Foundations of quantum mechanics: Connection with stochastic processes

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(Received 30 July 1999; published 14 April 2000)

In this paper we explore the mathematical and epistemological connections between the stochastic derivation of the Schrödinger equation and the one proposed by ourselves in previous papers. It will be shown that these connections are accomplished by means of the fluctuation-dissipation theorem, to which we may unambiguously relate the symbols and physical references of both approaches. As a by-product of our investigation, it will be possible to interpret the time-energy dispersion relation on sounder grounds. It will also be possible to discuss the superposition principle and to interpret it on a quite simple basis. The origin of the stochasticity and its relation to stability will be also addressed and the bridge to an axiomatic formulation of stochastic electrodynamics will be constructed.

PACS number(s): 03.65.Bz, 02.50.Ey, 02.50.Kd, 03.65.Ca

I. INTRODUCTION

We have already derived the Schrödinger equation from two slightly different axiomatizations, each one with three postulates [1,2]. These derivations have shown themselves formally equivalent but epistemologically complementary, since insights concealed in one are supplied by the other. It is always interesting to have several ways of developing a theory, especially when one can connect them mathematically and epistemologically. Indeed, our first axiomatization of quantum mechanics used a more mathematical approach [1], which simplified the calculations while hiding much of the underlying physics. In our second axiomatization [2], however, the blanks left by the first one were filled using the central concept of entropy and we also proved these two axiomatizations to be mathematically equivalent. The second derivation method, by using the concept of entropy, allowed us to generalize the derivation of the Schrödinger equation to embrace some generalized versions of the Boltzmann-Gibbs entropy, in particular that formulated by Tsallis, thus generalizing the Schrödinger equation [3].

In this paper we will show how the stochastic derivation may be connected with the one we have used in paper II [2] (and, consequently, also with the one of paper I [1]). This will help us to unravel some of the mathematical and epistemological features stimulated by that derivation. In this sense, the concept of a stochastic force, central in stochastic theory, will be connected with the concept of entropy, fundamental to one of our previous derivations.

The search for a stochastic support for quantum mechanics has taken place since the early 1950s [4] and became a fertile research field in the following two decades [5–13]. It is still an important field for investigation of the mathematical and epistemological foundations of quantum mechanics.

This approach can be illustrated by the mathematical derivation of the quantum mechanical formalism (Schrödinger equation) using only the formal apparatus of classical statistical mechanics, together with a kind of ''Brownian movement'' theory [14]. In this case, the kinematic description of the Brownian movement assumes a movement with no friction, an approach that has been used in the Einstein-Smoluchovski theory [15,16].

The model created by this approach is one in which the particles of a system, interacting via mutual forces, remain in dynamic equilibrium because of the balance of these forces with a stochastic force responsible for random movement [17]. The problem here, however, is to explain the *origin* of such stochasticity—and this is precisely the weakness of the stochastic approach. Moreover, in carrying out the derivation of the Schrödinger equation, one needs to introduce by postulate the stochastic velocity and acceleration and relate them, also by postulate, with the generalized stochastic force. Thus, the argument against the stochastic derivation may be that, by postulating all of these features, one is just forcing the result (Schrödinger equation). By accepting stochastic behavior without knowing its origin, the derivation of the Schrödinger equation from this line of reasoning, instead of just postulating it as usual, is only a matter of taste, for we are replacing the unknown by the obscure.

The important point here is that in such a theory, where $\mathbf{x}(t)$ is considered a stochastic process, it is not possible to define a total time derivative d/dt, since the movement is discontinuous, and we have to search for substitutes to this operator that might be used to formulate another *new* ''Newtonian'' theory, formally equivalent to the mathematical structure of quantum mechanics, as given by the Schrödinger equation. This is amply reported in the literature and will also be shown schematically in the present paper.

We have already shown in some of our previous papers [1,2] that it is possible to derive the quantum formalism (Schrödinger equation) from three basic postulates. In the first axiomatization the postulates were [1] (A1) the general validity of Newton's laws for the individual systems composing the *ensemble*; (A2) the general validity of Liouville's equation for the *ensemble*; and (A3) the possibility of connecting the joint probability density function on phase space $F(\mathbf{x},\mathbf{p};t)$ to a characteristic function $Z_Q(\mathbf{x},\delta\mathbf{x};t)$, defined upon configuration space, by means of an infinitesimal transformation given by

$$Z_Q(\mathbf{x},\delta\mathbf{x};t)=\int e^{i\mathbf{p}\cdot\delta\mathbf{x}/\hbar}F(\mathbf{x},\mathbf{p};t)\,d\mathbf{p}, \tag{1}$$

where $\delta\mathbf{x}$ is considered an infinitesimal displacement.

In the second axiomatization [2], the first two postulates were kept while the third was replaced by (A3′) the applicability of quantum mechanical formalism for situations of statistical equilibrium with respect to the entropy function *written in configuration space*, where the relation between the fluctuations in coordinate and momenta obtained from this entropy function is given by

$$\overline{(\delta x)^2}\;\overline{(\delta p)^2}=\frac{\hbar^2}{4}. \tag{2}$$

This second derivation allowed an interpretation of the variable δx, whose significance was concealed by the previous one.

By examining the epistemological and mathematical connection between our approach and the stochastic method, we sought to gain new insights into fundamental questions. Indeed, from the proof that quantum mechanics may be understood as a stochastic process, we felt that it would be rather interesting to know at what point in our approach this stochastic character effectively appears and what its ontological status is. If this goal could be achieved, the point where stochasticity appears in our derivation should explain its physical significance. Because of these considerations, in our previous derivations we struggled to avoid leaving any symbol of the theory without a physical reference. It is the objective of the present paper to resolve these questions.

To achieve this goal we will briefly develop, in the second section, the derivation of the Schrödinger equation from the stochastic point of view. We will follow the pioneering papers of Nelson [14], Kershaw [9] and De La Peña [18], and the review paper of De La Peña [17] (this latter very closely) as our guides in the development of the related formalism and interpretation. In the third section, we will show again, and very schematically, our derivation of the Schrödinger equation using the configuration space entropy concept. In the fourth section we will show how the stochastic derivation is connected with the one presented in the third section. That is, we will show where, within our own derivation, the stochastic character of matter was revealed by the formalism. The fifth section will be devoted to clarifying the role played within our approach by the superposition principle, from which we may draw, as we will show, a quite simple interpretation. In the sixth section we will finally address the problem related to the origin of the fluctuations and its relation to the system's mechanical stability. The last section is devoted to our final conclusions.

II. STOCHASTIC DERIVATION

As we mentioned in the previous section, $\mathbf{x}(t)$ is a stochastic process and we cannot define a time derivative d/dt for it. This means that the velocity related to this process cannot be obtained by direct derivation, for $\mathbf{x}(t)$ is not, in general, differentiable.

In this case we have to introduce a finite time interval Δt, small compared with the characteristic times of the systematic movement (that related to Newton's equation), but large enough compared with the correlation time of the *fluctuating force* (the reader is strongly encouraged to look at Ref. [17] for details) in a process known as coarse graining. Using this finite time interval we may define the forward time derivative (see [14] for details) as

$$D\mathbf{x}(t)=\lim_{\Delta t\to 0^+} E_t\left(\frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t)}{\Delta t}\right)=\left\langle\frac{\delta\mathbf{x}(\Delta t)}{\Delta t}\right\rangle_t, \tag{3}$$

where the average $E_t[\ ]$ or $\langle\ \rangle_t$ is taken over the Δt distribution, which means that it is the conditional average in the interval Δt and reflects a statistical distribution of the displacements $\delta\mathbf{x}$ [9]. We may also define the backward derivative as

$$D_*\mathbf{x}(t)=\lim_{\Delta t\to 0^+} E_t\left(\frac{\mathbf{x}(t)-\mathbf{x}(t-\Delta t)}{\Delta t}\right)=\left\langle\frac{\delta\mathbf{x}(-\Delta t)}{-\Delta t}\right\rangle_t, \tag{4}$$

where, in general, $D\mathbf{x}(t)\neq D_*\mathbf{x}(t)$. From these two derivatives we form the systematic and stochastic derivatives

$$D_c=\frac{D-D_*}{2};\quad D_s=\frac{D+D_*}{2}. \tag{5}$$

Since Δt is a very small time interval, we may write the expansion

$$\begin{aligned}\frac{1}{\Delta t}&[f(\mathbf{x}(t+\Delta t),t+\Delta t)-f(\mathbf{x}(t),t)]\\ &\approx\left(\frac{\partial}{\partial t}+\frac{1}{\Delta t}\sum_i\,[x_i(t+\Delta t)-x_i(t)]\frac{\partial}{\partial x_i}\right.\\ &\quad+\frac{1}{2\Delta t}\sum_{ij}\,[x_i(t+\Delta t)-x_i(t)]\\ &\quad\left.\times[x_j(t+\Delta t)-x_j(t)]\frac{\partial^2}{\partial x_i\partial x_j}\right)f(x;t).\end{aligned} \tag{6}$$

The general velocity of the process is given by

$$\mathbf{c}=\lim_{\Delta t\to 0} E_t\left(\frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t)}{\Delta t}\right)=D\mathbf{x}(t), \tag{7}$$

which we may split into two components: the systematic component $\mathbf{v}$ and the stochastic one $\mathbf{u}$,

$$\mathbf{c}=D_c\mathbf{x}+D_s\mathbf{x}=\mathbf{v}+\mathbf{u}. \tag{8}$$

In the ''Newtonian'' limit we have $D\mathbf{c}\to d\mathbf{v}/dt$.

To build a dynamic theory we must now postulate the following relation between the stochastic acceleration and the stochastic derivative of $\mathbf{c}$:

$$m\mathbf{a}=mD\mathbf{c}, \tag{9}$$

as a substitute for Newton's equation, but giving Newton's second law in the Newtonian limit. This implies that we must have

$$D_c\mathbf{v}+D_s\mathbf{u}=m\mathbf{a},$$
$$D_c\mathbf{u}+D_s\mathbf{v}=0. \tag{10}$$

Now let $\mathbf{f}$ be the total force acting upon the system. This force also has a stochastic component and may be equally written as a linear combination of systematic plus stochastic forces. In this case we may write

$$\mathbf{f}=\mathbf{f}_0+m(1+\lambda)\mathbf{a}_s, \tag{11}$$

where $\mathbf{f}_0$ is the part of the force derivable from a potential and λ is a (yet unknown) constant. With all these results we finally get the system

$$\mathbf{f}_0=m(D_c\mathbf{v}-\lambda D_s\mathbf{u}),$$
$$D_c\mathbf{u}+D_s\mathbf{v}=0, \tag{12}$$

which may be written explicitly as

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-\nu_-\nabla^2\mathbf{v}-\lambda(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}-\lambda\nu_+\nabla^2\mathbf{u}=\mathbf{f}_0/m,$$
$$\partial\mathbf{u}/\partial t+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}+\nu_+\nabla^2\mathbf{v}-\nu_-\boldsymbol{\nabla}\mathbf{u}=0. \tag{13}$$

Assuming that ν_+ and ν_- will depend only upon time, that the velocities are rotation-free, and that the external forces are derivable from a potential V, we may rewrite the system of equations (13) as

$$\partial\mathbf{v}/\partial t+\boldsymbol{\nabla}(\mathbf{v}^2/2-\nu_-\boldsymbol{\nabla}\cdot\mathbf{v}-\lambda\mathbf{u}^2/2-\lambda\nu_+\boldsymbol{\nabla}\cdot\mathbf{u})$$
$$=-\boldsymbol{\nabla}V/m,$$
$$\partial\mathbf{u}/\partial t+\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{u}+\nu_+\boldsymbol{\nabla}\cdot\mathbf{v}-\nu_-\boldsymbol{\nabla}\cdot\mathbf{u})=\mathbf{0}. \tag{14}$$

To obtain the Schrödinger equation from the nonlinear equation (14) we have only to make the ansatz

$$\mathbf{v}=2D_0\boldsymbol{\nabla}S;\quad \mathbf{u}=2D_0\boldsymbol{\nabla}R, \tag{15}$$

with $\nu_+=D_0$, $\nu_-=0$, and $\psi_\pm=\exp(R\pm iS/\sqrt{-\lambda})$, where R and S are real functions depending upon $\mathbf{x}(t)$ and t. After some algebra we get

$$\mp 2imD_0\sqrt{-\lambda}\,\frac{\partial\psi_\pm}{\partial t}=-2m\lambda D_0^2\nabla^2\psi_\pm+V\psi_\pm, \tag{16}$$

where V, as was said above, is the potential function related to the external force $\mathbf{f}_0$.

Since the parameters λ and D_0 appear in Eq. (16) only through the product $D_0\sqrt{-\lambda}$, it is clear that we may adjust the scale through D_0 and take $|\lambda|=1$ [17]. If $\lambda=-1$ we get the equation

$$\mp 2mD_0\partial\psi_\pm/\partial t=2mD_0^2\nabla^2\psi_\pm+V\psi_\pm, \tag{17}$$

having as its solution

$$\psi_\pm=e^{R\pm S}=\rho^{1/2}e^{\pm S};\quad \rho=\psi_+\psi_-. \tag{18}$$

This equation is of the parabolic type [17] and describes the irreversible time evolution of the (real) amplitudes ψ_- and ψ_+.

If $\lambda=+1$, then Eq. (16) becomes

$$\mp 2imD_0\partial\psi_\pm/\partial t=-2mD_0^2\nabla^2\psi_\pm+V\psi_\pm, \tag{19}$$

having as its solution

$$\psi_\pm=e^{R\mp iS}=\rho^{1/2}e^{\mp iS};\quad \rho=\psi_+\psi_-. \tag{20}$$

This is a hyperbolic type equation [17] and describes the reversible evolution of the (complex) amplitudes $\psi_-=\psi$ and $\psi_+=\psi^\dagger$. If we put into Eq. (19)

$$D_0=\frac{\hbar}{2m}, \tag{21}$$

where $\hbar$ is Planck's constant, we finally get the Schrödinger equation.

With the definition

$$\rho=\psi^\dagger\psi=e^{2R}, \tag{22}$$

we have, because of the second relation in Eq. (15),

$$\mathbf{u}=D_0\frac{\boldsymbol{\nabla}\rho}{\rho};\quad D_0=E_t\left(\frac{[\delta\mathbf{x}(\Delta t)]^2}{2\Delta t}\right)=\frac{\hbar}{2m}. \tag{23}$$

In the same way, since we are considering $\lambda=1$, Eq. (11) gives

$$\mathbf{f}=\mathbf{f}_0+2m\mathbf{a}_s, \tag{24}$$

where [17]

$$\mathbf{a}_s=\boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2+\frac{\hbar}{2m}\boldsymbol{\nabla}\cdot\mathbf{u}\right) \tag{25}$$

is the stochastic acceleration giving a stochastic force $\mathbf{f}_s=m\mathbf{a}_s=-\boldsymbol{\nabla}\phi_{stoc}$ related to the ''potential''

$$\phi_{stoc}=-\frac{\hbar^2}{4m}\left(\frac{\nabla^2\rho}{\rho}-\frac{1}{2\rho^2}(\boldsymbol{\nabla}\rho)^2\right). \tag{26}$$

The results (23) and (26) will be crucial for the comparison between this derivation and the one based upon the configuration space entropy concept, to be presented in the next section.

The process of derivation of the equation related to a Brownian-type movement and the one related to the quantum formalism leaves no doubt about the irreducibility of one type of phenomenon into the other. Indeed, since the very beginning, we have said that the quantum mechanical process has to be understood as one where there is no room for friction, which distinguishes it from the usual Brownian process [17]. We will return to this question later on.

III. SECOND DERIVATION

Now we will present again the mathematical derivation of the Schrödinger equation from the Liouville equation and Newton's laws using the concept of entropy, thus facilitating references to be made later on.

In our previous paper [2] (to which the reader is also strongly referred) we begin with the Liouville equation

$$\frac{\partial F(x,p;t)}{\partial t}+\frac{p}{m}\frac{\partial F(x,p;t)}{\partial x}-\frac{\partial V(x)}{\partial x}\frac{\partial F(x,p;t)}{\partial p}=0. \tag{27}$$

Using the definitions

$$\int F(x,p;t)dp=\rho(x;t);\quad \int pF(x,p;t)dp=p(x;t)\rho(x;t),$$

$$M_2(x;t)=\int p^2F(x,p;t)dp,$$

$$\overline{(\delta p)^2}\rho(x;t)=M_2(x;t)-p(x;t)^2\rho(x;t) \tag{28}$$

$$=\int [p-p(x;t)]^2F(x,p;t)dp,$$

where $\rho(x;t)$ is the probability density on configuration space, $p(x;t)$ is the so called macroscopic momentum, $M_2(x;t)$ is the second order momentum statistical moment, and $\overline{(\delta p)^2}$ is the momentum fluctuation (projected upon configuration space), we (a) directly integrate the Liouville equation over the momentum space and (b) multiply it by p and integrate it over the momentum space to find the two equations

$$\frac{\partial\rho(x;t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p(x;t)}{m}\rho(x;t)\right)=0, \tag{29}$$

representing a continuity equation for the probability density $\rho(x;t)$, and

$$\frac{1}{m}\frac{\partial}{\partial x}[M_2(x;t)-p^2(x;t)\rho(x;t)]+\rho(x;t)\left[\frac{\partial p(x;t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p^2(x;t)}{2m}\right)+\frac{\partial V(x)}{\partial x}\right]=0. \tag{30}$$

It remains for us to find a functional expression for $\overline{(\delta p)^2}$. To achieve this goal, we first consider the relation between the entropy and the probability density of an isolated system as given by [19]

$$\rho(x;t)=e^{S(x;t)/k_B}, \tag{31}$$

where k_B is the Boltzmann constant.

We are searching for the densities representing the *statistical equilibrium* situations of the system, compatible with *spontaneous local fluctuations in positions*. We expect that the entropy, calculated within some region $(x-\delta x/2, x+\delta x/2)$, will be a maximum. We thus write

$$S=S_{eq}+\frac{1}{2}\left(\frac{\partial^2S_{eq}}{\partial x^2}\right)(\delta x)^2, \tag{32}$$

where $S_{eq}(x)$ stands for the statistical equilibrium configuration entropy, and where we used the fact that the entropy must be a maximum, giving

$$\left(\frac{\partial S_{eq}(x)}{\partial x}\right)_{\delta x=0}=0. \tag{33}$$

In this step we used the assumption of local equilibrium. This makes it possible to define a local entropy, which is the same function of the local thermodynamics variables (here the positions) as the equilibrium entropy is a function of the equilibrium statistical parameters: the Callen concept of instantaneous entropy [20]. We thus use Eqs. (32) and (31) to get, for the position fluctuations,

$$\overline{(\delta x)^2}=\frac{\int_{-\infty}^{+\infty}(\delta x)^2e^{-\gamma(\delta x)^2}d(\delta x)}{\int_{-\infty}^{+\infty}e^{-\gamma(\delta x)^2}d(\delta x)}=\frac{1}{2\gamma}, \tag{34}$$

where we put

$$\frac{1}{2\gamma}=k_B\left|\frac{\partial^2S_{eq}(x;t)}{\partial x^2}\right|^{-1} \tag{35}$$

and where we considered the natural sample space $[-\infty, +\infty]$ for the fluctuations. The relation (34) is just a statement of the fluctuation-dissipation relation due to Einstein.

We have, *a priori*, no relation between these displacement fluctuations and those related to the momenta. We then impose the restriction that, in this statistical equilibrium situation, we must have

$$\overline{(\delta p)^2}\,\overline{(\delta x)^2}=\frac{\hbar^2}{4}, \tag{36}$$

meaning that, if the fluctuations in the positions are too broad, they have to occur sufficiently slowly in order to leave the system enough time to accommodate itself (adiabatic processes). If, on the other hand, the fluctuations in the momenta are too severe, then they have to be confined to very small portions of the system in such a way that they are too localized (and have too high a frequency) to disturb the statistical equilibrium situation. We then get, using Eqs. (34) and (36),

$$\overline{(\delta p)^2}=-\frac{\hbar^2}{4}\left(\frac{\partial^2\ln\rho(x;t)}{\partial x^2}\right), \tag{37}$$

which allows us to write

$$\frac{1}{\rho}\frac{\partial}{\partial x}[\overline{(\delta p)^2}\rho(x;t)] = -\frac{\hbar^2}{4}\frac{\partial}{\partial x}\left[\frac{\partial^2 \ln \rho(x;t)}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \ln \rho(x;t)}{\partial x}\right)^2\right]. \tag{38}$$

Now, putting

$$p(x;t) = \frac{\partial s(x;t)}{\partial x}, \tag{39}$$

we may rewrite Eq. (30) as

$$\rho(x;t)\frac{\partial}{\partial x}\left\{\frac{\partial s(x;t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial s(x;t)}{\partial x}\right)^2 + V(x) - \frac{\hbar^2}{4m}\left[\frac{\partial^2 \ln \rho}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \ln \rho}{\partial x}\right)^2\right]\right\} = 0. \tag{40}$$

Now, writing

$$\rho(x;t) = R(x;t)^2 \tag{41}$$

we get from Eq. (40) the equation

$$R^2(x;t)\frac{\partial}{\partial x}\left[\frac{\partial s(x;t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial s(x;t)}{\partial x}\right)^2 + V(x) - \frac{\hbar^2}{2mR(x;t)}\frac{\partial^2 R(x;t)}{\partial x^2}\right] = 0, \tag{42}$$

which we have called the *modified Schrödinger equation* for it is, as we have already shown [1], together with Eq. (29), equivalent to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x;t)}{\partial x^2} + V(x)\psi(x;t) = i\hbar\frac{\partial \psi(x;t)}{\partial t}, \tag{43}$$

with

$$\psi(x;t) = R(x;t)e^{is(x;t)/\hbar}. \tag{44}$$

Note that the derivative in Eq. (42) is unessential since it implies that the expression inside the square brackets is an arbitrary function of time $f(t)$. We may write this function within the term $\partial s/\partial t$ in a simple way [1], implying a redefinition of the energy level. Note also that our definition of the probability amplitude in the expression (44) is slightly different from the stochastic one, but in a totally unessential way since comparisons will be based upon the probability density $\rho(x;t)$, which is the same in both approaches.

Finally, it is worth stressing that the application of the entropy concept, defined upon configuration space, to the study of random processes as a means to give a statistical description of the underlying dynamics is a common procedure already established in the literature [21].

IV. COMPARISON BETWEEN THE DERIVATIONS

Now that both derivations have been presented, there remains the question of what connection exists between them. The unraveling of such a connection may be of great importance to clarify some of the interpretations of the quantities appearing in both formalisms, and also their place in the underlying epistemology.

We begin by stressing that Eq. (40) can be cast into a much more intuitive format by writing

$$\frac{\partial p(x;t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{p(x;t)^2}{2m} + V(x) - \frac{\hbar^2}{4m}\left[\frac{\partial^2 \ln \rho}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \ln \rho}{\partial x}\right)^2\right]\right\}, \tag{45}$$

which may be written as

$$\frac{\partial p(x;t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{p(x;t)^2}{2m} + V(x) - \frac{\hbar^2}{4m\rho}\times\left[\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2\rho}\left(\frac{\partial \rho}{\partial x}\right)^2\right]\right\}. \tag{46}$$

This resembles Hamilton's equation

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \tag{47}$$

but with the momentum p replaced by an average momentum $p(x;t)$, and the Hamiltonian H replaced by an average effective Hamiltonian

$$h_{eff}(x;t) = \frac{p(x;t)^2}{2m} + V(x) - \frac{\hbar^2}{4m\rho}\left[\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2\rho}\left(\frac{\partial \rho}{\partial x}\right)^2\right], \tag{48}$$

the stochastic potential appearing in the last term on the right, as can be seen by comparing it with expression (26) of the second section. We call

$$V_{eff} = V(x) - \frac{\hbar^2}{4m\rho}\left[\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2\rho}\left(\frac{\partial \rho}{\partial x}\right)^2\right] \tag{49}$$

the effective potential. Moreover, this ''potential,'' according to the derivation of the previous section, comes from the momentum fluctuations, as may be seen from expression (38), which gives the contribution of these fluctuations to the overall dynamics of the average momentum, as implied by Eq. (46).

It is well known that it is the fluctuating force that is responsible for holding the system within its statistical equilibrium situation, dissipating the eventual spontaneous fluctuations, thus leading to the fluctuation-dissipation formula (34). This is the point that allows us to link the *symbols* of both approaches (a necessary condition, since we are claim-

ing that they are equivalent). To do this we return to expression (8), where the stochastic velocity was defined by

$$\mathbf{u}=D_s\mathbf{x}, \tag{50}$$

and expression (23), where its analytic expression was given as

$$\mathbf{u}=\frac{\hbar}{2m}\frac{\boldsymbol{\nabla}\rho}{\rho}. \tag{51}$$

Now, we rewrite the above expression as

$$\mathbf{u}=\frac{\hbar}{2mk_B}\boldsymbol{\nabla}(k_B\ln\rho)=\frac{\hbar}{2mk_B}\boldsymbol{\nabla}S, \tag{52}$$

where S is the entropy, as defined in the third section, and remembering that, for our fluctuating system, the linear Onsager relations are given by

$$\frac{d}{dt}(\delta\mathbf{x})=\hat{\alpha}\cdot\boldsymbol{\nabla}S \tag{53}$$

(see Ref. [19], p. 597), where $\hat{\alpha}$ is the so called friction coefficient and is a tensor given by

$$\hat{\alpha}=\frac{1}{k_B}\int_{-\infty}^{0}\langle\delta\dot{\mathbf{x}}(0)\,\delta\dot{\mathbf{x}}(s)\rangle_0 ds, \tag{54}$$

the ''cross-correlation function,'' thus relating the two fluctuating velocities.

With these expressions, it is readily seen that

$$\mathbf{u}=D_s\mathbf{x}=\frac{d}{dt}(\delta\mathbf{x}) \tag{55}$$

and

$$\frac{\hbar}{4}\hat{\mathbf{1}}=\frac{1}{2m}\int_{-\infty}^{0}\langle\delta\mathbf{p}(0)\,\delta\mathbf{p}(s)\rangle_0 ds, \tag{56}$$

where $\delta\mathbf{p}=md(\delta\mathbf{x})/dt$, and $\hat{\mathbf{1}}$ is the unit matrix. We have already used the fact that the momentum fluctuations in different directions are independent—this last result was proven by us in paper II [2], Appendix D [see also Ref. [17], Eq. (8)]. Thus we must have, since $\langle\delta p_i\rangle=0$ and $\langle\delta p_i^2\rangle=\langle\delta p_k^2\rangle$ by isotropy, the diagonal matrix equation in Eq. (56). Expression (56) thus says that Planck's constant comes from the kinetic energy correlations [see also the expression (23)]. It is needless to say that, with the relations (55) and (56), and the connection

$$\mathbf{v}=D_c\mathbf{x}=\mathbf{p}(\mathbf{x};t)=\int\mathbf{p}F(\mathbf{x},\mathbf{p};t)d\mathbf{p}, \tag{57}$$

where $\mathbf{v}$ is the systematic velocity and $\mathbf{p}(x;t)$ is the average momentum, all the other symbols of theory are connected unambiguously, since the primitive symbols of the stochastic derivation are $\mathbf{v}$, $\mathbf{u}$, D_c, and D_s and the other symbols are defined from them. Moreover, the quantities appearing in the Hamilton-type equation are average quantities, something that is concealed by the stochastic derivation.

At this point we may also explain the origin of the parabolic-type equation (17), which is obtained when we make the choice $\lambda=-1$. We begin by noting that the mathematical formalism of the derivation made in the third section is suitable to *stable equilibrium* situations, since we have used the relation (33). Now, it would be equally possible to consider the positive value of the entropy second derivative (*unstable equilibrium*), which is equivalent to choosing $\lambda=-1$ in the stochastic approach, Eq. (11). Thus, the stochastic force will tend to reinforce the action of the external force, since, in this case, these equations read

$$\mathbf{f}_0=m(\mathbf{a}_c+\mathbf{a}_s)\quad\text{and}\quad\mathbf{f}=\mathbf{f}_0, \tag{58}$$

and cannot be considered a dissipation function in the sense mentioned above—that is, equilibrium will never be attained.

Another aspect of the stochastic derivation that becomes clear by the above analysis is its Markovian character, coming from the use of only the first three statistical momentum moments in the derivation process [18].

The fact is that the fluctuations, when related to individual systems, entangle their description by means of classical Newtonian dynamics and classical statistical theory (the Liouville equation). To go onto quantum mechanics, from these two classical frameworks, it is necessary only to *restrict* the fluctuations of the systems, which might be any in the classical statistical framework, by the expression

$$\overline{(\delta x)^2}\;\overline{(\delta p)^2}=\frac{\hbar^2}{4}, \tag{59}$$

which not only does not violate this framework, but is, we stress, a mere restriction made upon the behavior of these systems' position and momentum fluctuations. This is exactly what we have done using our three postulates (A1), (A2), and (A3′), mentioned in the first section. Moreover, it is an experimental fact that restriction (59), made upon the whole class of classically allowed fluctuations, has a ubiquitous character, thus giving Planck's constant its known universal character.

The result of the previous discussion, however, has far-reaching consequences, since the approach we followed led to an equation (the Schrödinger equation) that restricts the energies (e.g., to discrete values) that an individual system may have, while introducing the fluctuating (stochastic) force. This feature is not present in Newton's laws, where the energy is not restricted as long as it is positive. This behavior came, clearly, from the inclusion of postulates (A2) and (A3′), related to Liouville's equation and the class of classically allowed fluctuations.

There is a final comment we could make regarding the ergodic connection between individual systems considered within a finite time interval and ensembles. For the above formalism to be applicable for individual systems, the time interval has to be sufficiently large (and therefore finite) for these individual systems to behave like an ensemble and be describable by the Schrödinger equation [this is precisely the

restriction of finite time interval related to the stochastic derivation made at the beginning of the second section, before Eq. (4)]. This is nothing more than a qualitative statement of the quantitative expression

$$\overline{(\delta E)^2}\,\overline{(\Delta t)^2}=\frac{\hbar^2}{4}, \tag{60}$$

which says that the time interval within which the quantum formalism applies is the one that allows the energy to fluctuate in the manner given by Eq. (60). This expression, obviously, leads to the Heisenberg inequality

$$\Delta E\Delta t\geqslant\frac{\hbar}{2}, \tag{61}$$

in the same sense that Eq. (59) leads to the position-momentum one. Note, however, that Δt here is a time *interval*, since there can be no fluctuations in time, which is a parameter within the present approach. This result comes quite naturally from the present approach; to see this we begin by defining the *energy correlation function* given by

$$K(s)=\overline{\delta E(t)\,\delta E(t+s)}, \tag{62}$$

which relates the energy fluctuations at times t and $t+s$ by an ensemble average. This definition implies that the mean square fluctuation in the energy is given by

$$\overline{(\delta E)^2}=K(0), \tag{63}$$

since $\overline{(\delta E)}=0$. Let us now introduce the Wiener-Khintchine relations (see Ref. [19], p. 585), given by

$$\begin{aligned}K(s)&=\int_{-\infty}^{+\infty}J(E)e^{iEs/\hbar}dE,\\ J(E)&=\frac{\hbar}{2\pi}\int_{-\infty}^{+\infty}K(s)e^{-iEs/\hbar}ds,\end{aligned} \tag{64}$$

leading to

$$\overline{(\delta E)^2}=\int_{-\infty}^{+\infty}J(E)dE, \tag{65}$$

where $J(E)$ is called the spectral density. Since $\delta E(t)$ is stationary and ergodic (by assumption), $K(s)$ is time independent, and thus *we may use a time average in place of the ensemble average*; hence

$$\begin{aligned}K(s)&=\overline{\delta E(t)\,\delta E(t+s)}=\{\delta E(t)\,\delta E(t+s)\}\\&=\frac{1}{2\Theta}\int_{-\Theta}^{+\Theta}dt'\,\delta E(t')\,\delta E(t'+s),\end{aligned} \tag{66}$$

where $2\Theta=\Delta t$ is *the minimum characteristic time* that allows us to use a time average instead of the ensemble average. By defining the new function

$$\delta E_\Theta(t)=\begin{cases}\delta E(t), & -\Theta<t<\Theta\\ 0, & \text{otherwise}\end{cases} \tag{67}$$

(see Ref. [19], p. 582), and making the spectral decomposition of this time-dependent function as

$$\delta E(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}C(E)e^{iEt/\hbar}dE, \tag{68}$$

we finally find, substituting these results in Eq. (66), the relation

$$K(s)=\frac{\hbar}{2\Theta}\int_{-\infty}^{+\infty}C(-E)C(E)e^{iEs/\hbar}dE. \tag{69}$$

With the result

$$C(-E)=C^*(E) \tag{70}$$

(see [19], p. 583), we get

$$\overline{(\delta E)^2}=K(0)=\frac{\hbar}{2\Theta}\int_{-\infty}^{+\infty}|C(E)|^2dE, \tag{71}$$

giving, because of Eq. (65), the spectral density as

$$J(E)=\frac{\hbar}{2\Theta}|C(E)|^2. \tag{72}$$

The expression (71) may be written as

$$\overline{(\delta E)^2}\Delta t=\hbar\int_{-\infty}^{+\infty}|C(E)|^2dE, \tag{73}$$

which is a restatement of the time-energy Heisenberg dispersion relation if we put

$$\int_{-\infty}^{+\infty}|C(E)|^2dE=\frac{\hbar}{4}, \tag{74}$$

in a way quite similar to the substitution done in the expression (56). It is important to stress that the time *interval* appears in a very natural way using this approach.

This clarifies the meaning of the time-energy Heisenberg relation as a fully objective one and is another improvement in our understanding of the quantum formalism that was brought about by the comparison of the stochastic derivation with our previous statistical one.

V. SUPERPOSITION PRINCIPLE

We are now ready to elucidate the role played by the superposition principle within the conceptual structure of the present framework. To proceed with this task it will be necessary to turn to the derivation given in paper I [1], whose postulates were already cited in the Introduction of this paper under the symbols (A1)–(A3). In very general terms (the reader is referred to that paper for details) the process of derivation begins with the axioms (A1)–(A3) where a characteristic function (in momentum space) is defined as

$$Z_Q(x,\delta x;t)=\int e^{ip\,\delta x/\hbar}F(x,p;t)dp, \tag{75}$$

and where the above Fourier transform is applied upon the Liouville equation, giving an equation satisfied by the function Z_Q. We now *impose* upon the characteristic function the formal appearance

$$Z_Q(x,\delta x;t)=\psi^*\left(x-\frac{\delta x}{2};t\right)\psi\left(x+\frac{\delta x}{2};t\right), \tag{76}$$

which is formally equivalent (as we have shown in paper II [2]) to the constraint (3) of the derivation of the third section. By writing the above amplitudes as

$$\psi(x;t)=R(x;t)e^{iS(x;t)/\hbar} \tag{77}$$

and taking the imposition (76) on the equation satisfied by the characteristic function, we are able to deduce the Schrödinger equation.

The feature deserving our attention now is that since expression (75) is a Fourier transform, and the characteristic function is defined as the product in Eq. (76), we must have the classical Liouville probability density function given by the convolution integral

$$F(x,p;t)=\int \phi^*(x,2p-p';t)\phi(x,p';t)dp', \tag{78}$$

where the functions $\phi(x,p;t)$ may be called the phase space probability amplitudes. These are related to the configuration space probability amplitudes given in Eq. (77) by

$$\psi(x+\delta x;t)=\int e^{ip\,\delta x/\hbar}\phi(x,p;t)dp, \tag{79}$$

as we have already shown in paper II. With the use of this last expression we were able to derive, in paper II, the Bohr-Sommerfeld quantization rules and the Feynman quantization prescription. As a final comment, it is important to note that the average values of operators were defined by the limiting process

$$\begin{aligned}\langle f(x,p)\rangle&=\int f(x,p)F(x,p;t)dp\,dx\\&=\lim_{\delta x\to 0}\int \hat{f}\left(\hat{x},-ih\frac{\partial}{\partial(\delta x)}\right)Z_Q(x,\delta x;t)dx,\end{aligned} \tag{80}$$

where no ambiguity concerning the order of the operators appears, since they commute, when related to the characteristic function [1]. With these results on hand we may now return to our main theme.

Since in the following it will be important to distinguish between pure states and statistical mixtures, we will now show how these two cases fit within our approach. The pure state is given simply by the above developments; the reader may be aware that our characteristic function is nothing more than the density ''matrix'' (with one element) written in the coordinate representation

$$\rho(x'',x';t)=\psi^*(x'';t)\psi(x';t), \tag{81}$$

where $x''=x-\delta x/2$ and $x'=x+\delta x/2$ [see Eq. (76)]. The average process in the expression (80) is thus simply

$$\langle f(x,p)\rangle=\lim_{x''\to x'}\mathrm{Tr}[\hat{f}(\hat{x},\hat{p})\hat{\rho}(x'',x';t)], \tag{82}$$

where $\mathrm{Tr}(f\hat{\rho})$ means the trace of the (one-element) matrix $\hat{\rho}$, and p is the momentum operator acting upon the probability amplitudes [1]. The limiting process is thus clearly equivalent to the process of taking the trace of the corresponding 1×1 matrix.

If we have a statistical mixture, however, there will be systems occupying many possible states and many probability amplitudes will be present. In this case, the density matrix will be written, in coordinate representation, as the matrix

$$[\hat{\rho}_{nm}(x'',x';t)]=[\psi_n^*(x'';t)\psi_m(x';t)], \tag{83}$$

where $[a_{nm}]$ represents the matrix with elements a_{nm}. The final state of the system will be given by the average process

$$\langle f(x,p)\rangle=\mathrm{Tr}[\hat{f}(\hat{x},\hat{p})\hat{\rho}_{nm}(x'',x';t)], \tag{84}$$

exactly as above. This averaging process, however, may be easily translated within our approach as the superposition

$$\begin{aligned}\langle f(x,p)\rangle&=\mathrm{Tr}[\hat{f}(\hat{x},\hat{p})\hat{\rho}_{nm}(x'',x';t)]\\&=\sum_{n=1}^{N}c_n\lim_{\delta x\to 0}\int \hat{f}\left(\hat{x},-ih\frac{\partial}{\partial(\delta x)}\right)\\&\quad\times Z_{Q(n,n)}(x,\delta x;t)dx,\end{aligned} \tag{85}$$

where $Z_{Q(n,n)}(x,\delta x;t)=\psi_n^*(x-\delta x/2;t)\psi_n(x+\delta x/2;t)$, and N is the dimension of the matrix (eventually infinite).

There are thus two kinds of superpositions: (a) one that comes from the linearity of the Schrödinger equation and may be related even to pure states (that are considered as entangled states) and (b) the other that takes into account the fact that we are dealing with a statistical mixture. It is clearly case (a) that produces controversy, since it is *assumed* that classical statistical theory, related to the Liouvillian function $F(x,p;t)$, does not present such a feature, this being the alleged main distinction between the conceptual frameworks of the classical and quantum theories. Therefore, we shall begin by discussing case (a).

There might be a misconception in what we have just written. Indeed, the argument just presented compares the classical probability *density* with the quantum probability *amplitudes*, which is clearly a comparison between two very distinct mathematical and physical objects. If we are to compare them, we must do so by comparing classical probability amplitudes with quantum mechanical amplitudes and classical probability densities with quantum mechanical densities.

Since the formalism we have just derived has furnished the classical probability amplitudes [the ϕ's in Eq. (78)], we are in position to carry out the required comparison. Indeed, we may assume that our system is in the superposition state

$$\psi_S = \sum c_n \psi_n , \tag{86}$$

but, now, according to the expression (79) we will have

$$\begin{aligned}\psi_S(x+\delta x;t) &= \sum c_n \psi_n(x+\delta x;t) \\ &= \sum c_n \int e^{ip\,\delta x/\hbar} \phi_n(x,p;t)dp \\ &= \int e^{ip\,\delta x/\hbar} \phi_S(x,p;t)dp,\end{aligned} \tag{87}$$

where

$$\phi_S(x,p;t) = \sum c_n \phi_n(x,p;t). \tag{88}$$

Thus, we expect to find the same superposition phenomenon in the classical level represented by the phase space probability amplitudes $\phi_n(x,p;t)$, which means that it is a mistake to say that this superposition has no classical resemblance. The error, as we have already pointed out, comes from the comparison of the quantum probability amplitudes with the classical probability density, which are different mathematical and conceptual objects.

If we are in the situation (b) above, where a statistical mixture appears, then there is no problem in writing the average process as

$$\begin{aligned}\langle f(x,p)\rangle &= \int f(x,p)F_n(x,p;t)dp\,dx \\ &= \sum_{n=1}^{N} c_n \int f(x,p;t)F_n(x,p;t)dx\,dp \\ &= \sum_{n=1}^{N} c_n \lim_{\delta x \to 0} \int \hat{f}\left(\hat{x}, -ih\frac{\partial}{\partial(\delta x)}\right) \\ &\quad \times Z_{Q(n,n)}(x,\delta x;t)dx,\end{aligned} \tag{89}$$

where

$$Z_{Q(n,n)}(x,\delta x;t) = \int e^{ip\,\delta x/\hbar} F_n(x,p;t)dp. \tag{90}$$

There is thus no difference, either mathematical or conceptual, between the two frameworks. The confusion appears because, whenever we use the classical probability density in the usual classical framework, we usually keep using the density $F(x,p;t)$ while the convolution in Eq. (78) is kept hidden, together with its related ''interference'' phenomena—which is obviously identical to the quantum mechanical case, as becomes clear from what we have just shown in expressions (78) and (88).

Some widespread interpretations interpret the quantum system as a single system *being considered in some instant of time t*. If there is a superposition, then this superposition implies that the quantum system may be in more than one state at the *same time* (Schrödinger's cat paradox)—it is needless to say that this interpretation implies that, in the classical limit, there can be no superposition (for this would really imply in Schrödinger's paradox of a dead *and* alive macroscopic cat).

Our interpretation could not be much farther from this one than it is. As we have already stressed on other occasions (see paper II and the previous section), our approach is able to refer to both single systems and ensembles, their connection being made by the ergodic hypothesis, which is plainly applicable within the realm of quantum theory [2]. This means that, when considering a single system, we must consider this single system within a time interval sufficiently large to allow the fluctuations to let the system fill its accessible states. The states appearing in the expansion (86) of the superposed probability amplitude are just those accessible states that have their energy in the interval of the energy fluctuation of the system—in the case of degenerate states even the energy fluctuation is unnecessary, since all the states have the same energy content and are equally accessible. Our single system will thus, by means of the fluctuations, occupy, successively in time, states with a probability greater or smaller depending on the specific problem; indeed, if we are considering a problem where we have a single system with an energy dispersion given by

$$\langle E\rangle - \Delta E < E < \langle E\rangle + \Delta E, \tag{91}$$

meaning that the fluctuating energies are within the above range, then we expect those states that have energy closest to the average value $\langle E\rangle$ to be the most probable, this probability decreasing as we move to the range extrema. This is why we may have different coefficients in the expansion (for the degenerate case, whenever there is no field splitting the degeneracy, these coefficients are obviously equal). The analysis for *ensembles* is similar, with the difference that we do not need to consider finite time intervals, and the coefficients are thus telling us how many systems in the ensemble are (most probably) occupying the underlying state at each instant of time [22].

The explanation of the superposition principle follows the same lines as Ballentine's statistical explanation [22]. Indeed, in the ensemble picture, we may think of the ensemble as having each of its component systems occupying a single specific quantum state at the same time, without the need of considering each system as occupying a set of states at the same time, which is implied by the superposition. The ergodic assumption, which takes us from the ensemble picture to the single fluctuation system, implies that we are considering the same single system at successive instants of time; this single system may occupy the quantum states implied by the underlying superposition at distinct instants of time, thus resolving the paradox. The same formal machinery that allowed us, in paper II, to avoid the duality problem is the one allowing us to avoid the (pseudo)paradox of the Schrödinger cat.

The great mathematical advantage of the quantum mechanical formalism is that it has a prescription [given by the restriction (76) or (78)] to *select* (reveal, uncover), among all the classically allowed probability densities, those having underlying amplitudes satisfying the principle of superposition. As a last comment, we stress that we have not made any impositions on the *size* of the system. The cautious reader, however, may feel uneasy with that, since the quantum formalism seems to be applicable, at least in general, to microscopic, instead of macroscopic, phenomena. The reason is that macroscopic systems are usually quite insensible to very small fluctuations, although this does not need to be always true. This, in turn, explains why it is so hard to observe interference phenomena in macroscopic objects. Thus, it is not true that the quantum formalism is applicable only to microscopic (small) systems, but, instead, that it is more adequate for studying these systems, since these are the ones that are more sensitive to the fluctuations it encompasses.

VI. ORIGIN OF STOCHASTICITY

Our approach may also help us in clarifying the important issue concerning the *origin* of the stochasticity. As it is probably better to do this by means of an example, we shall discuss the case of the hydrogen atom (as a working case), stressing, however, that the present considerations may be easily generalized to any other physical system. We will again use the approach related to the infinitesimal Wigner-Moyal transform of the previous section.

It is easy to show that, to derive the Schrödinger equation and include the electromagnetic field (nonrelativistically), it is only necessary to put

$$Z_Q(\mathbf{x},\mathbf{p};t)=\int e^{(i/\hbar)[\mathbf{p}+(e/c)\mathbf{A}]\cdot\delta\mathbf{x}}F(\mathbf{x},\mathbf{p};t)d\mathbf{p} \tag{92}$$

and the potential

$$V_T(\mathbf{x})=V(\mathbf{x})+\Phi(\mathbf{x},t), \tag{93}$$

where $\mathbf{A}(x;t)$ and $\Phi(\mathbf{x},t)$ are the vector and scalar electromagnetic potentials—this means that we just have to replace the mechanical momentum with the total canonical momentum. By applying the same steps mentioned above we finally end with the usual Schrödinger equation

$$\left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla}+\frac{e}{c}\mathbf{A}(\mathbf{x};t)\right)^2+[V(\mathbf{x})+\Phi(\mathbf{x},t)]\right]\psi(\mathbf{x};t) = i\hbar\frac{\partial\psi(\mathbf{x};t)}{\partial t}. \tag{94}$$

This, however, is not the equation we must solve when considering, for example, the hydrogen atom. Indeed, the equation is just

$$\left(-\frac{\hbar^2}{2m}\nabla^2+[V(\mathbf{x})+\Phi(\mathbf{x})]\right)\psi(\mathbf{x};t)=E\psi(\mathbf{x};t), \tag{95}$$

in which only the *static* Coulomb potential $\Phi(\mathbf{x})$ appears. The question then is: is there a reason to choose Eq. (95) instead of Eq. (94)? In the following we will see that the answer to this question leads to the profound role played by the fluctuations within the quantum formalism.

It is known that one of the major historic reasons for developing the quantum mechanical formalism was the fact that, when considering the full electromagnetic potentials, instead of only the electrostatic one, Newton's equations furnished rapidly decaying (unstable) solutions because of the radiation of electromagnetic energy (of course, the same Newton's equations will give stable solutions when only the electrostatic potential is considered). However, if we consider Eq. (94) with the vector potential as given by the Lienard-Wiechert formulas (which is time dependent), the solution will not be stationary anymore, and the process will be as unstable within the realm of quantum mechanics as it is from the perspective of Newtonian mechanics (and for the same reasons). If one accepts that this disappearance of the vector potential from the quantum equation justifies the stability, then one must also allow the same prerogative for the Newtonian approach.

This is why it is an old misconception, which is still believed by many of us, that the usual Schrödinger quantum theory explains the stability of the atoms; this is surely not the case, for the stability is being imposed rather than explained. [What is indeed correct to say is that the quantum formalism explains the quantization of the spectrum by assuming the stability, which is obvious from the stationary format of Eq. (95) and the inexplicable disappearance of the time-dependent vector potential.] We will see in the following that by eliminating the vector potential from Eq. (94), we will perceive the key to understanding the origin of the fluctuations (stochasticity).

From the expression (92) for the characteristic function we may derive, following the same steps as given in paper II, the Bohr-Sommerfeld relation

$$\oint_C\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\cdot d\mathbf{r}=nh, \tag{96}$$

now involving also the electromagnetic vector potential, and where $d\mathbf{r}$ is the vector taken over the orbit C. This relation, however, is just telling us that there is an *adiabatic invariance* of the magnetic flux through the orbit of the particle [23]. This adiabatic invariance of the magnetic flux has far-reaching consequences that have not been very much exploited in the literature. Indeed, we may recall that the crucial step, when moving from the electrostatic and magnetostatic theories toward the unified electromagnetic theory, is given by the link established by the Faraday integral equation. This equation, in its integral form, says that

$$\oint_C\mathbf{E}\cdot d\mathbf{r}=-\frac{d}{dt}\int_S\mathbf{B}\cdot d\mathbf{S}=-\frac{d}{dt}\oint_C\mathbf{A}\cdot d\mathbf{r}, \tag{97}$$

where C is the closed curve defining the limits of the open surface S. If the magnetic flux is an adiabatic invariant, the right hand side of Eq. (97) will be just zero and the electric

and magnetic fields will uncouple, which means that we remain within the realm of a static theory, since

$$\oint_C \mathbf{E}\cdot d\mathbf{r}=\mathbf{0}\Rightarrow \boldsymbol{\nabla}\times\mathbf{E}=\mathbf{0} \tag{98}$$

(for arbitrary C) and thus

$$\mathbf{E}=-\boldsymbol{\nabla}\Phi(\mathbf{x}). \tag{99}$$

In this case, it is adequate to remove the vector potential from Eq. (94) and work with the static electric potential in Eq. (95). Thus, now, and only now, we may say that quantization (which implies constant magnetic fluxes) leads to stability.

One more aspect may be mentioned with respect to Eq. (97); the constancy *and quantization* of the path integral of **A** are possible only when the potential being integrated has singularities. These singularities prevent us from passing freely from the integral format of the Maxwell equations to their differential format, from which the Lienard-Wiechert vector potential is derived. But, as we have shown in paper II, the fluctuations play precisely the role of introducing an effective potential that does have singularities, and this is the profound reason that explains and justifies the use of the sole electrostatic potential within the Schrödinger equation. The question here addresses the priority that one might give to the integral over the differential format of Maxwell's equations; this choice of the prevalence of the first over the latter may be sustained by remembering the historical development of the theory, which was a phenomenological theory based upon circuits, etc. If this choice is accepted, then some care must be taken when passing from the integral format to the differential one. Whenever singularities appear within the domain of integration, the integral theorems (Stokes, divergence, etc.) do not reduce the integral equations into the differential ones in a simple manner (we may recall that Dirac built upon this fact to propose his magnetic monopole, which is nothing but a singularity of the vector potential that furnishes the quantization of the ratio of the electric and magnetic charges; it is also such a phenomenon that is involved in the Bohm-Aharonov effect). These singularities, within the present framework, may be seen if we look at the *effective* potential, defined in the expression (49) [see also Eq. (45)]. The last term is the derivative of the entropy, which is $\ln\rho(x)$; thus, whenever we have zeros in the probability density function, infinities will appear in the effective potential. As we have seen, this term represents precisely the momentum fluctuations and thus confirms that it is the stochasticity that introduces such singularities, which, in turn, quantize the vector potential decoupling Maxwell's equations and making the system stable.

The next question thus refers to the meaning of the above adiabatic invariance. As is well known, adiabatic invariance implies that, whenever the system undergoes variations in the relevant parameters that are slow compared with its periods (adiabatic variation), the related action integrals remain invariant. Thus, Eq. (97) says that, although there may be *local variations* of the vector potential in a cycle, these variations are such as to compensate and to maintain the *total* magnetic flux invariant, and it is the total flux that is to be considered in Eq. (97). The analysis of Eq. (96) thus furnishes the result that we were searching for: according to this equation, since the sum of the magnetic flux and the mechanical momentum around the orbit is a constant, any local variation in the magnetic flux (fluctuation in the vector potential) will be transferred to the mechanical momentum, which means a transfer from electromagnetic to mechanical energy. However, this local variation will be quickly compensated (the fluctuations) and there will be a flux of energy in the opposite direction in such a way as to keep the magnetic flux constant—this flow of energy may be understood, in the particular example here discussed, as the initial absorption of a virtual photon by the electron in the process of electromagnetic to mechanical energy transfer and then its release by the electron in the inverse process.

This is precisely the point. It is well known that ''the equilibrium state, static from the viewpoint of classical thermodynamics, is incessantly dynamic. Local inhomogeneities continually and spontaneously generate, only to be attenuated and dissipated . . . ''(Ref. [20], p. 210), and, what is most important for our present analysis, ''the 'subsystem' may, in fact, be a small portion of a larger system, the remainder of the system then constituting the 'reservoir'. In that case the fluctuations are *local fluctuations* within a nominally homogeneous system'' (Ref. [20], p. 423). In the present example, the Schrödinger equation refers to the *electron* energy; the electromagnetic field energy is not considered and plays the role of the ''reservoir'' of the system. The total energy is obviously conserved in the process, since we are within a closed system—the electron and field energies will fluctuate separately because of the exchange process mentioned above. This is the main difference between a closed system and an open system interacting with its surroundings; in the closed system total energy is conserved (exactly as with the Schrödinger formalism) while in the open system it is not [24].

It is obvious from our analysis in the third section [see Eq. (33)] that here we are interested in local fluctuations. What we have said above allows us to treat the hydrogen atom (and, in general, any other quantum system) as a locally fluctuating system. For the specific case of the hydrogen atom, the usual stochastic electrodynamic approach (which consists in postulating the hydrogen atom as an open system [25] with a ubiquitous zero-point electromagnetic field as its reservoir) is unnecessary—thanks to the distinction between local and global fluctuations. The above mentioned postulate may be withdrawn without any consequences to the stochastic electrodynamic main concept, which consists in using the classical Maxwell equations with the boundary condition associated with fluctuating potentials and fields. In this case, Eq. (96) may furnish a powerful insight into the connections between the stochastic electrodynamic approach and the quantum field theory, something exhaustively searched for in the literature [25]. At this point it is interesting to note that Boyer, on a somewhat intuitive basis, had already pointed out the deep connection between the adiabatic invariants and stochastic electrodynamics. Indeed, we may quote him:

''here we wish to point out a curious sidelight found in the further development of the theory. It turns out that the action-angle variables provide a convenient description of mechanical systems in random classical radiation ... [for] it is easy to show that the zero-point radiation is the unique spectrum of random classical radiation, which leaves the adiabatic invariants of a nonrelativistic periodic mechanical system with *no harmonics* [my italics] as still adiabatically invariant in the presence of radiation. Furthermore, an adiabatic invariant $J_l = \oint p_l dp_l$ in classical zero-point radiation takes the average value $\langle J_l \rangle = h/2$.'' [26].

Thus our developments suggest the replacement of the interpretation postulate of a ubiquitous all-pervading external zero-point electromagnetic energy field (heat bath) of constant magnitude $\hbar/2$ by the interpretation, following solely from the axioms, of a closed system with local fluctuations depending on the actual state of the field. In keeping the stochasticity of the boundary conditions intact, while changing the fixed value of $\hbar/2$ by the field-state-dependent value $(n+1/2)\hbar/2$, this implies an alteration that may introduce many new results within the framework of stochastic electrodynamics, in particular with respect to its explanation of the stability of excited state orbits. It also seems, at least to this author, that the idea of local fluctuations of the electromagnetic field of the system is a much more acceptable concept than a constant electromagnetic field filling all of space (this would imply, for instance, an infinite energy that is detectable by the gravitational field).

The relations between the stochastic approach to electrodynamics and quantum field theory were only sketched here within a rather intuitive and somewhat qualitative perspective. We leave the unraveling of all their interconnections to a future paper.

VII. CONCLUSIONS

In the previous sections we have shown how the stochastic derivations of the Schrödinger equation reduce to our previous derivations; this equivalence was established not only with respect to the formalism but also with respect to the interpretation of the symbols of both theories. This reduction of the stochastic derivation into ours was important in order to unravel many of the obscurities or (apparent) arbitrariness of the former. It was also important to clarify some features of our own derivation; it is precisely this interplay that enriches the interpretation by furnishing many perspectives from which the same problem may be viewed. Indeed, we were able to show the following.

(1) The stochastic behavior comes from the spontaneous *local* fluctuations of the system.

(2) The stochastic force is the damping fluctuating force of the fluctuation-dissipation theorem.

(3) The quantities appearing in the stochastic expressions should be considered as *average* values [e.g., the average momentum $p(x;t)$] and are not equivalent to the quantities appearing in Newton's law, except when the limit of no fluctuation is considered ($\mathbf{u}=\mathbf{0}$)—when both formalisms reduce to Newton's equations.

(4) Contrary to the usually accepted opinions [see the comment in Ref. [18] before Eq. (3)], the stochastic velocity $\mathbf{u}$ does have a classical origin, since it is nothing but the fluctuating velocity appearing in the linear Onsager relation, coming from the classically defined entropy.

(5) It is possible to show that there is a superposition principle in the deepest realm of classical statistical mechanics which corresponds in every sense to the superposition principle of quantum mechanics. This principle can be interpreted in very simple terms by referring it to local fluctuations and by using the ergodic assumption; the role of the superposition principle is governed by the relevance of the fluctuations for the overall behavior of the system, implying that, for macroscopic systems, these fluctuations will be much more difficult to observe.

(6) Contrary to the universally accepted opinion among adepts of the stochastic view, the stochastic forces do not come from an interaction of the system with its surroundings (at least in a necessary way)—this is because, within our derivation, it was not necessary to include the surroundings in any sense, since the system is a closed one composed, however, of a subsystem linked to a reservoir giving rise to *spontaneous local fluctuations*. The *statistical character* of the theory comes from the fact that we adopt the strategy of treating the local fluctuations as hidden internal degrees of freedom (we treat them statistically [21])—this prescription is allowed whenever the Heisenberg relations remain fulfilled.

(7) The effective potential, derived in paper II, may be easily interpreted by using the notions of stochasticity. A comparison of the derivation process given in that paper with the present one is capable of unraveling the dynamical process underlying the quantum mechanical statistical one (see the discussion below).

Another important achievement was the clarification of the stochastic postulates. Indeed, when developing the stochastic derivation, we were faced with many (well-reasoned) impositions, such as the postulate of the behavior of the stochastic velocity upon time reversal [17], which becomes quite natural if we remember that this velocity is related to the damping force of the Onsager relations and that, being a damping force, it should behave this way upon time inversion. Other postulates may also be clarified. Indeed, the stochastic derivation always had in mind the reduction of its formalism to Newton's in the limit of non-stochasticity, such as the postulate related to expression (9), where Newton's second law is generalized to include the stochastic behavior. This expression, however, is nothing but our expression (45), and comes from the fact that the Liouville equation naturally assumes this format *for the average values* when fluctuating systems are being considered.

On the other hand, the stochastic approach, when linked with ours, has also furnished the key to the interpretation of many features of quantum theory, such as the time-energy dispersion relations, since it shows that we should consider the symbol δE as an energy *fluctuation* and the symbol Δt as the characteristic minimum time that allows us to consider the individual system as an *ensemble* (the time interval within which the system fluctuates enough so as to validate the ergodic assumption). It was also useful to understand the

origin of the fluctuations and their relation to stability, coming from the analysis of the Bohr-Sommerfeld relations.

Another important advantage of showing the equivalence between the stochastic derivation and ours is that, by means of this equivalence, we automatically append to our own framework all the many results achieved by the stochastic approach within recent decades. Examples are the relativistic extensions giving both the Klein-Gordon and the Dirac equation [27–29], their radiative corrections [30,31], a path integral formulation [32] together with its application to the problem of barrier penetration [33], to cite but a few. In particular, a comparison between the stochastic derivation of the Feynman path integral formulation [32] with ours [2] furnishes another beautiful example of the equivalence of the two approaches.

We also presented (in the particular case of the hydrogen atom) the physics governing the interpretation of the process described by the Schrödinger equation. For a general physical system this interpretation is as follows. Within the present framework we are considering one single system where a force field [with a physical potential function $V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$] is responsible for the interaction of the N particles composing the system. This system is a closed one, since no other external force field is present in the exact Newtonian equations governing the movement of each particle. The total energy of the system is, therefore, conserved (which is generally not the case with open systems). Now we make the decision to treat the closed single system as composed of two subsystems, the particles and the force field, each one capable of keeping some amount of energy (this is, obviously, only a distinction in words). We also choose to describe the parameters of the subsystem composed of the particles (e.g., the energy), while ignoring those related to the force field. With the adoption of this strategy of description, statistical physics tells us that fluctuations will appear that are responsible for the exchange of energy between the two subsystems. In this case, for instance, the energy of the particles, being taken into account explicitly, will fluctuate, sometimes being lowered by transferring energy to the force field, sometimes being increased by taking energy from the force field. This means that the average potential governing the average movement of the particles will not be given simply by $V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, but we will have to correct it to take into account, as an average, the energy fluctuations. This correction is given precisely by the extra term appearing as the ''potential'' $\phi_B(x)$ [Eq. (26)] in the second section, which is equivalent to the so called Bohm ''potential'' [one has to recall the ansatz used in the expression (22) and make the change $\rho = R'^2 = e^{2R}$ to find the common expression $\phi_B = -(\hbar^2/2mR')\nabla^2 R'$]—it is important to note that it has a kinetic origin, not a true potential one.

We may still ask which physical process would give rise to such fluctuations in the exchange of energy between the two subsystems. The answer is that, when considering a field, we are implicitly assuming the existence of the ''quasiparticles'' responsible for the interaction process (e.g., photons for the electromagnetic field). Thus, when a particle of the subsystem under explicit consideration is to interact with another particle of this same subsystem, it must interchange some amount of quasiparticles responsible for carrying the physical interaction. However, when it emits these quasiparticles, it transfers energy to the force field and its own energy is lowered (the fact that the quasiparticles have a finite velocity implies the effectiveness of this phenomenon). When the quasiparticles are absorbed by the other particle participating in the interaction process, the energy transferred from the particle subsystem to the force field is again transferred back to the particle subsystem—thus generating a fluctuation in the force-field energy. This process (the number of quasiparticles present in each interaction, etc.) depends upon the state of the field. This is the main difference between the present approach and the usual Newtonian one: in the latter the force field is always supposed to be fixed, in the sense that the potential function $V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ does not fluctuate. However, it is important to stress that, although this is a departure from the Newtonian mechanistic mathematical approach to the study of physical systems, it does not represent any departure from a classical description of Nature, since all the *concepts* here developed are part of the conceptual framework of classical statistical mechanics.

We would like to stress that this paper is a continuation of a series of other papers [1,2] in which we have shown how the various derivation methods of the Schrödinger equation may be reduced (or rephrased) to only one. This may be understood as the first step in the endeavor to establish a consistent interpretation of quantum formalism based upon its own postulating foundations, and not, as seems to be usual, upon wild guesses and/or philosophical presuppositions.

Thus, we have derived the Schrödinger equation by a more mathematical approach in paper I [1], which allowed us to unravel the origin of the operator structure of quantum mechanics and, as a by-product, allowed us to show how to quantize in generalized coordinates, a shameful void of the theory previously. We then derived the Schrödinger equation from a more physical set of postulates in paper II [2], which allowed us to make evident the meaning of the symbols in the approach made in paper I. The concept of entropy was then introduced. With the derivation of paper I, which is easier for mathematical manipulation, we were able, also in paper II, to derive and generalize the Bohr-Sommerfeld quantization rules and show that they are consistent with the same set of postulates giving rise to the Schrödinger equation, since they could be derived from this very same set. Moreover, the Feynman path integral derivation of quantum mechanics was shown to be equivalent to our previous one and closely related to the Bohr-Sommerfeld quantization rules and the results of paper I. In this paper we have shown how the stochastic derivation is also equivalent to ours. We are thus close to achieving our goal.

Indeed, since our derivation is a fully axiomatic one, in which the interpretation is much easier to carry out, *for it must be contained only in the (few) postulates*, we sincerely hope that at the end of this series we will be in a suitable position to present a coherent interpretation of quantum mechanics. Moreover, we are being careful to fix the physical reference of every relevant symbol of the theory—and this is one of the important features in showing the equivalence

between all of these derivations, since each one looks more deeply at a different aspect of reality, and thus uses one set of physical references more than another [e.g., operator structure (paper I), entropy, fluctuations, and adiabatic invariance (paper II), stochasticity (the present paper), etc.]. The important aspect of this program is that, after showing the *formal and epistemological* equivalence between these derivations (world views), the interpretation developed for one will be necessarily the interpretation of them all. It has also the advantage of being capable of giving us the means to reject other widespread interpretations, based solely upon the relations between the formalism and its symbolic references.

Although many of the underlying interpretation features have already been discussed, whenever allowed by the development of the formalism, the complete interpretation will be left for a future paper, where they will be stated in a more systematic manner.

[1] L.S.F. Olavo, Physica A **262**, 197 (1999).
[2] L.S.F. Olavo, Physica A **271**, 260 (1999).
[3] L.S.F. Olavo, A.F. Bakuzis, and E. Amilcar, Physica A **271**, 303 (1999).
[4] P. Braffort and C. Tzara, C. R. Hebd. Seances Acad. Sci. **239**, 157 (1954).
[5] I. Keynes, Z. Phys. **132**, 81 (1952).
[6] W. Weizel, Z. Phys. **134**, 264 (1953).
[7] W. Weizel, Z. Phys. **135**, 270 (1953).
[8] W. Weizel, Z. Phys. **136**, 582 (1954).
[9] D. Kershaw, Phys. Rev. **136**, B1850 (1964).
[10] G.G. Comisar, Phys. Rev. **138**, B1332 (1965).
[11] P. Braffort, M. Surdin, and A. Taroni, C. R. Hebd. Seances Acad. Sci. **261**, 4339 (1965).
[12] T. Marshall, Proc. Cambridge Philos. Soc. **61**, 537 (1965).
[13] R.C. Bourret, Can. J. Phys. **43**, 619 (1965).
[14] E. Nelson, Phys. Rev. **150**, 1079 (1966).
[15] A. Einstein, *Investigation on the Theory of the Brownian Movement*, translated by A.D. Cowper (Methuen and Company, London, 1926).
[16] M. V. Smoluchowski, *Abhandlungen über die Brownische Bewegung und verwandte Erscheinungen* (Akademische Velagsgesellschaft, Leipzig, 1923).
[17] L. De La Peña, Found. Phys. **12**, 1017 (1982).
[18] L. De La Peña, J. Math. Phys. **10**, 1620 (1969).
[19] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), p. 290.
[20] H B. Callen, *Thermodynamics* (Wiley, New York, 1960).
[21] S.G. Eubank and J.D. Farmer, in *Non-Linear Physics*, edited by Lui Lam, (Springer-Verlag, New York, 1997), p. 106.
[22] L.E. Ballentine, Rev. Mod. Phys. **42**, 358 (1970).
[23] W.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 419.
[24] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics*, (Springer-Verlag, Berlin, 1995), p. 5.
[25] T.H. Boyer, Phys. Rev. D **11**, 790 (1975), and references therein.
[26] T.H. Boyer, Phys. Rev. A **18**, 1238 (1978).
[27] L. De La Peña, Phys. Lett. **31A**, 403 (1970); J. Math. Phys. **12**, 453 (1971).
[28] L. De La Peña, Rev. Mex. Fis. **19**, 133 (1970).
[29] J.P. Vigier, Lett. Nuovo Cimento Soc. Ital. Fis. **24**, 265 (1979).
[30] L. De La Peña and A.M. Cetto, Phys. Rev. D **3**, 795 (1971).
[31] A.M. Cetto and L. De La Peña, Rev. Mex. Fis. **20**, 191 (1971).
[32] M. Berrondo, Nuovo Cimento Soc. Ital. Fis., B **18**, 95 (1973).
[33] D.L. Weaver, Phys. Rev. Lett. **40**, 1473 (1978).