Heisenberg-limit interferometry with four-wave mixers operating in a nonlinear regime

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A model of a four-wave mixer operated in a nonlinear regime, studied by Yurke and Stoler [Phys. Rev. A **35**, 4846 (1987)], is reexamined. Yurke and Stoler have shown that this device, under a certain condition, acts as an even-odd filter, switching even-photon-number states from the pump mode to the signal mode. An initial coherent state in the pump is converted into an entanglement of even and odd coherent states with vacuum states in the output signal and pump modes. We point out that under a different condition and with an even-photon-number state initially in the pump and a vacuum in the signal, the device creates a maximally entangled state between the number state and the vacuum. Using the device to replace the first beamsplitter of a Mach-Zehnder interferometer, phase uncertainties at the Heisenberg limit ($\Delta\phi = 1/n$) can be obtained. Since number states are difficult to generate, we point out that an even coherent state obtained from the output of one device can be used as input to a second to achieve the phase uncertainties $\Delta\phi = 1/n_e$, where n_e is the average photon number of the even coherent state.

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One of the goals in the quest for the generation of nonclassical states of light is to increase the sensitivity in the measurement of the relative phase shifts between the two paths of an interferometer. This is of considerable importance in attempts to measure the weak signals expected in gravity-wave detectors [1]. In a detector involving laser beams and beamsplitters, such as a Mach-Zehnder (MZ) interferometer, the zero-point fluctuations in the laser beam and in the unused port of the input beamsplitter will produce phase-difference fluctuations large enough to mask the true phase-difference change resulting from an incident gravity wave. In a MZ interferometer with two 50/50 beamsplitters and with a coherent laser field injected into one of the input ports and the vacuum into the other, the phase-difference uncertainty $\Delta\phi$ between the two paths varies as $1/\sqrt{\overline{n}}$, where $\overline{n}$ is the average number of photons supplied by the laser during the time interval of the measurement. Increasing the sensitivity of the interferometer requires an increasingly powerful laser source. An attractive alternative is to use states of light with distinctly nonclassical properties that may produce, as a matter of principle, greater sensitivity for a given average photon number. In Ref. [1] it was shown that if a squeezed state is injected into the previously unused port of the beamsplitter, then the phase uncertainty can be reduced to $\Delta\phi = e^{-r}/\sqrt{\overline{n}}$, where the squeezing parameter $r > 0$. The limiting case for phase uncertainties, on rather general grounds, is $\Delta\phi = 1/\overline{n}$ and this is known as the Heisenberg limit [2]. A number of schemes have been proposed to approach this limit. Holland and Burnett [3] showed that accuracies approaching $\Delta\phi = 1/n$ are possible if the input states at the first beamsplitter are Fock states of equal photon number n. Yet another scheme involves the replacement of one or both of the beamsplitters with active optical elements such as four-wave mixers. Hillery and Mlodinow [4] have shown that SU(2) minimum-uncertainty states for a two-mode field can be used to achieve phase-shift uncertainties of $\Delta\phi = 1/N$, where N is the total number of photons at the input port of a MZ interferometer. Such an accuracy can be obtained with or without the first beamsplitter being replaced by a four-wave mixer. Of course, one must somehow generate the SU(2) minimum-uncertainty state, a nontrivial task. On the other hand, schemes involving an SU(1,1) interferometer, which consists of a MZ interferometer with both beamsplitters replaced by four-wave mixers [5], have been considered with minimum-uncertainty SU(1,1) states as the inputs [6]. High accuracy, on the order of $\Delta\phi \sim 1/\overline{n}$, is possible, but again a special field state is required to be generated for the input.

Most of the above-mentioned schemes actually show only asymptotic phase uncertainty proportional to $1/n$ (or $1/\overline{n}$), with a proportionality constant greater than zero. On the other hand, Bollinger *et al.* [7] have pointed out that if the state after the first "beamsplitter" of the interferometer is a maximally entangled state of the form

$$\frac{1}{\sqrt{2}}(|n\rangle_a|0\rangle_b + e^{i\vartheta}|0\rangle_a|n\rangle_b), \tag{1}$$

where the subscripts a and b stand for the two output modes of the "beamsplitter," the phase uncertainty is exactly *equal* to $1/n$. The problem, of course, is that the usual beamsplitters cannot produce such a state, hence the quotation marks. In the case of the usual passive beamsplitter, an input number state $|n\rangle_a|0\rangle_b$ becomes an SU(2) coherent state with the n photons binomially distributed over the two output modes [8]. In this paper, we present a scheme that does yield states of the form of Eq. (1). Our proposal involves the replacement of the first beamsplitter with a nonlinear medium in which two competing processes are acting—four-wave mixing and a two-mode Kerr interaction. With an appropriate choice of relative coupling constant, and with the use of the Schwinger realization of the angular momentum operators in terms of a pair of boson operators, the model of the interaction can be made identical to a model of a four-wave mixer operated in a nonlinear regime studied some years ago by Yurke and Stoler [9]. The device was shown to act, under a certain condition, as an even-odd filter with respect to photon number. We review this below. We then point out that the same device, under a slightly different condition, essentially a different interaction time, with n photons at one input and the vacuum at the other, can produce states of the form of Eq. (1), if n is even. Since an n-photon state for large even n is very hard to generate, we then suggest that it might be possible to use the output of one of the devices acting as a even-odd filter to generate from an initial coherent state an even coherent state, an example of a Schrödinger cat state [10], which is then injected into a second device set up to generate a superposition of the maximally entangled states of the form of Eq. (1). The phase uncertainty is numerically shown to be $1/\overline{n}$ where $\overline{n}$ is the average number of photons in the even coherent state.

The interaction Hamiltonian for a degenerate four-wave mixer (FWM) is given by

$$H^{I}_{\mathrm{FWM}}=\hbar\frac{\Omega}{4}(a^{\dagger 2}b^{2}+a^{2}b^{\dagger 2}), \tag{2}$$

where Ω is proportional to the nonlinear susceptibility $\chi^{(3)}_{\mathrm{FWM}}$ of the four-wave mixing process in the medium and a and b are the operators of the two degenerate modes. The Kerr interaction of the medium is given by

$$H^{I}_{\mathrm{Kerr}}=\hbar\frac{K}{2}a^{\dagger}ab^{\dagger}b, \tag{3}$$

where we have assumed that self-modulation terms of the form $(a^{\dagger}a)^{2}$ and $(b^{\dagger}b)^{2}$ can be ignored by choosing the resonances of the medium in an appropriate way [11]. The constant K is proportional to the Kerr susceptibility $\chi^{(3)}_{\mathrm{Kerr}}$. We now assume that both processes are present in our medium, which henceforth we refer to as a nonlinear four-wave mixer (NFWM), and furthermore we assume that the condition $K=\Omega$ can be attained, perhaps through the enhancement of Kerr nonlinearities using electromagnetically induced transparency as suggested by Schmidt and İmamoḡlu [12]. Then the complete interaction Hamiltonian for the medium takes the form

$$\begin{aligned} H^{1}&=\hbar\frac{\Omega}{4}(a^{\dagger 2}b^{2}+a^{2}b^{\dagger 2})+\hbar\frac{\Omega}{2}a^{\dagger}ab^{\dagger}b \\ &=\hbar\frac{\Omega}{4}(a^{\dagger}b+ab^{\dagger})^{2}. \end{aligned} \tag{4}$$

Assuming that modes a and b are degenerate with frequency ω, the full Hamiltonian is

$$H=\hbar\omega(a^{\dagger}a+b^{\dagger}b)+\hbar\frac{\Omega}{4}(a^{\dagger}b+ab^{\dagger})^{2}. \tag{5}$$

We shall refer to the a and b modes as ''pump'' and ''signal'' modes, respectively.

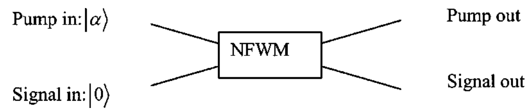

FIG. 1. The nonlinear four-wave mixing device with a coherent state in the input pump mode and a vacuum in the signal mode. When operated as a π device the output state will be of the form of Eq. (25).

We now introduce the Schwinger realization of the angular momentum operators [13]:

$$\begin{aligned} J_{1}&=\tfrac{1}{2}(a^{\dagger}b+ab^{\dagger}), \\ J_{2}&=\frac{1}{2i}(a^{\dagger}b-ab^{\dagger}), \\ J_{3}&=\tfrac{1}{2}(a^{\dagger}a-b^{\dagger}b), \end{aligned} \tag{6}$$

satisfying the commutation relations

$$[J_{i},J_{j}]=i\epsilon_{ijk}J_{k}. \tag{7}$$

The square of the angular momentum is given by

$$\overline{J}^{2}=J_{1}^{2}+J_{2}^{2}+J_{3}^{2}=J_{0}(J_{0}+1), \tag{8}$$

where

$$J_{0}=\tfrac{1}{2}(a^{\dagger}a+b^{\dagger}b) \tag{9}$$

commutes with all the operators in Eq. (3). The Fock states of the two modes $|n_{a}\rangle_{a}|n_{b}\rangle_{b}$ are related to the angular momentum states $|j,m\rangle$, which satisfy $J_{3}|j,m\rangle=m|j,m\rangle$ and $J_{0}|j,m\rangle=j|j,m\rangle$, according to

$$|j,m\rangle=\frac{(a^{\dagger})^{j+m}(b^{\dagger})^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle_{a}|0\rangle_{b}, \tag{10}$$

where $j=N/2$, $N=n_{a}+n_{b}$, and $m=(n_{a}-n_{b})/2$. For a particular total photon number N, $|j,-j\rangle=|N/2,-N/2\rangle$ corresponds to the number states $|0\rangle_{a}|N\rangle_{b}$, whereas $|j,j\rangle=|N/2,N/2\rangle$ corresponds to $|N\rangle_{a}|0\rangle_{b}$. Henceforth we shall drop the subscripts on the number states with the understanding that the ordering is pump followed by signal.

Evidently, using the identities in Eqs. (6) and (9), the Hamiltonian of Eq. (5) can be written in the form (with $\hbar=1$)

$$H=2\omega J_{0}+\Omega J_{1}^{2}. \tag{11}$$

Since $[J_{0},J_{1}]=0$, we henceforth work in the interaction picture where the dynamics is governed by the interaction Hamiltonian $H_{1}=\Omega J_{1}^{2}$.

We consider first the initial state containing n photons in the pump mode with the signal mode in the vacuum: $|\mathrm{in}\rangle=|n\rangle|0\rangle=|j,j\rangle$ for $j=n/2$ (see Fig. 1). The output state of the device is then

$$|\text{out},t\rangle = \exp(-it\Omega J_1^2)|\text{in}\rangle, \tag{12}$$

where t, the interaction time, is determined by the dimensions of the medium and the index of refraction. Following Yurke and Stoler [9] by using the mathematics of the rotation group [14], we write

$$|\text{out},t\rangle = \exp\left(-i\frac{\pi}{2}J_2\right)\exp(-it\Omega J_3^2)\exp\left(i\frac{\pi}{2}J_2\right)|j,j\rangle \tag{13}$$

where essentially a rotation about the ''2'' axis has transformed J_1 into J_3:

$$\exp\left(i\frac{\pi}{2}J_2\right)J_1\exp\left(-i\frac{\pi}{2}J_2\right) = J_3. \tag{14}$$

Inserting appropriately a complete set of states, Eq. (13) can be written as

$$|\text{out},t\rangle = \exp\left(i\frac{\pi}{2}J_2\right)\sum_{m=-j}^{j}\exp(-it\Omega m^2)d_{m,j}^{(j)}\left(\frac{\pi}{2}\right)|j,m\rangle, \tag{15}$$

where the

$$d_{m,j}^{(j)}\left(\frac{\pi}{2}\right) = \langle j,m|\exp\left(-i\frac{\pi}{2}J_2\right)|j,j\rangle \tag{16}$$

are the matrix elements for a rotation of $\pi/2$ about the ''2'' axis.

We now consider special interaction times. If $t=8\pi/\Omega$ (or any integral multiple), it follows that $\exp(-i\Omega tm^2)=1$ and thus the output is identical to the input. If $t=2\pi/\Omega$ (or an integral multiple) then $\exp(-i\Omega tm^2)=\exp(-2\pi im^2)=1$ for j integer and $-i$ for j a half-odd integer. But if $t=\pi/\Omega$ we have $\exp(-i\Omega tm^2)=\exp(-i\pi m^2)=(-1)^m$ and thus

$$|\text{out},\pi/\Omega\rangle = \exp\left(i\frac{\pi}{2}J_2\right)\sum_{m=-j}^{j}(-1)^m d_{m,j}^{(j)}\left(\frac{\pi}{2}\right)|j,m\rangle. \tag{17}$$

Now for j equal to a half-odd integer (n odd) we have $(-1)^m=\exp(-i\pi/4)$ and thus

$$|\text{out},\pi/\Omega\rangle = \exp\left(-i\frac{\pi}{4}\right)|n\rangle|0\rangle. \tag{18}$$

If j is integer (n even) we have, following Yurke and Stoler [9] by writing

$$(-1)^m d_{m,j}^{(j)}\left(\frac{\pi}{2}\right) = \langle j,m|\exp(-i\pi J_3)\exp\left(-i\frac{\pi}{2}J_2\right)|j,j\rangle, \tag{19}$$

that

$$|\text{out},\pi/\Omega\rangle = \exp\left(i\frac{\pi}{2}J_2\right)\sum_{m=-j}^{j}|j,m\rangle \times\langle j,m|\exp(-i\pi J_3)\exp\left(-i\frac{\pi}{2}J_2\right)|j,j\rangle = \exp(i\pi J_1)|j,j\rangle. \tag{20}$$

But it can be shown that

$$\exp(i\pi J_1)|j,j\rangle = (-1)^j|j,-j\rangle \tag{21}$$

and thus it follows that

$$|\text{out},\pi/\Omega\rangle = \begin{cases} i^n|0\rangle|n\rangle, & n \text{ even},\\ \exp\left(-i\frac{\pi}{4}\right)|n\rangle|0\rangle, & n \text{ odd}.\end{cases} \tag{22}$$

Evidently, at time $t=\pi/\Omega$, the device acts as an even-odd filter, the photons exchanging modes if n is even or not if it is odd. A NFWM acting in this way will be referred to as a π device.

Suppose now the initial state is $|\psi\rangle|0\rangle$, maintaining the previous ordering of the states, where

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle \tag{23}$$

is some arbitrary single-mode pure state initially in the pump mode. Then from the preceding results we have

$$|\text{out},\pi/\Omega\rangle = \exp(-i\pi J_1^2)|\psi\rangle|0\rangle = \sum_{n,\text{even}} C_n i^n|0\rangle|n\rangle + \exp\left(-i\frac{\pi}{4}\right)\sum_{n,\text{odd}} C_n|n\rangle|0\rangle. \tag{24}$$

As a particular, and important, example, suppose that $|\psi\rangle$ is a coherent state $|\alpha\rangle$ for which $C_n=\exp(-|\alpha|^2/2)\alpha^n/\sqrt{n!}$. Then the output takes the form

$$|\text{out},\pi/\Omega\rangle = \tfrac{1}{2}[|0\rangle_a(|i\alpha\rangle_b+|-i\alpha\rangle_b)+(|\alpha\rangle_a-|-\alpha\rangle_a)|0\rangle_b], \tag{25}$$

where we have restored the mode indices for emphasis. The state $|i\alpha\rangle+|-i\alpha\rangle$ is an even coherent state while $|\alpha\rangle-|-\alpha\rangle$ is an odd coherent state, examples of Schrödinger cat states [10]. Of course, Eq. (25) is an entangled state. Detection of the vacuum state in the pump mode reduces the signal mode to an even coherent state, which in normalized form is given by

$$|i\alpha\rangle_e = N_e(|i\alpha\rangle+|-i\alpha\rangle), \tag{26}$$

where the normalization factor is given by

$$N_e=\frac{1}{\sqrt{2}}[1+\exp(-2|\alpha|^2)]^{-1/2}. \tag{27}$$

It is important to note that the vacuum state of the pump mode is uniquely correlated with the even coherent state of the signal mode since the vacuum state is not contained in the odd coherent state of the pump mode. However, the vacuum state of the signal mode is *not* uniquely correlated with the odd coherent state of the pump mode as there is a finite probability of it being correlated with the pump vacuum state, as can be seen from Eq. (24). That is, if $C_0\neq 0$, there will be a contribution from the vacuum state of the pump mode if a vacuum of the signal mode is detected. But if $C_0\approx 0$, as would be the case for a coherent state for $|\alpha|$ large, then to a high degree of accuracy, detecting a vacuum in the signal mode projects the pump mode into the odd coherent state

$$|\alpha\rangle_o=N_o(|\alpha\rangle-|-\alpha\rangle) \tag{28}$$

where the normalization factor is

$$N_o=\frac{1}{\sqrt{2}}[1-\exp(-2\alpha|^2)]^{-1/2}. \tag{29}$$

We now go back to Eq. (15) and consider the state at the time $t=\pi/2\Omega$. For n even, hence m integer, it can be shown that

$$\exp(-i\Omega t m^2)=\exp\left(-i\frac{\pi}{2}m^2\right)$$
$$=\frac{\exp(-i\pi/4)+\exp(i\pi/4)(-1)^m}{\sqrt{2}}. \tag{30}$$

Then, using Eqs. (19)–(21) it follows that

$$|\mathrm{out},\pi/2\Omega\rangle=\frac{1}{\sqrt{2}}[|n\rangle_a|0\rangle_b+\exp(-i\Phi_n)|0\rangle_a|n\rangle_b]$$
$$=\frac{1}{\sqrt{2}}[|j,j\rangle+\exp(-i\Phi_{2j})|j,-j\rangle], \tag{31}$$

where $\Phi_n=(n+1)\pi/2$ and where an overall irrelevant phase factor has been suppressed.

Interestingly, the state in Eq. (31) is a particular version of a maximally entangled state that has been much discussed recently in the context of Ramsey spectroscopy with two-level trapped ions. In that case, the angular momentum states are the Dicke states [15] and $|j,j\rangle$ is the product state with all the atoms in the excited state e, i.e., $|j,j\rangle=|e_1,e_2,\ldots,e_n\rangle$, and the state $|j,-j\rangle$ is the state where all the atoms are in the ground state g, i.e., $|j,-j\rangle=|g_1,g_2,\ldots,g_n\rangle$. Thus the atomic state corresponding to Eq. (31) is

$$\frac{1}{\sqrt{2}}[|e_1,\ldots,e_n\rangle+\exp(-i\Phi_n)|g_1,\ldots,g_n\rangle]. \tag{32}$$

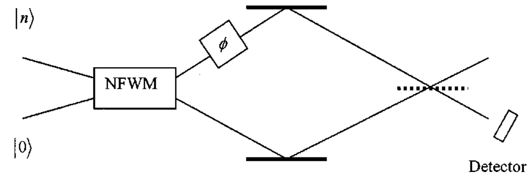

FIG. 2. A Mach-Zehnder interferometer with the first beamsplitter replaced by a nonlinear four-wave mixer operating as a $\pi/2$ device. With a number state as the input of the pump mode and a vacuum in the signal, the output state is of the form of Eq. (31). Only one of the beams of the output beamsplitter is detected with the result taken as the exponent of -1.

This is a maximally entangled state of the n two-level atoms, in fact, an n-particle Greenberger-Horne-Zeilinger state [16]. It is also a special case of an atomic Schrödinger cat state [17]. Bollinger *et al.* [7] and Huelga *et al.* [18] have shown that the resonant frequency ω_0 between the states $|e\rangle$ and $|g\rangle$ can be determined, using the Ramsey method, with an uncertainty of $\delta\omega_0\sim 1/n$, a fact that is of great importance in the pursuit of ultrahigh-resolution frequency standards and in the improvement of atomic clocks [19]. Proposals for generating such states in the context of n trapped ions have been given by a number of authors [19–22]. Some of the schemes involve addressing each of the ions individually with a well-focused laser beam, applying the interactions sequentially to each ion [20], while others involve global interactions with all the ions [19,21]. Møller and Sørensen [22] have proposed a method that should work for hot trapped ions, and this involves the engineering of an interaction whose Hamiltonian is proportional to J_1^2, where the operator J_1 acts on the collective internal states of the ions. This is, of course, the same interaction form used in the present case to model a nonlinear four-wave mixer. In fact, their discussion establishes the identity given in Eq. (30). We shall refer to NFWM devices that produce states of the type in Eq. (31) as $\pi/2$ devices.

In view of the earlier remarks on interferometry, we propose here a MZ interferometer with the first beamsplitter replaced by a $\pi/2$ NFWM device as illustrated in Fig. 2. The upper arm is taken to be the output pump beam and contains the phase shifter represented by the operator $U(\phi)=\exp(-i\phi a^\dagger a)$ [4]. The beamsplitter (assumed to be 50/50) at the output of the interferometer, in the language of the angular momentum operators, is represented by a $\pi/2$ ''rotation'' about the ''1'' axis [5,8]: $U_{\mathrm{BS}}=\exp(-i(\pi/2)J_1)$. Thus the output state of the interferometer is given by

$$|\mathrm{out}\rangle_{\mathrm{MZ}}=U_{\mathrm{BS}}U(\phi)|\mathrm{out},\pi/2\Omega\rangle$$
$$=\exp\left(-i\frac{\pi}{2}J_1\right)\frac{1}{\sqrt{2}}[\exp(-i\phi n)|n\rangle|0\rangle$$
$$+\exp(-i\Phi_n)|0\rangle|n\rangle]. \tag{33}$$

In a typical MZ interferometer experiment involving only passive beamsplitters, one measures the difference in photon numbers of the two output ports of the second beamsplitter, essentially the expectation value of the operator $J_3=(a^\dagger a - b^\dagger b)/2$. But for states of the type given in Eq. (33), $\langle J_3\rangle = 0$, so Bollinger *et al.* [7] suggested measuring the operator

$$O=(-1)^{b^\dagger b}=\exp[i\pi(J_0+J_3)]. \tag{34}$$

As they point out, detecting the operator O is equivalent to measuring the number of photons N_b in the output of the b mode of the interferometer and assigning to the measurement the value $(-1)^{N_b}$. The operator O is essentially a parity operator. The above procedure is equivalent to having measured the moments of the number operator $b^\dagger b$ as would be evident upon expanding the exponential in Eq. (34). From Eq. (33) we have

$$\begin{aligned}\langle 0\rangle &= {}_{\mathrm{MZ}}\langle \mathrm{out}|\exp[i\pi(J_0+J_3)]|\mathrm{out}\rangle_{\mathrm{MZ}}\\ &=(-1)^{n/2}\,{}_{\mathrm{MZ}}\langle \mathrm{out}|\exp(i\pi J_3)|\mathrm{out}\rangle_{\mathrm{MZ}}\\ &=(-1)^{n/2}\cos(n\phi-\Phi_n).\end{aligned} \tag{35}$$

In deriving Eq. (35) we have used the facts that

$$\exp\left(i\frac{\pi}{2}J_1\right)J_3\exp\left(-i\frac{\pi}{2}J_1\right)=J_2 \tag{36}$$

and that

$$\exp(i\pi J_2)|j,j\rangle=|j,-j\rangle,\quad 2j\ \text{ even.} \tag{37}$$

The uncertainty in the phase, $\Delta\phi$, is given by

$$\Delta\phi=\Delta O\bigg/\left|\frac{\partial\langle O\rangle}{\partial\phi}\right|, \tag{38}$$

where $\Delta O=(\langle O^2\rangle-\langle O\rangle^2)^{1/2}$. Since $O^2=1$, $\Delta O=\sin(n\phi-\Phi_n)$, and it follows that $\Delta\phi=1/n$, again for even n, which is the Heisenberg limit.

Recall that in the scheme of Holland and Burnett [3] using passive beamsplitters, phase uncertainties *approaching* the Heisenberg limit are obtained for identical photon number states as inputs. Such states may be hard to generate, especially for large n. In the present scheme, the initial state of the $\pi/2$ device is an even-number state in the pump mode and a vacuum-state signal. This is perhaps a bit easier to generate as pairs of number states are not required, but it is still an experimental challenge. With this in mind, we propose a scheme to obtain optimal phase uncertainties using as input to the $\pi/2$ device an even coherent state. The even coherent state is obtained, via state reduction, from the output of a π device.

The schematic for the proposed scheme is given in Fig. 3. NFWM1 is a π device whose input is the state $|\alpha\rangle|0\rangle$ and whose output is the state of Eq. (25). If state reduction is performed on the pump mode by the detection of zero photons (see Fig. 3), then the signal mode is projected into the even coherent state $|i\alpha\rangle_e$ of Eq. (26). This output is then taken as the input to the pump mode in the $\pi/2$ device, NFWM2, its signal mode initially in the vacuum state. The output of the NFWM2 is the state (suppressing irrelevant phase factors)

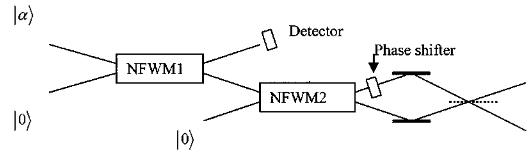

FIG. 3. Schematic for a proposed method to use even coherent states to achieve Heisenberg-limit phase uncertainty. NFWM1 is a π device that, with the input indicated, creates the state of Eq. (25). Detection of the vacuum state in the output pump mode of NFWM1 projects the signal mode into the even coherent state of Eq. (26). This state is then injected into NFWM2, a $\pi/2$ device acting just as in Fig. 2.

$$|\mathrm{out}\rangle_{\mathrm{NFWM2}}=\frac{1}{\sqrt{2}}N_e\exp(-|\alpha|^2/2)\sum_{n=0,2,4,\ldots}^{\infty}\frac{(i\alpha)^n}{\sqrt{n!}}\times[|n\rangle|0\rangle+\exp(-i\Phi_n)|0\rangle|n\rangle], \tag{39}$$

a superposition of the form of Eq. (31). With this state, the expectation value of the operator O is given by

$$\langle O\rangle=N_e^2\exp(-|\alpha|^2)\sum_{n=0,2,4,\ldots}^{\infty}(-1)^{n/2}\frac{|\alpha|^{2n}}{n!}\cos(n\phi-\Phi_n). \tag{40}$$

From this we once again calculate the phase uncertainty according to Eq. (38). However, this time no closed form seems to exist. We have checked numerically that $\Delta\phi$ is independent of ϕ. Furthermore, we have numerically shown that this phase uncertainty is virtually indistinguishable from $1/\overline{n}_e$, where $\overline{n}_e=|\alpha|^2\tanh(|\alpha|^2)$ is the average photon number in the even coherent state of Eq. (26) [23]. Thus with an even coherent state entering the $\pi/2$ NFWM we may achieve the Heisenberg limit in terms of the average photon number of the state.

So far, we have ignored the effects arising from any source of decoherence, such as a dissipative interaction with a heat reservoir (i.e., the environment consisting of the parts of the interferometer). Decoherence effects will, of course, degrade the sensitivity of the phase-shift measurements. Consider the case for which there is a definite number of photons N, where N is even. We know from the work of Huelga *et al.* [18] (in the context of trapped ions) that in the presence of decoherence, standard Ramsey interferometry on uncorrelated ions attains the same resolution as optimal measurements on maximally entangled states. This result should apply here as well. Of course, if decoherence is minimized, the maximally entangled state yields a higher resolution. But in the case involving a coherent state and the production of an even cat state, there is the further complication that could result from the decoherence of the even cat state into a statistical mixture. Decoherence has the effect of populating the odd-photon-number states and would thus further degrade the effectiveness of the proposed scheme. However, it must be stressed that decoherence effects will degrade all schemes proposed to enhance interferometric measurements of phase shifts. It might therefore be necessary to make detailed comparisons of all such schemes with the incorporation of decoherent interactions.

There is one other source of decoherence, other than environmental, that must be addressed. Recall that a crucial step in conditionally generating the input even coherent state to the second NFWM is the detection of a vacuum state in the output *a* mode of the first NFWM. Until now we have unrealistically assumed that the photon detectors are of 100% efficiency. In order to consider detectors of lower efficiency we introduce the so-called positive operator measure (POM) associated with the detection of photons in the output of the *a* mode. We are interested only in whether there is no photon (a NO ''click'') or any number of photons (a YES ''click''). To this end we follow Paris [24] and introduce the two-valued POM

$$\Pi_N=\sum_{p=0}^{\infty}(1-\eta)^p|p\rangle_{a\,a}\langle p|,\quad \Pi_Y=I-\Pi_N, \tag{41}$$

for the respective NO or YES detection of photons, where η is the quantum efficiency of the detector and I is the identity operator. Note that as the quantum efficiency of the detector approaches unity, Π_N approaches the projection operator onto the vacuum and Π_Y approaches the projection operator onto the orthogonal subspace of all the nonzero-photon-number states. With the output state of the first NFWM given by $|\mathrm{out},\pi/\Omega\rangle$ of Eq. (25), suppressing the π/Ω label, the probability of observing a NO in the output *a* mode is given by

$$\begin{aligned}P_N(\eta)&=\mathrm{Tr}_{ab}(|\mathrm{out}\rangle\langle\mathrm{out}|\Pi_N)\\&=\tfrac{1}{2}\{1+\exp(-2|\alpha|^2)+2\exp(-2|\alpha|^2)\\&\quad\times\sinh[|\alpha|^2(1-\eta)]\}.\end{aligned} \tag{42}$$

The conditional output state in the *b* mode is then

$$\begin{aligned}\rho_{N,b}&=\frac{1}{P_N}\mathrm{Tr}_a(|\mathrm{out}\rangle\langle\mathrm{out}|\Pi_N)\\&=\frac{1}{P_N}\left(\frac{1}{N_e^2}|i\alpha\rangle_{ee}\langle i\alpha|+4\exp(-|\alpha|^2)\right.\\&\quad\times\sinh[|\alpha|^2(1-\eta)]|0\rangle\langle 0|\\&\quad\left.+c|0\rangle_e\langle i\alpha|+c^*|i\alpha\rangle_e\langle 0|\right)_b\end{aligned} \tag{43}$$

where $|i\alpha\rangle_e$ is given by Eq. (26) and where

$$c=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{(1-\eta)^N\alpha^n}{\sqrt{n!}}[1-(-1)^n]. \tag{44}$$

Two things are noteworthy here. First, only even states appear in Eq. (43). Measurement of the output of the *a* mode produces even number states (but in general nonpure) even if the detector efficiency is less than ideal. The second point is that even if η is not too close to unity, for a sufficiently large value of $|\alpha|$ only the first term survives in Eq. (43) and thus to a good approximation the state entering NFWM2 is a pure even coherent state. Thus an increase in the amplitude of the initial coherent state may compensate for the detector inefficiency.

There is one other problem that must be mentioned in regard to detector efficiency. The detector after the beamsplitter must be sensitive to photon number at the level of one photon in order to measure the operator O of Eq. (34). Ordinary photon detectors are not yet available at such resolutions. But since O is just a parity operator, it might be possible to simply use another π device after the beamsplitter in place of the photon detector, with the other input just the vacuum. The mere presence of *any* photons in either of the output modes of the device yields the parity according to Eq. (22). Indeed, Yurke and Stoler [25] have already discussed this method of performing parity measurements.

The last, but by no means the least, obstacle to overcome are the conditions required for obtaining the π and $\pi/2$ nonlinear devices. Such concerns must also be addressed in regard to the proposals of Refs. [9] and [25]. To obtain the conditions $\Omega t=\pi,\pi/2$ requires either long times or high nonlinearities. Currently available fibers have low nonlinearities and would induce severe decoherent effects at the lengths required for sufficiently long times. But as already mentioned, high nonlinearities might be generated through the use of electromagnetically induced transparency effects that modify the index of refraction of the medium [12]. Indeed, such effects have already been demonstrated in the laboratory [26]. The possibility of using such effects for the entanglement of ultraslow photons has recently been discussed [27].

Finally we mention that Ansari *et al.* [28] have previously discussed the use of even and odd coherent states as alternative input states to the squeezed states in gravity-wave detectors in the form of a Michelson interferometer. However, their discussion does not consider active optical elements.

In summary, we have shown that a four-wave mixer, operated in the nonlinear regime, which has previously been shown to act as an even-odd filter [9], can also act as a device to generate maximally entangled states (out of even-photon-number states). Such states yield phase uncertainties at the Heisenberg limit when used in interferometers. Since even-number states are hard to generate, we showed that an even coherent state obtained from another nonlinear four-wave mixer can be used to obtain the Heisenberg limit in terms of the average photon number of that state.

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