec{\nabla}\varphi_c(x)]^2}{2m}+\frac{1}{4m}\left[\frac{1}{2}\left(\frac{\vec{\nabla}\rho_c}{\rho_c}\right)^2-\frac{\vec{\nabla}^2\rho_c}{\rho_c}\right]\\&+U(x)+\mu,\end{aligned}\tag{101}$$

$$\frac{\partial\rho_c(x)}{\partial t}+\vec{\nabla}\cdot\left(\frac{\rho_c\vec{\nabla}\varphi_c}{m}\right)=0.\tag{102}$$

The above equations are the equations describing a quantum fluid known as a Madelung fluid [11,12]. In this case the hydrodynamic quantization approach gives an exact solution. In this sense our method is equivalent to recent papers dealing with the hydrodynamic reformulation of the Schrödinger theory directly from a variational approach [12] .

The second term on the right-hand side of Eq. (101) is known as the quantum potential and it is obtained from the functional variation of the term

$$\frac{1}{2m}\{\vec{\nabla}[\sqrt{\rho(x)}]\}^2\tag{103}$$

in Eq. (70).

It was shown recently by Recami and Salesi [5] that the quantum potential is a consequence of spin and Zitterbewegung. In this way, it has a well-understood origin. In the case of spin-$\frac{1}{2}$ particles, the nonrelativistic limit of Dirac equation lead to the following decomposition of the velocity,

$$\vec{V}=\frac{\vec{\nabla}-e\vec{A}}{m}+\frac{\vec{\nabla}\times\rho\vec{S}}{m\rho},\tag{104}$$

where the last piece is the Zitterbewegung velocity which is associated to the spin *s* of the particle,

$$\vec{V}_S\equiv\frac{\vec{\nabla}\times\rho\vec{S}}{m\rho},\tag{105}$$

so that

$$\vec{V}_S^2=\frac{(\vec{\nabla}\rho)^2\vec{S}^2}{m^2\rho^2}-\frac{(\vec{\nabla}\rho)\cdot\vec{S}}{(m\rho)^2}.\tag{106}$$

This result shows, in the $S=1/2$ case, the existence of a connection of the quantum potential with spin [5].

B. Condensation–noninteracting particles

Let us now consider the occupation of a state in the case of particles that do not present self-interaction. The wave equation is therefore

$$\left(-\frac{\sum \vec{\nabla}_i^2}{2m}+U(x_i)\right)\psi(x_1,\ldots,x_N)=\frac{i\partial}{\partial t}\psi(x_1,\ldots,x_N), \tag{107}$$

having the solution

$$\left(-\frac{\sum \vec{\nabla}_i^2}{2m}+\sum U(x_i)\right)\left(\prod_{i=1}^{N}\psi_c(x_i)\right)=\frac{i\partial}{\partial t}\left(\prod_{i=1}^{N}\psi_c(x_i)\right), \tag{108}$$

which holds true for $\psi_c(x_i)$ satisfying Eqs. (101) and (102), where $\psi_c(x)$ is defined in Eq. (99).

Clearly the wave function, in this case, is

$$\psi(x_1\ldots x_n)=\prod_{i=1}^{N}\psi_c(x_i), \tag{109}$$

which describes a system of N particles occupying the state $\psi_c(x)$. Again, our approach provides the right answer for the condensation.

C. Two-body interaction

As for a more realistic application, let us consider the case of a system having a two-body interaction given by the potential $V(x_i-x_j)$. In this case the condensate wave functions $\psi_c(x)$ satisfy the equations

$$-\frac{\partial\varphi_c(x)}{\partial t}=\frac{[\vec{\nabla}\varphi_c(x)]^2}{2m}+\frac{1}{4m}\left[\left(\frac{1}{2}\frac{\vec{\nabla}\rho_c(x)}{\rho_c(x)}\right)^2-\frac{\vec{\nabla}^2\rho_c(x)}{\rho_c(x)}\right]-\mu+U(x)+\int V(x-x')\rho_c(x'), \tag{110}$$

$$\frac{\partial\rho_c(x)}{\partial t}+\vec{\nabla}\cdot\left(\frac{\rho_c\vec{\nabla}\varphi_c}{m}\right)=0. \tag{111}$$

The above set of Eqs. (111) and (112) is equivalent to

$$i\frac{\partial\psi_c(x)}{\partial t}=\left(-\frac{\vec{\nabla}^2}{2m}+U(x)-\mu+\int dx U(x-x')\psi_c^*\psi_c(x')\right)\psi_c(x), \tag{112}$$

a variant of the Gross-Pitaevski equation [13].

D. Hard-sphere interaction

Within the hard-sphere approximation the above equation becomes

$$i\frac{\partial\psi_c(x)}{\partial t}=\left(-\frac{\vec{\nabla}^2}{2m}+U(x)-\mu+\lambda\psi_c^*\psi_c(x')\right)\psi_c(x), \tag{113}$$

which is again the Gross-Pitaevsky equation [13] for local self-interaction.

VIII. INCOMPRESSIBLE CONDENSATES—SUPERFLUIDITY

In close analogy with the usual fluids, we define an incompressible condensate as the one for which

$$\rho_c(x)=\rho_b=\text{const}; \tag{114}$$

that is, the condensate is a uniform one. We shall see later that Bogoliubov's condensate is just a special case of incompressible condensates.

From now on, in this section, we shall work with quasi-incompressible condensates. That is, we will not consider uniform condensates but condensates for which $\rho_c(x)$ varies slowly as a function of x. By this we mean that $\vec{\nabla}^2\sqrt{\rho_c}$ is negligible when compared with ρ_c. The condition is

$$\frac{\vec{\nabla}^2\sqrt{\rho_c(x)}}{\sqrt{\rho_c}}\cong 0. \tag{115}$$

These quasi-incompressible condensates are the ones for which one can neglect the quantum velocity contribution to the Hamiltonian density:

$$V_{\text{quantum}}^2=[\vec{\nabla}\ln\sqrt{\rho_c(x)}]^2. \tag{116}$$

The arguments of Salesi and Recami [5] strongly suggests that this approximation might be valid on quite general grounds; that is, without evoking weak external fields.

Under approximation (115) we obtain, basically, the equation of a superfluid. In this way, for nearly uniform condensates one expects that the condensate will move as a superfluid.

In fact, Eq. (102) is the continuity equation, whereas Eq. (101) can be written as

$$\frac{\partial\varphi_c(x)}{\partial t}=\frac{[\vec{\nabla}\varphi_c(x)]^2}{2m}+h(x)\rho_c(x)+U(x)+\mu. \tag{117}$$

First of all, one notes that this equation is just an extended version of Bernoulli's equation. In fact, for stationary solution we write, for $U(x)=0$,

$$\varepsilon'=\frac{\vec{P}^2(x)}{2m}+\varepsilon+\frac{P(x)}{\rho(x)}, \tag{118}$$

so that

$$\frac{m\vec{V}^2}{2}+\frac{P(x)}{\rho(x)}=\text{const.} \tag{119}$$

For nonstationary solutions, taking the gradient of Eq. (117), we obtain

$$\frac{d\vec{V}}{dt}=-\vec{\nabla}\left(\frac{m\vec{V}^2(x)}{2}\right)-\vec{\nabla}h(x)+\vec{F}_{\text{ext}} \tag{120}$$

which corresponds to an ideal fluid equation. We notice that Eq. (117) has the following form, when in the presence of external electromagnetic fields:

$$\frac{d\vec{P}}{dt}=-\vec{\nabla}\frac{[\vec{P}(x)-\vec{A}(x)]^2}{2m}-\vec{\nabla}h(x)+\vec{F}_{\mathrm{ext}}, \tag{121}$$

which allows the determination of $\vec{P}(x)$, or subsequently $\varphi(x)$.

An important point to be stressed is that, as suggested by Landau [6], there exist states associated with potential flow. In the absence of magnetic field and for slowly varying external fields, we can write

$$\hat{\vec{P}}(x)\psi_c[\rho,\psi_c]=[-i\vec{\nabla}\psi_c(x)]\psi_c[\rho,\psi_c], \tag{122}$$

so that, for slowly varying condensates,

$$\vec{P}_c(x)=-\vec{\nabla}\varphi_c, \tag{123}$$

and, as a consequence,

$$\vec{\nabla}\times\vec{V}_c(x)=\vec{0}, \tag{124}$$

which is associated with Landau's potential flow.

It is then tempting to propose a two-fluid picture for all condensates, as done by Landau for superfluid ^{4}He. Phenomenologically one can write

$$\rho=\rho_c+\rho_n, \tag{125}$$

$$\vec{V}=\vec{V}_c+\vec{V}_n \tag{126}$$

where ρ_c is the condensate density and ρ_n is associated with excited states (particles not in the condensate).

Although the result is interesting, the validity of the approximation given by Eq. (115) remain to be investigated, since they are essential for the results obtained. In the case where the approximation is valid, condensation is associated with superfluid, always.

IX. BOGOLIUBOV'S CONDENSATE

We define Bogoliubov's condensate as an incompressible condensate ($\rho_c=$const) with zero-momentum particles. For this case the problem is reduced to the variation problem

$$\left.\frac{\delta H^I(\rho)}{\delta\rho}\right|_{\rho=\rho_c(x)}=\mu, \tag{127}$$

where μ is the chemical potential. The first obvious result for the case where there is no interparticle interaction is that, from Eq. (127),

$$\mu=0. \tag{128}$$

which is the well-known result in the free ideal gas case.

Within this approximation, for the hard-sphere two-body interacting case, one can write

$$\mathcal{H}[\rho]=-\mu\rho_0+\frac{\lambda}{2}\rho_0^2, \tag{129}$$

where λ is related to the so-called scattering length (a) by the relation

$$\lambda=\frac{4\pi a}{m}, \tag{130}$$

and it contains all the information of the interparticle potential.

We are now ready to make estimates for an N-body system that undergoes Bose-Einstein condensation in the $\vec{p}=\vec{0}$ state. These results are fairly general, and we shall compare with the ^{4}He case [14]. We are interested in the following features of the system: ground-state energy, density of the condensed, zero-temperature pressure, zero-temperature equation of state, and sound speed.

In the case of two-body interactions, condition (127) becomes

$$\mu=\int V(x-x')\rho_c(x')dx', \tag{131}$$

so that the problem of finding $\rho_c(x')$ is analogous to finding the equipotentials of the two-body interaction. Since a typical two-body potential has a minimal corresponding to a depth ΔE, and since we are looking for the minimum of the energy, we shall take the lowest value for this parameter μ. From Eq. (130) a good guess for μ is

$$\mu=\Delta E, \tag{132}$$

where ΔE is the depth of the interparticle potential.

Taking this phenomenological parameter we reduce our variational problem of finding $\rho_c(x')$ to

$$\Delta E=\int V(x-x')\rho_c(x')dx'. \tag{133}$$

For the case in which ρ_c is constant, we have the uniform condensate approximation. For this case, the following properties can be obtained.

A. Ground-state energy

From Eq. (132) it follows that

$$\Delta E=\lambda\rho_b, \tag{134}$$

where ρ_b is the Bogoliubov incompressible condensate.

From Eq. (133), ρ_b is given by

$$\rho_b=\frac{\Delta E}{\lambda}=\frac{\Delta E m}{4\pi a}. \tag{135}$$

The Hamiltonian density in this case is

$$\mathcal{H}[\rho_b]=-\frac{\lambda}{2}\rho_b^2=-\frac{2\pi a}{m}\rho_b^2, \tag{136}$$

and therefore the energy of the ground state is given by

$$H=\int \mathcal{H}dv=-\frac{\lambda\rho_b}{2}(\rho_b V). \tag{137}$$

B. Density in the condensed phase

Admitting that all N particles are in a condensate,

$$\rho_b=\frac{N}{V}=\frac{\Delta E}{\lambda}. \tag{138}$$

C. Binding energy per particle

From Eq. (136), it follows that

$$\varepsilon=-\frac{\lambda\rho_b}{2}=-\frac{2\pi a}{m}\rho_b\,. \tag{139}$$

Applying this result for ^{4}He, where

$$m=3.5\times10^9\ \ \text{eV}, \tag{140}$$

$$a^{-1}=(2.6\ \ \text{Å})^{-1}, \tag{141}$$

$$\Delta E=8.81\times10^{-4}\ \ \text{eV}, \tag{142}$$

we obtain the following values for the density of the total condensate, and the binding energy per particle:

$$\rho_b=\frac{N}{V}=2.45\times10^{22}\ \ \text{atom/cm}^3, \tag{143}$$

$$\varepsilon=-4.40\times10^{-4}\ \ \text{eV/atom}. \tag{144}$$

For comparison we note that in ^{4}He the experimental values are

$$\rho_{\text{He}}=\frac{N}{V}=2.17\times10^{22}\ \ \text{atom/cm}^3, \tag{145}$$

$$\varepsilon=-5.5\times10^{-4}\ \ \text{eV/atom}. \tag{146}$$

thus showing that the calculated values are in good agreement with the experimental results.

D. Pressure at zero temperature

The pressure at $T=0$ (in the ground state) will be

$$P=\rho\left(\frac{h}{2}\right)=\frac{\lambda\rho_b^2}{2}=\frac{2\pi a}{m}\rho_b^2\,. \tag{147}$$

E. Sound speed

The sound speed, defined as

$$v_s=\sqrt{\frac{\partial P}{\partial\rho}\,\frac{1}{m}}, \tag{148}$$

can be evaluated by using Eq. (146); one can predict

$$v_s=\sqrt{\frac{\lambda\rho_b}{m}}=\sqrt{\frac{4\pi a}{m^2}\rho_b}. \tag{149}$$

F. Equation of state

For a two-body interacting system, one can predict from Eqs. (146) and (138), the following equation of state:

$$\frac{P}{\rho}=-\varepsilon. \tag{150}$$

Again, in the case of ^{4}He, our predictions are

$$P(0)=17\ \text{atom}, \tag{151}$$

whereas the experimental result is

$$P^{\text{expt}}(0)\cong 10\quad \text{atom}. \tag{152}$$

The sound speed we predict is

$$v_{\text{sound}}\equiv 150\ \ \text{m/s}, \tag{153}$$

whereas, experimentally,

$$v_{\text{sound}}^{\text{expt}}\cong 210\ \ \text{m/s} \tag{154}$$

where the experimental values were obtained from Refs. [15] and [14].

X. EXCITED STATES

Having established some properties of the ground state of a condensed system, we should attempt with build wave functionals associated with excited states. In particular, states in which a certain number of particles are free (not in the trap). This would help in a calculation of condensate excitations.

Based in the idea that other states can be generated by applying some functional to the ground state, as described in Sec. III of this paper, the best guess for the wave functional associated with an excited state in which one excitation is present could be [see Eq. (46)]

$$\psi^{(1)}[\rho]=\rho_k^{(1)}[\rho]\psi_c[\rho]=\int e^{ikx}\rho(x)dx\,\psi_c[\rho]. \tag{155}$$

The wave function associated to (155) is therefore

$$\psi^{(1)}(x_1,\ldots,x_N)=\sum_{i=1}^{N} e^{ikx_i}\psi_c(x_1,\ldots,x_N). \tag{156}$$

Wave function (156) becomes Feynman's wave function if one replaces ψ_c by $\psi_0(x_1,\ldots,x_N)$ where ψ_0 is the ground-state wave function.

The generalization is easy if one neglects self-interactions [4]. Suppose that $\chi_0[\rho]$ is a solution of Eq. (91) associated with $E=0$. Under these circunstances the equation of the wave function of the condensate is

$$\left(-\frac{\vec{\nabla}^2}{2m}+U(x)-\mu\right)\psi_c(x)=0. \tag{157}$$

It is easy to check that the functional

$$\chi[\rho]=\int dx f(x)\rho(x)\chi_0[\rho] \tag{158}$$

is a solution of Eq. (91) with eigenvalue E if

$$\left(-\frac{\vec{\nabla}^2}{2m}+U(x)-\mu\right)[f(x)\psi_c(x)]=E[f_c(x)\psi_c(x)]. \tag{159}$$

The wave function

$$\psi'(x_1,\ldots,x_n)=\sum_{j=1}^{n} f(x_j)\prod_{i=1}^{N}\psi_c(x_i) \tag{160}$$

corresponds to the wave functional (158). From Eq. (159) it follows that $\psi'(x_1,\ldots,x_n)$ represents a state in which one particle is in the excited state $f(x)\psi_c(x)$. The free-particle excited state [Eq. (155)] is just a simple example of Eq. (158) in the free particle case $E=k^2/2m$.

The functional which corresponds to excited states $f_1(x)\psi(x),\ldots,f_n(x)\psi(x)$ has a general expression

$$\chi[\rho]=\int dx_1\cdots dx_n f_1(x_1)\cdots f_n(x_n)\rho_n(x_1,\ldots,x_n)\chi_0[\rho], \tag{161}$$

where $\rho_n(x_1,\ldots,x_n)$ is given by Eq. (46). The symmetric wave function associated with (161) is

$$\psi''(x_1,\ldots,x_n)=\sum_{j_i\neq\cdots\neq j_{n=1}}^{n} f_1(x_{j_1})\cdots f_n(x_{j_n})\prod_{i=1}^{N}\psi_c(x_i). \tag{162}$$

One can go further in the parallel with Feynman's approach by using our expression substituting ψ_c with the ground-state wave function. We obtain explicitly,

$$\psi^{(1)}[\rho]=\int dx e^{ikx}\rho(x)\exp\left[i\int dx'\phi_c(x')\rho(x)\right]\chi_0[\rho]\psi_0, \tag{163}$$

where now

$$\chi_0[\rho]=\exp\int dx'\ln\rho_c(x)\rho(x). \tag{164}$$

Once again we see the similarity with the Feynman-Cohen [16] wave function. In fact, by using Eq. (164) we obtain for $\psi^{(1)}$,

$$\psi^{(1)}(\vec{r}_1,\ldots,\vec{r}_N)=\exp i\sum_{i=1}^{N}\varphi_c(\vec{r}_i)\prod\rho(\vec{r}_i)\sum_{i=1}^{N}e^{i\vec{k}\cdot\vec{r}_j}. \tag{165}$$

So that the only difference between our wave function and Feynman-Cohen's wave function is in the replacement

$$\sum_{j=1}^{N}\prod_{i=1}^{N}\rho_c(\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_j}\to\sum_{j=1}^{N}h_i(\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_j}, \tag{166}$$

where $h_i(\vec{r}_i)$ is Feynman and Cohen's guess to represent a localized excitation.

However, looking at Eq. (165), we see that the term

$$\sqrt{\rho_c}(\vec{r})e^{i\vec{k}\cdot\vec{r}_i} \tag{167}$$

leads to a current of the form

$$\vec{j}(\vec{r})=\rho_c(\vec{r})\frac{\vec{k}}{m}, \tag{168}$$

and is similar to the Feynman-Cohen prediction. The total current is then

$$\vec{J}=\rho_c(\vec{r})\frac{\vec{k}}{m}+\frac{\vec{\nabla}\varphi_c(\vec{r})}{m}, \tag{169}$$

as conjunctured by Feynman and Cohen.

In this context $\vec{\nabla}\varphi_c(\vec{r})/m$ is just their famous backflow current [16]

$$\vec{j}'_b=\frac{\vec{\nabla}\varphi_c(\vec{r})}{m}. \tag{170}$$

For ^{4}He Feynman and Cohen suggested that for large r, $\varphi_c(\vec{r})$ has the form of the velocity potential of a dipole, which is

$$\phi_c(\vec{r})\underset{r\to\infty}{\sim}\frac{\vec{\mu}\cdot\vec{r}}{r^3}. \tag{171}$$

Finally we can introduce a correlation between the excitation and the particles in the condensate. In this case we assume phase correlation and write

$$\psi'^{(1)}[\rho]=\int dx\exp(ikx)\rho(x)\times\exp\left(i\int dx\varphi_c(x'-x)\rho(x')\chi_0[\rho]\right). \tag{172}$$

The physical interpretation for this is that in Eq. (166) we ensure a backflow around the atom with momentum $\vec{k}$. This is another version of the Feynman-Cohen wave function, that was originally written as

$$\psi'^{(1)}(x_1,\ldots,x_n)=\sum_{i=1}^{n}\exp(ikx_i)\exp\left(i\sum_{i\neq j}\varphi_c(\vec{r}_i-\vec{r}_j)\chi_0(\vec{r}_1,\ldots,\vec{r}_N)\right), \tag{173}$$

thus showing a great similarity with our expressions for the excited states, as well as for the ground state. In our approach we do have an explicit form for the ground state in terms of the wave function of the condensate.

XI. CONCLUSIONS

In this paper we have applied the framework provided by the hydrodynamic quantization in order to study condensation phenomena. This is done in close analogy with the well-known phenomena of Bose-Einstein condensation in the $\vec{p}=0$ state.

This approach provides a unifying picture to all theories related to superfluid ^{4}He. In view of the common features of condensation in the ground state, this method should be useful in the study of observed Bose-Einstein condensation at very low-temperature gases, as recently demonstrated.

We have seen that condensed states are well described by coherent states of the pure phase component of the field. The coherent-state *Ansatz* wave functional reduces the problem of finding the wave function of the condensate ψ_c to a variation problem. The interesting aspect here is that the equations of the density and phase of the wave function can be cast under the forms of equations of motion of a quantum liquid. From this it follows that whenever one can neglect the quantum velocity term (within a quasiuniform approximation), the equations for the quantum fluid become the equations of a classical fluid. We can thus predict that condensates might become superfluid, as we observe in superfluid ^{4}He. Furthermore a two-fluid picture might be useful in understanding properties of condensed states.

We have shown that our framework can lead to an exact result in the case of a single particle. For particles that do not interact with each other, we have seen that our coherent states describe condensed states. We discuss some possible nonuniform condensates.

We have discussed zero-momentum condensation within the uniform density approximation. We have shown that many predictions about the condensed state can be made by taking this simple approximation. We obtain simple estimates for the uniform density, the binding energy of the ground state, the zero-temperature pressure, the zero-temperature equation of state, and the sound speed. The comparison with ^{4}He shows that these predictions are in fair agreement with the observed results.

It is fairly simple to construct excited states for a condensed system. Under simple assumptions one can build wave functions very similar to Feynman and Cohen's (with or without backflow). Presumably some of these wave functions will be useful in the general problem of Bose-Einstein condensation. This is a possible line of research to be pursued further.

The finite-temperature version of this functional approach should also be acomplished in order to obtain the thermodynamic properties involved in the condensation. It should be pointed out that the approach presented here could complement recently published theoretical work concerning predictions of properties in condensated trapped atoms.

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