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Geometrical approach to two-level Hamiltonians

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Two-level systems were shown to be fully described by a single function, known sometimes as the Stueckelberg parameter. Using concepts from differential geometry, we give geometrical meaning to the Stueckelberg parameter and to other related quantities. As a result, a generalization of the Stueckelberg parameter is introduced, and a relation obtained between two-level systems and spatial one-dimensional curves in three-dimensional space. Previous authors used this Stueckelberg parameter to solve analytically several two-level models. We further develop this idea, and solve analytically three fundamental models, from which many other known models emerge as special cases. We present the detailed analysis of these models.

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I. INTRODUCTION

Two-level systems have been a subject of major research interest since the early days of quantum mechanics. The pioneering work of Rabi [1] analyzing (analytically) a two-level model is well known. Since then, many other models were proposed, of which only a few have a known analytical solution. The interest in these systems is due to several reasons. First, these are usually the simplest systems that really can be seen in nature. Maybe the most obvious examples are spin 1/2 and light polarization [2]. Second, many systems can be approximated as two-level systems, when most of the interaction occurs within a two-dimensional subspace of the Hamiltonian. This is the case, for example, in two-level atoms [3] or in two-neutrino oscillations [4]. Third, it is used to describe level crossing in otherwise adiabatic systems. The most famous example is probably the Born-Oppenheimer approximation in molecular physics [5].

In many cases two-level systems can be analyzed using various numerical estimations. But this approach has two major drawbacks. First, it is not always accurate enough [6], and second, it is difficult to gain insight about the dependence of the system on its parameters.

For these reasons, efforts were invested in seeking tools to facilitate two-level analytical evaluations. One of the most efficient tools was introduced by Delos and Thorson [7], who showed that only one real function, the Stueckelberg parameter [8], is needed to fully describe a Hamiltonian. Normally, a two-level Hamiltonian is characterized by four real parameters (functions), and therefore the immediate outcome of the Delos-Thorson approach is that all the two-level Hamiltonians can be grouped into three-parameter families, each associated with a single differential equation. This technique was used by Delos, Thorson, and others (e.g., [9] and [10]) to generalize several analytical models.

In the present paper we further develop the Delos-Thorson approach. We show that the function that characterizes the Hamiltonian is the Stueckelberg parameter only for real Hamiltonians, while an additional term should be added in the general case. The use of a single characteristic function to describe Hamiltonians is demonstrated to be very efficient in categorizing models, and reveals equivalence between apparently different ones. As a matter of fact, we solve three fundamental models and show how all the analytically solvable models known to us emerge as special cases of these models. Furthermore, it is shown how by solving a single Hamiltonian, a variety of other models, mathematically equivalent but of different shapes, can be produced. We also give a geometrical interpretation to the modified Stueckelberg parameter and to other related quantities, and so associate these three-parameter families with special spatial curves that we dub *canonical curves*. However, it should be kept in mind that although elegant and of potential usefulness, the geometrical interpretation is not necessary for the efficient implementation of the method.

In Sec. II we introduce basic notations and give brief reviews on two subjects that are fundamental for the rest of the paper. The first is the intrinsic representation of spatial curves, and the second is the analogy between the two-level Schrödinger equation and the three-dimensional precession equation, known as the Bloch equation. In Sec. III we develop the general Delos-Thorson approach and establish its geometrical interpretation. In Sec. IV we investigate two special spatial curves — the straight line and the planar curve. In Secs. V–VII we analyze three basic models, that will be shown to generalize all the known analytically solvable models. Section VIII is a summary.

II. PRELIMINARIES

The two-level Schrödinger equation was shown to be equivalent to a vectorial three-dimensional precession equation, mostly known in physics as the Bloch equation or the magnetization equation. This equivalence was demonstrated by Feynman *et al.* [11], and then further investigated by others (e.g., [12]). In this paper we present another equivalent equation to the two-level Schrödinger equation. This equation is the one known from differential geometry as the Frenet-Serret equation. It is a kind of ''precession'' equation that describes the shape of one-dimensional spatial curves in a three-dimensional space. We devote this section to a brief review of certain elements in this equation.

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Frenet-Serret equation [13]. Given a curve $\vec{r}(u)$, u being some general parameter, it is customary to define at each point a local orthonormal triad (sometimes referred to as the *Frenet triad*) — the tangent $\vec{t}(u)$, the principal normal $\vec{n}(u)$, and the binormal $\vec{b}(u)$. The Frenet-Serret equation describes the rotation of this triad when moving along the curve. This equation is usually expressed using the natural parameter s, which is the length of the curve and can be obtained from the general parameter u by

$$s=\int_{u_0}^{u}\left|\frac{d\vec{r}(u)}{du}\right|du. \tag{2.1}$$

The Frenet-Serret equation is a precession equation that can be written in matrix form as

$$\frac{dA}{ds}=DA, \tag{2.2}$$

with

$$A\equiv\begin{pmatrix} t_1 & t_2 & t_3\\ n_1 & n_2 & n_3\\ b_1 & b_2 & b_3\end{pmatrix},\quad D\equiv\begin{pmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}. \tag{2.3}$$

Here $\kappa(s)$ is the curvature (a non-negative variable by definition), and $\tau(s)$ is the torsion. Knowing them determines uniquely the curve up to translations and rotations, thus their name — the intrinsic parameters of the curve. We call D the *Frenet-Serret matrix* and A the *curve matrix*. Solving the Frenet-Serret equation, i.e., knowing $A(s)$, is equivalent to knowing the curve $\vec{r}(s)$. We define the pair (D,A) as the *geometrical problem*, where D defines the problem (system) and A its solution.

Two-level Schrödinger equation vs *Bloch equation.* The general two-level Hamiltonian is characterized by four real parameters. Throughout this paper we adopt two alternate representations for this Hamiltonian:

$$H=\begin{pmatrix} H_{11} & H_{12}\\ H_{12}^\star & H_{22}\end{pmatrix}=\begin{pmatrix} H_{11} & |H_{12}|e^{-i\eta}\\ |H_{12}|e^{i\eta} & H_{22}\end{pmatrix},$$

$$\text{or}\quad H=\begin{pmatrix} H_0+H_3 & H_1-iH_2\\ H_1+iH_2 & H_0-H_3\end{pmatrix}, \tag{2.4}$$

which are related by

$$H_0=\frac{1}{2}(H_{11}+H_{22}),$$

$$\vec{H}=\frac{1}{2}(H_{12}+H_{12}^\star,i(H_{12}-H_{12}^\star),H_{11}-H_{22}). \tag{2.5}$$

Notice that the second representation emerges out of expressing H in terms of the Pauli matrices $H=H_0+\vec{H}\cdot\vec{\sigma}$. The wave function will also be represented alternately by two representations,

$$|\psi\rangle=\begin{pmatrix} a\\ b\end{pmatrix}$$

$$\text{or}\quad |\psi\rangle=e^{i\alpha}\begin{pmatrix} \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\,e^{i\varphi}\end{pmatrix}, \tag{2.6}$$

with $aa^\star+bb^\star=1$. In analogy with the geometrical problem we define the pair (H,ψ) as the *physical problem*, where H defines the problem and ψ the solution.

The dynamics of the wave function is determined by the Schrödinger equation, while that of the corresponding density matrix ρ is determined by the Liouville-Von Neumann equation $d\rho/dt=-i[H,\rho]$. Writing ρ in the form $\rho=\frac{1}{2}(p_0+\vec{p}\cdot\vec{\sigma})$ gives the following equation of motion for the vector $\vec{p}$ (mostly referred to as the polarization vector): $d\vec{p}/dt=2\vec{H}\times\vec{p}$ [14]. This is the equation that is called the magnetization equation or the Bloch equation [15], and its matrix form is

$$\frac{d}{dt}\begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix}=\begin{pmatrix} 0 & -2H_3 & 2H_2\\ 2H_3 & 0 & -2H_1\\ -2H_2 & 2H_1 & 0\end{pmatrix}\begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix}. \tag{2.7}$$

The polarization vector $\vec{p}$ is of unit length and is equivalent to the wave function $|\psi\rangle$ up to a global phase. The relation between them is given by $\vec{p}=\langle\psi|\vec{\sigma}|\psi\rangle$, i.e.,

$$\vec{p}=(ab^\star+a^\star b,i(ab^\star-a^\star b),aa^\star-bb^\star)$$

$$\text{or}\quad \vec{p}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta). \tag{2.8}$$

III. SPATIAL CURVES AND TWO-LEVEL HAMILTONIANS

In this section we establish the relation between the physical problem and the geometrical problem. In the first two parts we define explicitly this relation, and in the last part we discuss the physical interpretation of the geometrical quantities.

A. The geometrical equivalent of the Hamiltonian

The equivalence is based upon identifying the Frenet-Serret equation,

$$\frac{d}{ds}\begin{pmatrix} t_1 & t_2 & t_3\\ n_1 & n_2 & n_3\\ b_1 & b_2 & b_3\end{pmatrix}=\begin{pmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix} t_1 & t_2 & t_3\\ n_1 & n_2 & n_3\\ b_1 & b_2 & b_3\end{pmatrix},$$

with the Bloch equation,

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 & -2H_3 & 2H_2 \\ 2H_3 & 0 & -2H_1 \\ -2H_2 & 2H_1 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

It is clear that three difficulties need to be overcome before these equations could be made mathematically identical.

(1) H_2 should be made zero.

(2) Suppose we succeed in making H_2 vanish. Then it is tempting to define $\kappa=-2H_3$ and $\tau=-2H_1$. However, such a definition is not valid since the curvature must be non-negative, while H_3 is an arbitrary function of time.

(3) The Bloch equation should be written in a matrix form instead of a vector form.

The first obstacle can be removed by transforming to a rotating coordinate system. Such a transformation is done by applying a time-dependent unitary transformation $U(t)$ to the wave function, which results in the Hamiltonian transforming according to

$$H' = UHU^\dagger - iU\dot{U}^\dagger.$$

A transformation such that the resultant Hamiltonian H' will be real can always be chosen. Actually, there is an infinite number of such transformations, for example,

$$U(t) = \begin{pmatrix} e^{i(\eta/2)} & 0 \\ 0 & e^{-i(\eta/2)} \end{pmatrix},$$

all of which give Hamiltonians that differ only by their trace (which is proportional to H_0'), and have the same $\vec{H}'$ component,

$$\vec{H}' = \left(|H_{12}|, 0, \frac{1}{2}\left(H_{11} - H_{22} - \frac{d\eta}{dt} \right) \right). \qquad (3.1)$$

Even in the primed frame we cannot identify κ with $-2H_3'$ since the latter remains an arbitrary function of time. However, it is obvious from (3.1) that H_1' is non-negative. We may associate the curvature with H_1' by defining a new unit vector,

$$\vec{Q}^1 = R\vec{p}\,', \qquad (3.2)$$

with

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

a transformation that interchanges the spatial axes 1 and 3. The equation of motion of the vector $\vec{Q}^1 = (p_3', p_2', p_1')$ is given by

$$\frac{d}{dt}\begin{pmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \end{pmatrix} = \begin{pmatrix} 0 & 2H_1' & 0 \\ -2H_1' & 0 & 2H_3' \\ 0 & -2H_3' & 0 \end{pmatrix}\begin{pmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \end{pmatrix}. \qquad (3.3)$$

Now we can safely define

$$\kappa \equiv 2H_1', \qquad (3.4)$$

$$\tau \equiv 2H_3'.$$

Combining this result with (3.1) and (2.5), we can write these equations in terms of the original Hamiltonian,

$$\kappa = 2|H_{12}| = 2\sqrt{H_1^2 + H_2^2}, \qquad (3.5)$$

$$\tau = H_{11} - H_{22} - \frac{d\eta}{dt} = 2H_3 + \frac{H_2^2}{H_1^2 + H_2^2}\frac{d}{dt}\left(\frac{H_1}{H_2}\right).$$

Now we obtain well-defined curvature and torsion. All that is left is to transform Eq. (3.3) into a matrix form. This can be carried out in two equivalent methods:

(1) The evolution operator for $\vec{Q}^1$ is a 3×3 orthonormal matrix, designated U_Q, obeying the Frenet-Serret equation

$$\frac{dU_Q}{dt} = DU_Q,$$

where D is defined in (2.3) and the curvature and torsion are given in (3.4).

(2) From the structure of the Frenet-Serret equation (2.2), each of the column vectors

$$\begin{pmatrix} t_i \\ n_i \\ b_i \end{pmatrix}, \quad i = 1,2,3$$

obeys an equation similar to (3.3). Therefore, assuming that we can define an orthonormal triad,

$$\vec{Q}^i = \begin{pmatrix} t_i \\ n_i \\ b_i \end{pmatrix}, \quad i = 1,2,3$$

such that any of the vectors Q^i obey (3.3), then the matrix

$$Q = \begin{pmatrix} Q_1^1 & Q_1^2 & Q_1^3 \\ Q_2^1 & Q_2^2 & Q_2^3 \\ Q_3^1 & Q_3^2 & Q_3^3 \end{pmatrix},$$

will also obey the Frenet-Serret equation (2.2). We found explicit expressions for those Q^i. The simplest way to write them is in terms of the wave function in a frame in which the Hamiltonian is not only real but also traceless. A transformation to such a frame is always possible using the unitary transformation

$$U(t)=\begin{pmatrix} \exp\left(i\left[\int H_0(t')dt'+\eta/2\right]\right) & 0 \\ 0 & \exp\left(i\left[\int H_0(t')dt'-\eta/2\right]\right)\end{pmatrix}.$$

We label quantities in this frame by the subscript rt (we will return to this frame later in this paper). One can verify that in terms of ψ_{rt} the three vectors

$$\vec{Q}^1=(\cos\theta_{rt},\sin\theta_{rt}\sin\varphi_{rt},\sin\theta_{rt}\cos\varphi_{rt}), \tag{3.6}$$

$$\vec{Q}^2=\begin{pmatrix} -\sin\theta_{rt}\cos(2\alpha_{rt}+\varphi_{rt}) \\ -\sin 2\alpha_{rt}\cos^2\frac{\theta_{rt}}{2}-\sin^2\frac{\theta_{rt}}{2}\sin(2\alpha_{rt}+2\varphi_{rt}) \\ \cos 2\alpha_{rt}\cos^2\frac{\theta_{rt}}{2}-\sin^2\frac{\theta_{rt}}{2}\cos(2\alpha_{rt}+2\varphi_{rt})\end{pmatrix},$$

$$\vec{Q}^3=\begin{pmatrix} -\sin\theta_{rt}\sin(2\alpha_{rt}+\varphi_{rt}) \\ \cos 2\alpha_{rt}\cos^2\frac{\theta_{rt}}{2}+\sin^2\frac{\theta_{rt}}{2}\cos(2\alpha_{rt}+2\varphi_{rt}) \\ \sin 2\alpha_{rt}\cos^2\frac{\theta_{rt}}{2}-\sin^2\frac{\theta_{rt}}{2}\sin(2\alpha_{rt}+2\varphi_{rt})\end{pmatrix}$$

obey all the above requirements (and of course $\vec{Q}^1$ is the same $\vec{Q}^1$ we have defined before).

Now the differential equations for the Schrödinger equation and the Frenet-Serret equation are mathematically identical, and their solutions will be the same as long as the initial conditions are the same. How can we find Q or U_Q given the solution of the related curve under arbitrary initial conditions? Let A be the curve matrix for certain initial conditions $A(t_0)$, and let $\tilde{A}$ be the matrix of the same curve, but for different initial conditions, say $\tilde{A}(t_0)$. We denote by $\tilde{R}$ the orthogonal rotation that at time t_0 rotates the Frenet triad of A to overlap the equivalent triad of $\tilde{A}$. Such a rotation is described by

$$\tilde{A}^T(t_0)=\tilde{R}A^T(t_0),$$

so that

$$\tilde{R}=\tilde{A}^T(t_0)A(t_0). \tag{3.7}$$

Since the curves are identical, this rotation will make them completely overlapping, so that $\tilde{A}(t)=A(t)\tilde{R}^T$ for all times. Substituting (3.7) gives

$$\tilde{A}(t)=A(t)A^T(t_0)\tilde{A}(t_0). \tag{3.8}$$

Taking $\tilde{A}$ to be Q we get

$$Q(t)=A(t)A^T(t_0)Q(t_0),$$

and taking $\tilde{A}$ to be U_Q we obtain

$$U_Q(t)=A(t)A^T(t_0)$$

(which is, of course, the evolution operator of Q), or explicitly

$$U_Q=\begin{pmatrix} \vec{t}\cdot\vec{t}(0) & \vec{t}\cdot\vec{n}(0) & \vec{t}\cdot\vec{b}(0) \\ \vec{n}\cdot\vec{t}(0) & \vec{n}\cdot\vec{n}(0) & \vec{n}\cdot\vec{b}(0) \\ \vec{b}\cdot\vec{t}(0) & \vec{b}\cdot\vec{n}(0) & \vec{b}\cdot\vec{b}(0)\end{pmatrix}.$$

Thus far we related spatial curves to two-level Hamiltonians. While a general Hamiltonian is characterized by four real parameters, a general curve is characterized by two parameters only. Therefore, we conclude that a two-parameter family of Hamiltonians is associated with every single curve. In other words, a solution of one geometrical problem yields the solutions of a two-parameter family of Hamiltonians. We can write explicitly the two-parameter family that is associated with the curve characterized by curvature κ and torsion τ. We designate a general member in this family by $H_{\kappa,\tau}(\xi,\zeta)$, where ξ and ζ serve as the two free parameters. Knowing that the trace does not affect the associated curve, it is natural to choose H_0 as one of the free parameters, say $\zeta=H_0$. We arbitrarily choose the other one to be H_3 so that $\xi=H_3$. Comparison of the two representations in (2.4) gives

$$\cos\eta=\frac{H_1}{\sqrt{H_1^2+H_2^2}},\quad \sin\eta=\frac{H_2}{\sqrt{H_1^2+H_2^2}}.$$

This result, together with (3.5), yields

$$H_1=\frac{1}{2}\kappa\cos\eta,\quad H_2=\frac{1}{2}\kappa\sin\eta.$$

η can be expressed in terms of ξ and τ using (3.5) together with (2.4),

$$\eta=\int(2\xi-\tau)dt,$$

so that

$$H_1=\frac{1}{2}\kappa\cos\int(2\xi-\tau)dt,\quad H_2=\frac{1}{2}\kappa\sin\int(2\xi-\tau)dt.$$

Substituting this in (2.4) gives

$$H_{\kappa,\tau}(\xi,\zeta)=\begin{pmatrix} \zeta+\xi & \frac{1}{2}\kappa\exp\left(-i\int(2\xi-\tau)dt\right) \\ \frac{1}{2}\kappa\exp\left(i\int(2\xi-\tau)dt\right) & \zeta-\xi \end{pmatrix}, \tag{3.9}$$

which is the general form of a member in the family associated with the curve whose curvature is κ and whose torsion is τ.

Within each family we can consider some interesting one-parameter subfamilies. The one-parameter subfamily of traceless Hamiltonians is given by

$$H_{\kappa,\tau}(\xi,\zeta=0)=\begin{pmatrix} \xi & \frac{1}{2}\kappa\exp\left(-i\int(2\xi-\tau)dt\right) \\ \frac{1}{2}\kappa\exp\left(i\int(2\xi-\tau)dt\right) & -\xi \end{pmatrix}.$$

The one-parameter subfamily of real Hamiltonians is

$$H_{\kappa,\tau}\left(\xi=\frac{1}{2}\tau,\zeta\right)=\begin{pmatrix} \zeta+\frac{1}{2}\tau & \frac{1}{2}\kappa \\ \frac{1}{2}\kappa & \zeta-\frac{1}{2}\tau \end{pmatrix}.$$

The intersection of these two subfamilies is a unique member of the family, which is both traceless and real,

$$H_{rt}=H_{\kappa,\tau}\left(\xi=\frac{1}{2}\tau,\zeta=0\right)=\begin{pmatrix} \frac{1}{2}\tau & \frac{1}{2}\kappa \\ \frac{1}{2}\kappa & -\frac{1}{2}\tau \end{pmatrix}.$$

(We have mentioned this kind of Hamiltonian earlier, and labeled it by the subscript rt.) It is a very common representation for Hamiltonians in physics, and many models are formulated in this frame. Yet another form that appears frequently in two-level models is the so-called Rosen-Zener Hamiltonian [16], which is also a unique member of the family, identified by zero diagonal elements,

$$H_{rz}=H_{\kappa,\tau}(\xi=0,\zeta=0)=\begin{pmatrix} 0 & \frac{1}{2}\kappa\exp\left(i\int\tau dt\right) \\ \frac{1}{2}\kappa\exp\left(-i\int\tau dt\right) & 0 \end{pmatrix}.$$

B. Definition of canonical curve

So far we did not distinguish between the physical time parameter and the natural parameter of the curve, and tacitly regarded them identical. But, considering the possibility of representing a curve by other parametrizations yields a powerful technique that enables the grouping of all possible curves into one-parameter families, where the members of each family share a common curve solution.

Let s be the natural parameter of some curve, and $s=s(u)$ a monotonically increasing function. The Frenet-Serret equation in terms of the variable u is

$$\frac{d}{du}A(s(u))=D(s(u))\frac{ds}{du}A(s(u)). \tag{3.10}$$

Let us define ''effective'' curvature and torsion

$$\kappa'(u)=\kappa(s(u))\frac{ds}{du}, \tag{3.11}$$

$$\tau'(u)=\tau(s(u))\frac{ds}{du},$$

and thus define the matrix

$$D'(u)=D(s(u))\frac{ds}{du}=\begin{pmatrix} 0 & \kappa'(u) & 0 \\ -\kappa'(u) & 0 & \tau'(u) \\ 0 & -\tau'(u) & 0 \end{pmatrix}.$$

All that is left is to denote $A'(u)=A(s(u))$, so that (3.10) can be rewritten as

$$\frac{dA'(u)}{du}=D'(u)A'(u). \tag{3.12}$$

However, this equation is nothing but the Frenet-Serret equation of another curve, characterized by $\kappa'(u)$ and $\tau'(u)$, with natural parameter u. If we know $A(s)$, the solution of the original curve, we can immediately write the solution of the second curve $A'(u)$ simply by substituting $s\rightarrow s(u)$ in

$A(s)$. We can pick any function $s(u)$, as long as $ds/du>0$. Any such function yields a different curve $A'(u)$. Therefore, one solution yields the solutions of an entire family of curves, characterized by the parameter $s(u)$. Combining this conclusion with our previous results concerning Hamiltonians, we conclude that one solved geometrical problem enables us to solve immediately for a one-parameter family of curves, or a three-parameter family of Hamiltonians. Thus, out of the four parameters that normally characterize a Hamiltonian, only one is really needed.

The problem is, given two curves, how to decide whether they belong to the same family. Our approach is to define a transformation, which we dub *the canonical transformation*, that when applied to a curve transforms it into a form, called *the canonical curve*, which is common to all curves within the same family. In other words, the canonical curve is a unique member in the family, and the canonical transformation maps any other curve in the family into the canonical form. We define the canonical curve to be characterized by a constant unit curvature at all points. Its torsion is a function of s, the natural parameter of the canonical curve, and denoted $\tau_c(s)$. This function can be taken as the parameter that characterizes each family of curves. Given a curve with natural parameter u, curvature $\kappa'(u)$, and torsion $\tau'(u)$, applying the canonical transformation yields the variables s and τ_c. $s(u)$ is obtained from the first of equations (3.11), using the fact that for the canonical curve $\kappa(s(u))=1$,

$$s(u)=\int^{u}\kappa'(u')du'. \tag{3.13}$$

Being monotonic, this function must have an inverse function, $u(s)$. Using this function, we can utilize the second of Eq. (3.11) to produce the canonical torsion,

$$\tau_c(s)=\frac{\tau'(u(s))}{\kappa'(u(s))}. \tag{3.14}$$

We can think of $\tau_c(s)$ as a single function that suffices to characterize any two-dimensional Hamiltonian. The space of two-dimensional Hamiltonians is therefore constructed of three-parameter families, where the members of each family share a common solution, and each family is identified by the canonical torsion.

Physically, changing the parametrization of a curve is equivalent to applying a time gauge to the system. We can obtain the same results starting from

$$i\frac{d|\psi(t)\rangle}{dt}=H(t)|\psi(t)\rangle,$$

and then applying the gauge $t=t(u)$, with $t(u)$ a monothonically increasing function. The Schrödinger equation for the variable u is

$$i\frac{d|\psi(t(u))\rangle}{du}=H(t(u))\frac{dt}{du}|\psi(t(u))\rangle.$$

Defining

$$|\psi'(u)\rangle\equiv|\psi(t(u))\rangle \quad H'(u)\equiv H(t(u))\frac{dt}{du},$$

we get the following equation:

$$i\frac{d|\psi'(u)\rangle}{du}=H'(u)|\psi'(u)\rangle, \tag{3.15}$$

which is nothing but the Schrödinger equation for the Hamiltonian $H'(u)$. Equation (3.15) is the physical problem equivalent to (3.12). Actually, it is easy to see that the curvatures and torsions associated with the Hamiltonians $H(t)$ and $H'(u)$, obey the relations (3.11).

C. Physical interpretation

The curvature, torsion, canonical torsion, and natural parameter are all functions of the elements of the Hamiltonian. Under what circumstances do they have physical meaning? It depends, of course, on the interpretation we give to the Hamiltonian itself. In the following we survey briefly some of the systems for which interesting physical interpretations may be obtained.

(1) Two-level atom. The physics is usually described by the pseudo spin $\vec{s}$, which is an analog of the polarization vector defined in (2.7). The dynamics of the system, in the rotating wave approximation (RWA), is described by the Bloch equation $d\vec{s}/dt=2\vec{\Omega}\times\vec{s}$, with $\vec{\Omega}=(-dE,0,\frac{1}{2}(\omega_0-\omega))$ [3]. Here E is the field amplitude, d is the electric dipole of the atom, ω_0 is the frequency gap between the two levels, and ω is the field frequency. In this case it is easy to see that

$$\kappa=2dE,$$

$$\tau=\omega_0-\omega.$$

These two functions are very familiar in this field (see, e.g., [3,17,18]). The curvature is known as *the field envelope*, and the torsion is known as *the detuning* — the frequency difference between the radiation and the resonance.

(2) Spin 1/2. The Hamiltonian is usually written as $H=-\frac{1}{2}\gamma\vec{B}\cdot\vec{\sigma}$, with γ the giromagnetic ratio, $\vec{B}$ the magnetic field, and $\vec{\sigma}$ the Pauli matrices. For this case,

$$\kappa=|\gamma|\sqrt{B_1^2+B_2^2},$$

$$\tau=-\gamma B_3-\frac{B_1^2}{B_1^2+B_2^2}\frac{d}{dt}\left(\frac{B_2}{B_1}\right),$$

so that the curvature is proportional to the projection of the magnetic field on the 1-2 plane. The torsion is harder to interpret, but if B_1/B_2 does not depend on time, the torsion becomes proportional to the projection of the magnetic field on the 3 axis.

(3) Double Stern-Gerlach experiment. This is the system that was treated by Rosen and Zener [16]. They analyzed the case in which one of the output beams of the Stern-Gerlach

experiment is passed through a rotating weak magnetic field and then is subject to a second Stern-Gerlach experiment. For their model the curvature is the rotation rate of the magnetic field, and the torsion is the frequency gap between the two levels of the beam.

(4) Electronic transitions. Two-level transitions due to collisions and level crossings were investigated intensively. One defines (e.g., [7]) the inelastic action function $\tilde{s}$ and the (dimensionless) classical action difference function Ξ as

$$\tilde{s}=\int H_{12}dt,$$

$$\Xi=\int (H_{11}-H_{22})dt,$$

and the Stueckelberg [8] parameter, $\tilde{t}$, is defined as

$$\tilde{t}=\frac{H_{11}-H_{22}}{2H_{12}}.$$

This is the function used by Delos and Thorson [7] as the single function that characterizes all the two-dimensional Hamiltonians, analogous to our canonical torsion. They also showed how basic solutions can be generalized using this function to describe three-parameter Hamiltonian families. Notice that they assumed real Hamiltonians, with positive off-diagonal elements. Under their assumptions, the Stueckelberg parameter is indeed identical to the canonical torsion, since for real and positive off-diagonal elements $\kappa=2|H_{12}|=2H_{12}$ and $\eta=0$ so that

$$\tau_c=\tilde{t}=\frac{H_{11}-H_{22}}{2H_{12}}.$$

Therefore, the canonical torsion is a generalization of the Stueckelberg parameter. Also, under these assumptions the inelastic action function becomes simply the natural parameter of the canonical curve $\tilde{s}=s$, while the classical action difference function is just $\Xi=\int\tau dt$.

D. Application of the formalism

Whether the formalism is viewed through the eyes of geometry, or simply as a mathematical tool, it has great advantages in the process of analyzing two-level models.

First we should note that two models that belong to the same family can be unrecognizably different. The above procedure is perfectly suited to reveal such cases. Any one who formulates new models, should first determine its canonical torsion and compare it to the known ones (see the following sections), thus instantly checking whether the solution for the model was actually already obtained. In this perspective, the canonical formalism is an efficient classification procedure.

Furthermore, given τ_c, Eqs. (3.13) and (3.14) can be utilized to generate new models that belong to the family. Simply pick $\kappa(u)$, and $\tau(u)$ follows from the formalism.

Finally, although not designed for this purpose, we were able to use the formalism to present two generalizations of known models—a generalization of the Landau-Zener model (in Sec. VI) and a generalization of the Demkov model (in Sec. VII).

IV. ZERO CURVATURE VS ZERO TORSION

For completeness, we analyze here the two singular cases, i.e., $\kappa=0$ and $\tau=0$.

A. Straight line

A straight line is the geometrical curve associated with zero curvature everywhere. Physically, the Hamiltonian is diagonal at all times, with time-dependent diagonal elements. This is, of course, a trivial case and we bring it only for completeness. The torsion in this case is not defined, so we can simply associate it with any arbitrary function that we pick. No canonical curve can be defined, since (3.13) is identically zero and (3.14) is not defined. The real and traceless Hamiltonian has the form

$$H_{rt}=\begin{pmatrix} \frac{1}{2}\tau & 0 \\ 0 & -\frac{1}{2}\tau \end{pmatrix},$$

with the torsion a general function of time, and the wave function is given by

$$\psi_{rt}=\begin{pmatrix} a_0\exp\left(-\frac{i}{2}\int_0^t\tau(t')dt'\right) \\ b_0\exp\left(\frac{i}{2}\int_0^t\tau(t')dt'\right) \end{pmatrix}.$$

B. Planar curve

Of more interest is the case $\tau=0$ everywhere. The geometrical meaning of the torsion is the amount of nonplanar twisting of the curve [13], and a curve with zero torsion everywhere is called planar. The Hamiltonian H_{rt} is

$$H_{rt}=\begin{pmatrix} 0 & \frac{1}{2}\kappa \\ \frac{1}{2}\kappa & 0 \end{pmatrix},$$

with the curvature an arbitrary function of time. The Schrödinger equation for this case can be solved analytically for any κ by other means (see, e.g., [3]), but the solution using the canonical transformation is immediate. From (3.13) and (3.14) one deduces that all the planar curves are members of the same family, characterized by the canonical torsion $\tau_c=0$, i.e., the canonical curve is just the unit circle. The canonical Hamiltonian, i.e., the one for which $\kappa=1$, is constant and the appropriate wave function is trivial,

$$\psi_{rt}^c=\begin{pmatrix} a_0\cos\frac{1}{2}s-ib_0\sin\frac{1}{2}s \\ -ia_0\sin\frac{1}{2}s+b_0\cos\frac{1}{2}s \end{pmatrix},$$

where the superscript c stands for ''canonical.'' To obtain the solution for general κ, all we need do is substitute $s \to \int \kappa dt$ to get

$$\psi_{rt} = \begin{pmatrix} a_0 \cos \frac{1}{2} \int \kappa dt - i b_0 \sin \frac{1}{2} \int \kappa dt \\ -i a_0 \sin \frac{1}{2} \int \kappa dt + b_0 \cos \frac{1}{2} \int \kappa dt \end{pmatrix}.$$

V. THE HYPERGEOMETRIC MODEL

Rosen and Zener [16] discussed the results of a double Stern-Gerlach experiment. In our notation, their model is

$$\kappa_{rz} = \alpha \operatorname{sech} \gamma t,$$

$$\tau_{rz} = \beta.$$

They solved this model analytically, using the gauge transformation $z = \frac{1}{2}(1 + \tanh \gamma t)$, replacing the time $t \in (-\infty, \infty)$ by the parameter $z \in [0,1]$. They showed that in terms of z, the Schrödinger equation becomes identical with the hypergeometric differential equation, thus the wave function components can be found analytically in terms of hypergeometric functions. Since their pioneering work, many other studies further generalized this model (see below). In the present section we find the most general model that can be solved using the hypergeometric equation, and demonstrate that many related models, including the Rosen-Zener model itself, emerge as special cases of our solution. In the derivation we follow the guidelines established by Rosen-Zener and others (see, e.g., [17,19]).

The differential equation for the component a of the wave function in the Rosen-Zener frame is

$$\ddot{a}_{rz} - \left(\frac{\dot{\kappa}}{\kappa} + i\tau \right) \dot{a}_{rz} + \left(\frac{\kappa}{2} \right)^2 a_{rz} = 0.$$

A similar equation but with $\tau \to -\tau$ holds for the component b. We transform to the argument z, such that

$$z(-\infty) = 0, \quad z(\infty) = 1, \quad \dot{z}(t) > 0.$$

The differential equation for $a_{rz}(z)$ is

$$a''_{rz} + \frac{a'_{rz}}{\dot{z}} \left[\frac{d}{dt} \ln \frac{\dot{z}}{\kappa} - i\tau \right] + \left(\frac{\kappa}{2\dot{z}} \right)^2 a_{rz} = 0, \tag{5.1}$$

with dot denoting time derivative and prime denoting the derivative with respect to z. This equation can be brought into the form of the hypergeometric equation (see Appendix) only if

$$\frac{\kappa}{\dot{z}} = \frac{2\sqrt{-pq}}{\sqrt{z(1-z)}} \tag{5.2}$$

$$\frac{\tau}{\dot{z}} = \frac{(p+q)z + \frac{1}{2} - r}{iz(1-z)}.$$

The requirement that the curvature and torsion be real imposes the constraints

$$-pq = \text{non-negative real},$$

$$p + q = \text{pure imaginary},$$

$$\operatorname{Re}(r) = \frac{1}{2}, \quad \operatorname{Im}(r) \neq 0,$$

from which we see that $q = -p^\star$. The canonical transformation can be applied to (5.2) to give

$$s - s_0 = \int \kappa(t) dt = \int \frac{\kappa(z)}{\dot{z}} dz = 4\sqrt{-pq} \tan^{-1} \sqrt{\frac{z}{1-z}},$$

so that

$$z = \sin^2 \frac{s - s_0}{4\sqrt{-pq}}.$$

The canonical torsion is therefore

$$\tau_c = \frac{\frac{1}{2} - r + (p+q)\sin^2 \frac{s-s_0}{4\sqrt{-pq}}}{i\sqrt{-pq} \left| \sin \frac{s-s_0}{2\sqrt{-pq}} \right|}.$$

To obtain a more compact form we choose $s_0 = -3\pi\sqrt{-pq}$ and then

$$\tau_c = \frac{\frac{1}{2}(1+p+q) - r - \frac{1}{2}(p+q)\sin \frac{s}{2\sqrt{-pq}}}{i\sqrt{-pq} \left| \cos \frac{s}{2\sqrt{-pq}} \right|}.$$

Defining now

$$p = -q^\star = \frac{\sqrt{\alpha^2 - \delta^2} + i\delta}{2\gamma}, \quad r = \frac{1}{2} - i\frac{\beta - \delta}{2\gamma},$$

with $\alpha, \beta, \gamma, \delta$ real constants, we can substitute the relations

$$\sqrt{-pq} = \frac{\alpha}{2\gamma}, \quad p + q = i\frac{\delta}{\gamma}, \quad \frac{1}{2}(1+p+q) - r = i\frac{\beta}{2\gamma},$$

into the expression for the canonical torsion, to get the form

$$\tau_c = \frac{\beta + \delta \sin \frac{\gamma s}{\alpha}}{\alpha \left| \cos \frac{\gamma s}{\alpha} \right|}. \tag{5.3}$$

This defines a three-parameter family of Hamiltonians, all associated with the hypergeometric equation, and thus we name this family *the hypergeometric family*. a_{rz} is a solution of the hypergeometric equation; thus (see Appendix) it is given by

$$a_{rz}=AF(p,q,r;z)+Bz^{1-r}F(p+1-r,q+1-r,2-r;z),$$

with constants A and B. b_{rz} is also a solution of the hypergeometric equation, but with different coefficients. Applying the transformation $\tau\rightarrow-\tau$ one finds that these coefficients are precisely the complex conjugates of those for a_{rz}, i.e., $p\rightarrow p^{\star}=-q$, $q\rightarrow q^{\star}=-p$ and $r\rightarrow r^{\star}=1-r$. Thus,

$$b_{rz}=CF(-p,-q,1-r;z)+Dz^{r}F(r-p,r-q,1+r;z),$$

with constants C and D. The dependence of C and D on A and B can be revealed using the relation $i\dot{a}_{rz}=\frac{1}{2}\kappa e^{i\int\tau dt}b_{rz}$ (which emerges out of the Schrödinger equation),

$$C=-2^{i\delta/\gamma}\frac{1-r}{\sqrt{pq}}B \quad D=-2^{i\delta/\gamma}\frac{\sqrt{pq}}{r}A.$$

The full solutions have the form

$$a_{rz}=AF(p,q,r;z)+Bz^{1-r}F(p+1-r,q+1-r,2-r;z), \tag{5.4}$$

$$b_{rz}=-2^{i\delta/\gamma}\frac{1-r}{\sqrt{pq}}BF(-p,-q,1-r;z) -2^{i\delta/\gamma}\frac{\sqrt{pq}}{r}Az^{r}F(r-p,r-q,1+r;z),$$

with A and B given in terms of the initial values a_0 and b_0, taken in time t_0 (or z_0),

$$A=\frac{1-r/\sqrt{pq}F(-p,-q,1-r;z_0)a_0+2^{-i\delta/\gamma}z_0^{1-r}F(p+1-r,q+1-r,2-r;z_0)b_0}{1-r/\sqrt{pq}F(p,q,r;z_0)F(-p,-q,1-r;z_0)-\sqrt{pq}/rz_0F(p+1-r,q+1-r,2-r;z_0)F(r-p,r-q,1+r;z_0)}$$

$$B=\frac{2^{-i\delta/\gamma}F(p,q,r;z_0)b_0+\sqrt{pq}/rz_0^{r}F(r-p,r-q,1+r;z_0)a_0}{\sqrt{pq}/rz_0F(p+1-r,q+1-r,2-r;z_0)F(r-p,r-q,1+r;z_0)-1-r/\sqrt{pq}F(p,q,r;z_0)F(-p,-q,1-r;z_0)}.$$

This is the most general solution of the hypergeometric model. Various special cases can be obtained using different choices of $\dot{z}$ and of the model parameters. The original model of Rosen and Zener [16] is the special case $\dot{z}=2\gamma z(1-z)$ and $\delta=0$, from which we obtain indeed

$$z_{rz}=\frac{1}{2}(1+\tanh\gamma t),$$

$$\kappa_{rz}=\alpha\,\mathrm{sech}\,\gamma t,$$

$$\tau_{rz}=\beta.$$

Hioe [17] presented a generalization of the Rosen-Zener model. In our notation, he removed the constraint $\delta=0$. Hence, the Hioe model is

$$z_H=\frac{1}{2}(1+\tanh\gamma t),$$

$$\kappa_H=\alpha\,\mathrm{sech}\,\gamma t,$$

$$\tau_H=\beta+\delta\tanh\gamma t.$$

Another model was proposed by Bambini and Berman [20]. They investigated cases in which the torsion is not zero and the curvature is an asymmetric function of time. Their model is described by taking

$$\dot{z}=\frac{z(1-z)}{\mu+\lambda z}, \quad \delta=0.$$

Two very significant generalizations of the model are worth mentioning here. Dinterman and Delos [9] found the general solution associated with the canonical torsion

$$\tau_c^{DD}=\frac{\beta}{\alpha\left|\cos\frac{\gamma s}{\alpha}\right|},$$

which is obviously the special case $\delta=0$ in (5.3). Hioe and Carroll [18] realized how to find all the models associated with the most general canonical torsion (5.3). They found a way to associate all the members of the hypergeometric family with a single function, analogous to the canonical torsion. Therefore, they were the first to solve analytically the hypergeometric model.

We mention two more well-known analytic models that are special members of the hypergeometric family. Both were developed using the RWA with the Bloch equation. The first is due to McCall and Hahn [21], who considered the transition of optical radiation in matter and formulated the quantum area theorem. To demonstrate their claims they built a model that may be obtained by choosing $\dot{z}=2\gamma z(1-z)$, $\delta=0$, and $\alpha=2\gamma$, to yield

$$\kappa_{MH} = 2\gamma \,\mathrm{sech}\, \gamma t,$$

$$\tau_{MH} = \beta.$$

For this model, the solutions (5.4) may be expressed in terms of simple analytic functions. The second model was developed by Allen and Eberly ([3], Sec. 4.6), and is obtained by choosing $\dot{z} = 2\gamma z(1-z)$, $\beta = 0$, and $\alpha = \gamma\sqrt{1+\delta^2/\gamma^2}$, to yield

$$\kappa_{AE} = \gamma\sqrt{1+\delta^2/\gamma^2}\,\mathrm{sech}\,\gamma t,$$

$$\tau_{AE} = \delta \tanh \gamma t.$$

For this model, too, it is possible to express the solutions (5.4) in terms of simple analytic functions.

VI. THE PARABOLIC-CYLINDRIC MODEL

Landau [22] and Zener [23] investigated the level structure of a two-atom molecule undergoing level crossing. This triggered the Stueckelberg analysis [8], which yielded the Stueckelberg parameter. Landau and Zener were interested in the final transition probability, for the model described by

$$\kappa_{LZ} = \gamma,$$

$$\tau_{LZ} = \delta t,$$

which is called the Landau-Zener model. This model was already solved analytically for all times, assuming certain initial conditions [24]. We show that the Landau-Zener model is a special case of a generalized model, which we solve analytically. This model emerges from identifying the Schrödinger equation with the parabolic-cylindric equation, and therefore we name it the parabolic-cylindric model.

The differential equation for the component a of the canonical wave function in the rt frame is

$$\frac{d^2 a_{rt}^c}{ds^2} + \left[\frac{1}{4}(1+\tau_c^2) + \frac{i\dot{\tau}_c}{2}\right] a_{rt}^c = 0.$$

A similar equation but with $\tau_c \to -\tau_c$ holds for the component b. This equation can be brought into the form of the parabolic-cylindric equation (see Appendix) only if

$$2i\dot{\tau}_c = 4ps^2 + 4qs + 4r - 1 - \tau_c^2. \tag{6.1}$$

This is the general definition of the parabolic-cylindric canonical torsion, but we do not know the general solution of this equation. However, we can find a special solution by assuming a finite polynomial for the canonical torsion $\tau_c = \Sigma_{j=0}^{n} a_j s^j$. Inserting this polynomial into the equation reveals that the only possible finite polynomial that solves it is

$$\tau_c = \alpha + \beta s. \tag{6.2}$$

This canonical torsion defines a subfamily of the parabolic-cylindric family. For this subfamily, the components of the wave function satisfy the equations

$$\frac{d^2 a_{rt}^c}{ds^2} + \left[\frac{1}{4}\beta^2 s^2 - \frac{1}{2}\alpha\beta s + \frac{1}{4}(1+\alpha^2) + \frac{1}{2}i\beta\right] a_{rt}^c = 0,$$

$$\frac{d^2 b_{rt}^c}{ds^2} + \left[\frac{1}{4}\beta^2 s^2 - \frac{1}{2}\alpha\beta s + \frac{1}{4}(1+\alpha^2) - \frac{1}{2}i\beta\right] b_{rt}^c = 0.$$

To obtain the standard form of the parabolic-cylindric equation (see Appendix), we transform to the variable $w = \sqrt{i\beta}s + \sqrt{i/\beta}\alpha$. In terms of this variable the equations become

$$\frac{d^2 a_{rt}^c}{dw^2} - \left(\frac{1}{4}w^2 + V_a\right) a_{rt}^c = 0,$$

$$\frac{d^2 b_{rt}^c}{dw^2} - \left(\frac{1}{4}w^2 + V_b\right) b_{rt}^c = 0,$$

with V_a and V_b given by

$$V_a = -V_b^{\star} = \frac{i}{4\beta} - \frac{1}{2}.$$

These equations are solved by (see Appendix)

$$a_{rt}^c = Ae^{\{-(1/4)w^2\}} M\left(f, \frac{1}{2}; \frac{1}{2}w^2\right) + Bwe^{\{-(1/4)w^2\}} M\left(f + \frac{1}{2}, \frac{3}{2}; \frac{1}{2}w^2\right),$$

$$b_{rt}^c = Ce^{\{-(1/4)w^2\}} M\left(f + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}w^2\right) + Dwe^{\{-(1/4)w^2\}} M\left(f + 1, \frac{3}{2}; \frac{1}{2}w^2\right),$$

where we denoted $f = i/8\beta$. We can use the relation $2i\,da_{rt}^c/ds = \tau_c a_{rt}^c + b_{rt}^c$ (which results from the Schrödinger equation) to express C and D in terms of A and B. We get

$$C = -2\sqrt{\frac{\beta}{i}}B \quad D = -4f\sqrt{\frac{\beta}{i}}A.$$

Now we can write down the full solution

$$a_{rt}^c = Ae^{\{-(1/4)w^2\}} M\left(f, \frac{1}{2}; \frac{1}{2}w^2\right) + Bwe^{\{-(1/4)w^2\}} M\left(f + \frac{1}{2}, \frac{3}{2}; \frac{1}{2}w^2\right), \tag{6.3}$$

$$b_{rt}^c = -4f\sqrt{\frac{\beta}{i}}Awe^{\{-(1/4)w^2\}} M\left(f + 1, \frac{3}{2}; \frac{1}{2}w^2\right) - 2\sqrt{\frac{\beta}{i}}Be^{\{-(1/4)w^2\}} M\left(f + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}w^2\right),$$

with A and B given in terms of the initial conditions

$$A=e^{\{(1/4)w_0^2\}}\frac{a_0M(f+\frac{1}{2},\frac{1}{2};\frac{1}{2}w_0^2)+\frac{1}{2}\sqrt{i/\beta}b_0w_0M(f+\frac{1}{2},\frac{3}{2};\frac{1}{2}w_0^2)}{2fw_0^2M(f+\frac{1}{2},\frac{3}{2};\frac{1}{2}w_0^2)M(f+1,\frac{3}{2};\frac{1}{2}w_0^2)-M(f,\frac{1}{2};\frac{1}{2}w_0^2)M(f+\frac{1}{2},\frac{1}{2};\frac{1}{2}w_0^2)},$$

$$B=e^{\{(1/4)w_0^2\}}\frac{2fa_0w_0M(f+1,\frac{3}{2};\frac{1}{2}w_0^2)+\frac{1}{2}\sqrt{i/\beta}b_0M(f,\frac{1}{2};\frac{1}{2}w_0^2)}{M(f,\frac{1}{2};\frac{1}{2}w_0^2)M(f+\frac{1}{2},\frac{1}{2};\frac{1}{2}w_0^2)-2fw_0^2M(f+\frac{1}{2},\frac{3}{2};\frac{1}{2}w_0^2)M(f+1,\frac{3}{2};\frac{1}{2}w_0^2)}.$$

The original Landau-Zener model is obtained for $\alpha=0$ and $s=\gamma t$, which yields

$$\kappa_{LZ}=\gamma,$$

$$\tau_{LZ}=\beta\gamma^2 t\equiv\delta t.$$

The solution is given by (6.3) with $w=\sqrt{i\beta}\gamma t$. One may obtain many other models by choosing different transformations. For example, we can generalize the Landau-Zener model by choosing $\kappa=\gamma t^n$, for which we get

$$\kappa=\gamma t^n, \tag{6.4}$$

$$\tau=\alpha\gamma t^n+\frac{\beta\gamma^2}{n+1}t^{2n+1},$$

with $w=\sqrt{i\beta}\gamma/(n+1)t^{n+1}+\sqrt{i/\beta}\alpha$. The constraint $\kappa>0$ will limit the solution for odd n to the region $t\in[0,\infty)$. The Landau-Zener model is a special case of this model with $n=0$.

VII. THE CONFLUENT-HYPERGEOMETRIC MODEL

We choose again Eq. (5.1) as a starting point, but now we try to write it in the form of the confluent-hypergeometric equation (see Appendix). For the sake of brevity, let us designate

$$F(z)\equiv\frac{bh'}{h}-h'-\frac{h''}{h'},$$

$$G(z)\equiv\frac{R}{z}+f',$$

$$H(z)\equiv\frac{R(R-1)}{z^2}+\frac{2Rf'}{z}+f''+f'^2-\frac{ah'^2}{h}=G'+G^2-\frac{ah'^2}{h}.$$

Equation (5.1) will have the form of the confluent-hypergeometric equation if

$$\frac{\kappa}{\dot z}=2\sqrt{FG+H},\quad \frac{\tau}{\dot z}=i(2G+F)+\frac{i}{2}\frac{F'G+G'F+H'}{FG+H}.$$

These expressions can be put in an even more compact form by adopting the notation

$$P\equiv FG+H,\quad Q\equiv 2G+F.$$

Then

$$\frac{\kappa}{\dot z}=2\sqrt{P}, \tag{7.1}$$

$$\frac{\tau}{\dot z}=iQ+\frac{i}{2}\frac{P'}{P}.$$

Any particular member in the confluent-hypergeometric family will be defined through the functions $G(z),H(z)$ and the parameters a and b. We demonstrate in the following lines how different choices of these quantities reproduce many known models.

Let us start by assuming that $G=0$. Then $P=H=-ah'^2/h$ and $Q=F=bh'/h-h'-h''/h'$, and thus

$$\frac{\kappa}{\dot z}=2i\sqrt{a}\frac{h'}{\sqrt{h}},\quad \frac{\tau}{\dot z}=i\frac{h'}{h}\left(b-\frac{1}{2}\right)-ih'.$$

Applying the canonical transformation we use (3.13) to get

$$h=-\frac{s^2}{16a}.$$

Substitution in the equation for the torsion yields

$$\frac{\tau}{\dot z}=\frac{2i}{s}\frac{\kappa}{\dot z}\left(b-\frac{1}{2}\right)-\frac{s}{8ia}\frac{\kappa}{\dot z}.$$

For the torsion to be real we must have $a=i\alpha$ and $b=\frac{1}{2}-i\beta$, with α,β real constants. Substituting the above results in the expression for the canonical torsion we find

$$\tau_c=\frac{2\beta}{s}+\frac{s}{8\alpha}. \tag{7.2}$$

The solution of the equation is given by (see Appendix)

$$a_{rz}^c=AM\left(i\alpha,\frac{1}{2}-i\beta;h\right)+Bh^{1/2+i\beta}M\left(\frac{1}{2}+i(\alpha+\beta),\frac{3}{2}+i\beta;h\right),$$

where we used the fact that $G=0$ yields $f'=-R/z$, or $f=-R\ln z+f_0$, and therefore $z^{-R}e^{-f(z)}=e^{f_0}=\text{Const.}$ b_{rz}^c is also a solution of the confluent-hypergeometric equation, whose coefficients are determined by the mapping $\alpha\to-\alpha$ and $\beta\to-\beta$. The result is

$$b_{rz}^c=CM\left(-i\alpha,\frac{1}{2}+i\beta;-h\right)+D(-h)^{1/2-i\beta}M\left(\frac{1}{2}-i(\alpha+\beta),\frac{3}{2}-i\beta;-h\right).$$

As usual, the coefficients C and D will be related to A and B through the Schrödinger equation that gives $ida_{rz}^c/ds=\frac{1}{2}e^{i\int\tau_c ds}b_{rz}^c$, so that

$$C=B\left(\frac{1}{2}+i\beta\right)e^{\{(i/16)\alpha\}}\frac{i}{\sqrt{\alpha/i}}\left(\frac{16\alpha}{i}\right)^{-i\beta},$$

$$D=A\frac{\alpha}{\frac{1}{2}-i\beta}\left(-\frac{16\alpha}{i}\right)^{-i\beta}e^{\{(i/16)\alpha\}}\frac{i}{\sqrt{\alpha/i}}.$$

Eventually we get

$$a_{rz}^c=AM(i\alpha,\tfrac{1}{2}-i\beta;h)+Bh^{1/2+i\beta}M(\tfrac{1}{2}+i(\alpha+\beta),\tfrac{3}{2}+i\beta;h), \tag{7.3}$$

$$b_{rz}^c=\left(\frac{16\alpha}{i}\right)^{-i\beta}e^{i/16\alpha}\frac{i}{\sqrt{\alpha/i}}\left[\frac{i\alpha}{\frac{1}{2}-i\beta}Ah^{1/2-i\beta}M(\tfrac{1}{2}-i(\alpha+\beta),\tfrac{3}{2}-i\beta;-h)+B(\tfrac{1}{2}+i\beta)M(-i\alpha,\tfrac{1}{2}+i\beta;-h)\right].$$

Writing A and B explicitly in terms of the initial conditions is possible, but the resulting expression is so long that we decided to eliminate it here. The model we developed here is obtained then by

$$G(s)=0,\quad h(s)=i\frac{s^2}{16\alpha},\quad a=i\alpha,\quad b=\frac{1}{2}-i\beta.$$

This model is a generalization of the Demkov model [25], originally proposed to describe atomic collisions. His model in our notations can be written as

$$\kappa_D=\kappa_0e^{-\gamma t},$$

$$\tau_D=\tau_0.$$

Applying the canonical transformation gives $s-s_0=-(\kappa_0/\gamma)e^{-\gamma t}$ so that

$$\tau_c^D=-\frac{\tau_0}{\gamma(s-s_0)}.$$

Taking $s_0=0$, and comparing with (7.2) shows that the Demkov model is the special case $\alpha\to\infty$ and $\beta=-\tau_0/2\gamma$. Using the behavior of the confluent-hypergeometric functions for $a\to\infty$ (see Appendix), we get from (7.3) the result

$$a_{rz}^c=As^{1/2+i\beta}I_{-1/2+i\beta}\left(\frac{is}{2}\right),$$

$$b_{rz}^c=BJ_{1/2-i\beta}\left(\frac{is}{2}\right).$$

Not letting α go to infinity, and assuming the same κ, we may easily generalize the Demkov model. The generalized model is

$$\kappa=\kappa_0e^{-\gamma t}, \tag{7.4}$$

$$\tau=\tau_0+\tau_1e^{-2\gamma t},$$

with $\tau_0=-2\beta\gamma$ and $\tau_1=-\kappa_0^2/8\alpha\gamma$. Surely for $\alpha\to\infty$ we get the simple Demkov model.

Another model that is a special member of the confluent-hypergeometric family is the Crothers model [10],

$$\kappa_{CR}=\kappa_0e^{-\gamma t},$$

$$\tau_{CR}=\tau_0+\kappa_1e^{-\gamma t}.$$

The canonical transformation gives $s-s_0=-(\kappa_0/\gamma)e^{-\gamma t}$ and

$$\tau_c^{CR}=-\frac{\tau_0-\dfrac{\gamma\kappa_1}{\kappa_0}(s-s_0)}{\gamma(s-s_0)}.$$

It can be shown that this model is obtained by the set

$$G_{CR}(t)=\gamma-i\tau_0-\frac{i}{2}(\kappa_1+\sqrt{\kappa_0^2+\kappa_1^2})e^{-\gamma t},$$

$$h_{CR}(t)=\frac{i}{\gamma}\sqrt{\kappa_0^2+\kappa_1^2}e^{-\gamma t},$$

$$a_{CR}=1-\frac{i\tau_0}{2\gamma}-\frac{\tau_0\kappa_1}{2i\gamma\sqrt{\kappa_0^2+\kappa_1^2}},$$

$$b_{CR}=2-i\frac{\tau_0}{\gamma}.$$

The Nikitin model [26],

$$\kappa_N=\kappa_0,$$

$$\tau_N=\tau_0+\tau_1e^{-\gamma t},$$

is yet another special member of the confluent-hypergeometric family. The canonical transformation yields $s-s_0=\kappa_0t$ and

$$\tau_c^N=\frac{\tau_0}{\kappa_0}+\frac{\tau_1}{\kappa_0}e^{\{-(\gamma/\kappa_0)(s-s_0)\}}.$$

This model is reproduced using the set

$$G_N(t)=-\frac{i}{2}(\tau_0+\sqrt{\kappa_0^2+\tau_0^2})-i\tau_1 e^{-\gamma t},$$

$$h_N(t)=\frac{i\tau_1}{\gamma}e^{-\gamma t},$$

$$a_N=1-\frac{i}{2\gamma}(\sqrt{\kappa_0^2+\tau_0^2}-\tau_0),$$

$$b_N=1-\frac{i}{\gamma}\sqrt{\kappa_0^2+\tau_0^2}.$$

As a final example we note that the model proposed by Petcov [4] to describe matter-enhanced neutrino oscillations is characterized by

$$\kappa_P=\kappa_0,$$

$$\tau_P=\tau_1 e^{-\gamma t},$$

thus being a special case of the Nikitin model with $\tau_0=0$.

VIII. DISCUSSION

This work established a relation between two-level Hamiltonians and spatial curves. Consequently, a relatively simple framework was obtained that can be utilized to analyze physical models. Defining the canonical torsion enables a simple and systematic approach that results in convenient classification and generalization methods. Another implication of this approach is that all two-level Hamiltonians are grouped into three-parameter families, the members of each family sharing a common solution.

When encountering a brand new model, one may apply the canonical transformation and see if it is a member of a known family. Only if this is not the case, the problem need be solved, either for the particular Hamiltonian or for any of its comembers in the family.

A different approach may be used to develop new analytically solvable models. The whole process may be reversed, starting from a general, arbitrarily chosen, curve, and proceeding backwards to find the associated three-parameter family, the solution of which is then guaranteed.

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APPENDIX

Several useful differential equations and some of their characteristics are summarized here; this material is particularly relevant to the derivations in Secs. V–VII. All the results are taken from [27].

The hypergeometric equation is defined as

$$y''+\frac{r-(p+q+1)z}{z(1-z)}y'-\frac{pq}{z(1-z)}y=0,$$

with p,q,r, and z complex numbers. This equation is solved by

$$y=AF(p,q,r;z)+Bz^{1-r}F(p+1-r,q+1-r,2-r;z),$$

with A,B constants and F the hypergeometric function, given by the series

$$F(p,q,r;z)=1+\frac{pq}{1\times r}z+\frac{p(p+1)q(q+1)}{1\times 2\times r(r+1)}z^2+\cdots,$$

which converges for $-1<|z|<1$ as long as $\mathrm{Re}(r-p-q)>-1$.

The parabolic-cylindric equation is defined as

$$\frac{d^2y}{dx^2}+(px^2+qx+r)y=0,$$

with p,q, and r complex constants. Usually this equation is brought into one of two standard forms,

$$\frac{d^2y}{dx^2}-(\tfrac{1}{4}x^2+V)y=0,$$

$$\frac{d^2y}{dx^2}-(\tfrac{1}{4}x^2-V)y=0,$$

with V a complex constant. The solution of the first equation is given by

$$y=Ae^{\{-(1/4)x^2\}}M\left(\frac{1}{2}V+\frac{1}{4},\frac{1}{2},\frac{1}{2}x^2\right)$$
$$+Bxe^{\{-(1/4)x^2\}}M\left(\frac{1}{2}V+\frac{3}{4},\frac{3}{2},\frac{1}{2}x^2\right),$$

and that of the second equation by a similar expression in which V is replaced by $-iV$ and x is replaced by $xe^{\{(1/4)i\pi\}}$. M is the confluent-hypergeometric function given by the series

$$M(a,b;z)=1+\frac{a}{1\times b}z+\frac{a(a+1)}{1\times 2\times b(b+1)}z^2$$
$$+\frac{a(a+1)(a+2)}{1\times 2\times 3\times b(b+1)(b+2)}z^3+\cdots.$$

For very large a these functions satisfy

$$\lim_{a\to\infty}M\left(a,b;\frac{z}{a}\right)=\Gamma(b)z^{\{(1/2)(1-b)\}}I_{b-1}(2\sqrt{z}),$$

$$\lim_{a\to\infty}M\left(a,b;-\frac{z}{a}\right)=\Gamma(b)z^{\{(1/2)(1-b)\}}J_{b-1}(2\sqrt{z}),$$

with I and J the appropriate Bessel functions.

The confluent-hypergeometric equation is

$$y''+\left(\frac{2R}{z}+2f'+\frac{bh'}{h}-h'-\frac{h''}{h'}\right)y'+\left[\left(\frac{bh'}{h}-h'-\frac{h''}{h'}\right)\left(\frac{R}{z}+f'\right)+\frac{R(R-1)}{z^2}+\frac{2Rf'}{z}+f''+f'^2-\frac{ah'^2}{h}\right]y=0,$$

with a, b, and R constants and f,h functions of z. The solution of this equation is given by

$$y=Az^{-R}e^{-f(z)}M(a,b;h(z))+Bz^{-R}e^{-f(z)}U(a,b;h(z)),$$

where M is the confluent-hypergeometric function and U is given by

$$U(a,b;z)=\frac{\pi}{\sin\pi b}\left[\frac{M(a,b;z)}{\Gamma(1+a-b)\Gamma(b)}-z^{1-b}\frac{M(1+a-b,2-b;z)}{\Gamma(a)\Gamma(2-b)}\right].$$

Substituting this definition, the solution may be written in terms of the confluent-hypergeometric function only:

$$y=Az^{-R}e^{-f(z)}M(a,b;h(z))+Bz^{-R}e^{-f(z)}[h(z)]^{1-b}M(1+a-b,2-b;h(z)).$$

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