Classification of multiqubit mixed states: Separability and distillability properties

W. Dür and J. I. Cirac
Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria

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We give a complete, hierarchic classification for arbitrary multiqubit mixed states based on the separability properties of certain partitions. We introduce a family of N-qubit states to which any arbitrary state can be depolarized. This family can be viewed as the generalization of Werner states to multiqubit systems. We fully classify those states with respect to their separability and distillability properties. This provides sufficient conditions for nonseparability and distillability for arbitrary states.

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I. INTRODUCTION

Entanglement is one of the basic concepts of quantum mechanics and an important feature of most applications of quantum information. It arises when a state of a multiparticle quantum system cannot be prepared by acting on the particles individually, i.e., is nonseparable. Despite the fact that we do not know yet how to classify and quantify entanglement in general, much progress has been made in recent years. In particular, the concept of entanglement distillation or purification [1] was introduced. This process, which is the creation of (few) maximally entangled states out of many not-maximally entangled ones, turned out to be one of the most important concepts in quantum information theory. When combined with teleportation [2], it makes it possible to send quantum information over noisy channels [2,3] and to convey secret information via quantum privacy amplification [4].

Particular important states of two qubits are the so called Werner states (WS) [5], which are mixtures of a maximally entangled state, e.g., $|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$, with the totally depolarized state. These states are fully characterized by the fidelity F, which measures the overlap of the maximally entangled state $|\Phi^+\rangle$ with the WS. They play an essential role in the understanding of the entanglement and distillability properties of two qubit systems [6]. On the one hand, it has been shown that WS are separable for $F \leqslant 1/2$ and nonseparable (entangled) for $F > 1/2$. On the other hand, Bennett *et al.* [1] showed that one can purify WS with arbitrary high fidelity out of many pairs with $F > 1/2$ by using local operations and classical communication. Furthermore, any arbitrary state can be depolarized to a WS without changing the fidelity F, which automatically provides a sufficient criterion for non-separability [7–9] and distillability [10] for arbitrary states.

The description of the entanglement and distillability properties of systems with more than two particles is still almost unexplored (see Refs. [11] and [12], however). In Ref. [13], some steps towards the understanding of three-particle entanglement of mixed states were taken. In particular, a complete classification of arbitrary three-qubit states was proposed and the distillability and separability properties of a family of states was obtained. In this paper, we generalize the ideas introduced in Ref. [13] to multiparticle quantum systems. We provide a complete classification of a family of states of N-qubit systems. These states are characterized in terms of 2^{N-1} parameters and play the role of WS in such systems, since any arbitrary state can be depolarized to this form. We fully analyze the separability and distillability properties of this family, thereby generalizing the purification procedure introduced in Ref. [13] to multiqubit systems. This automatically provides us—as in the bipartite case—with sufficient conditions for arbitrary multiqubit states. Among other things, this allows us to give the necessary and sufficient separability and distillability conditions of mixtures of a maximally entangled state and the completely depolarized state [14,15]. Furthermore, we introduce a hierarchic classification of general N-qubit states with respect to their entanglement properties.

The paper is organized as follows. We start by briefly reviewing some of the present knowledge about distillability and entanglement of bipartite quantum systems in Sec. II. In the following we generalize the results of Ref. [13] to multiqubit systems. We start by giving a classification of arbitrary N-qubit systems in Sec. III. Then we introduce a family of states that can be obtained via depolarization from an arbitrary one in Sec. IV. Here we also investigate the separability and distillability properties of this family. Section V gives examples to illustrate the results obtained in the preceding sections. In particular, we analyze in detail the simplest cases of 3 and 4 qubit systems. In Sec. VI, we apply our results to the case where we have a maximally entangled state of N qubits mixed with the totally depolarized state. Finally, we conclude and summarize in Sec. VII.

II. BIPARTITE SYSTEMS AND PARTIAL TRANSPOSITION

Let us start out by briefly reviewing the separability and distillability properties of bipartite systems. A bipartite mixed state ρ is called separable if it can be prepared locally, i.e., it can be written as a convex combination of (unnormalized) product states

$$\rho = \sum_i |a_i\rangle_{\text{party1}}\langle a_i| \otimes |b_i\rangle_{\text{party2}}\langle b_i|. \tag{1}$$

A state is called distillable if one can create out of (infinitely) many copies of ρ one maximally entangled state, e.g., $|\Phi^+\rangle$. In practice, it is difficult to decide whether a given state is separable or distillable respectively. As shown by Peres [7] and the Horedecki [8–10], the partial transposition of a density operator turns out to provide a simple, sufficient criterion for the classification of bipartite systems. Given an operator X acting on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$, the partial transposition with respect to the first subsystem in the standard basis $\{|1\rangle,|2\rangle,\ldots,|d_1\rangle\}$, X^{T_A}, is defined as follows:

$$X^{T_A}\equiv\sum_{i,j=1}^{d_1}\langle i|X|j\rangle|j\rangle\langle i|. \tag{2}$$

Clearly, the partial transposition of the operator X is basis dependent, but the eigenvalues are not. We say that a self adjoint operator has positive partial transposition ($X^{T_A}\geqslant 0$)—positive partial transportation (PPT)—iff all eigenvalues of X^{T_A} are nonnegative. On the opposite, we say an operator has nonpositive partial transposition (NPPT) iff at least one eigenvalue is negative. Sometimes NPPT is also called ''negative partial transposition'' (NPT).

For bipartite two-level systems ($d_1=d_2=2$) it was shown that positive partial transposition (PPT) is a necessary and sufficient condition for separability [7,9] while negative partial transposition (NPT) is a necessary and sufficient condition for distillability [10]. For higher dimensional systems, however, the partial transposition only provides necessary conditions for separability [9] and it seems that it provides only a necessary condition for distillability [16,17]. In $\mathbb{C}^2\otimes\mathbb{C}^d(d\geqslant 2)$ systems we have that a sufficient condition for separability is that $\rho=\rho^{T_A}$ [18], while the negativity of the partial transposition already ensures distillability of those systems [16].

In the following, we generalize the notion of separability and distillability to multiqubit systems. It turns out that in order to characterize an important family of multiqubit mixed states, it is useful to consider bipartite splits of multiparticle systems and their corresponding partial transpositions. Since a bipartite split of a multiqubit system can be viewed as a state in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$, the partial transposition of the density operator ρ is well defined in this case.

III. MULTIQUBIT SYSTEMS

We will give a classification of general N-qubit systems in terms of the separability properties of their partitions. In particular, we consider k-partite splits (that are partitions dividing a N-partite systems into $k\leqslant N$ parties), which gives rise to a whole hierarchy of classes.

A. Separability with respect to certain splits

Let us start by generalizing the notion of separability to the case of multiparticle systems. We consider N parties, each holding a system with dimension d_i, i.e., $\mathcal{H}=\mathbb{C}^{d_1}\otimes\ldots\otimes\mathbb{C}^{d_N}$. We call ρ fully separable if it can be written as a convex combination of (unnormalized) product states, i.e.,

$$\rho=\sum_i |a_i\rangle_{\text{party}1}\langle a_i|\otimes|b_i\rangle_{\text{party}2}\langle b_i|\otimes\ldots\otimes|n_i\rangle_{\text{party}N}\langle n_i|. \tag{3}$$

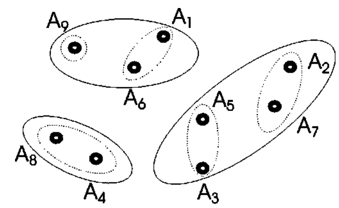

FIG. 1. Example of two partitions of a 9-qubit system into three sets [full line: $S_3=(A_1A_6A_9)$-$(A_2A_3A_5A_7)$-(A_4A_8)] and five sets [dotted line $-S_5=(A_1A_6)$-(A_9)-(A_2A_7)-$(A_3A_5)-(A_4A_8)$]. We have that S_5 is contained in S_3.

In the following, we will consider a system of N qubits, each held by one of the parties $A_1,A_2,\ldots,A_N$. In this case, $d_1=d_2=\cdots d_N=2$. Let us now consider a partition of the N-qubit system into $k\leqslant N$ sets, which we call a k-partite split of the system (see Fig. 1). That is we allow some of the parties to act together such that finally k parties remain. As a special case, we have 2-partite splits which we will also call bipartite splits. A state ρ is called k-separable with respect to this specific partition (or equivalently split) if it is fully separable in the sense that we consider ρ as a k-party system, i.e., as a state in $\mathcal{H}=\mathbb{C}^{d_1}\otimes\ldots\otimes\mathbb{C}^{d_k}$.

Considering all possible partitions (including all permutations) of the N-qubit system and determining the corresponding separability properties is sufficient to fully characterize the system in terms of its entanglement properties. However, the number of possible partitions grows rapidly with the number of parties involved (see Sec. III B 2), and it turns out that the information given by all these properties is redundant in some cases. We, thus, propose a hierarchic classification in terms of the separability properties with respect to the partitions, i.e., we consider all k-partite splits at level k of our classification. This turns out to be useful, since in some cases (see Sec. IV) the information of one level (in particular level 2) already implies all properties at the other levels. Furthermore, there are connections between the different levels, which can be used to reduce the effort to determine the full entanglement properties of the system. However, we learned in the case of 3-qubits that these connections are sometimes not obvious or are even counterintuitive. For example, we have that for a 3-qubit system separability with respect to all bipartite splits (i.e., partitions into two sets) are not sufficient to guarantee 3-separability (i.e., full separability when considering each system A_1, A_2, A_3 as a separate party) of the system [13,21].

B. Classification of arbitrary states ρ

In Ref. [13], the biseparability properties of the state ρ where used to classify those states completely, and all together (apart from permutations among the parties) 5 distinct classes were found (see Sec. V A for details). Here we generalize this classification to N-qubit systems, considering all possible partitions of the system.

1. Hierarchic classification

The basic idea of the hierarchic classification we propose here is to consider all possible k-partite splits of a N-partite system for all ($k \in \{N, N-1, \ldots, 2\}$) and determine for each split whether it is k-separable or not. For simplicity, we divide this procedure into levels, starting with $k=N$, continue with $k=N-1$, etc. until we reach $k=2$. This minimizes the necessary effort for a full classification, since, as mentioned in the previous section, there are connections between the different levels which will be explained in more detail in Sec. III B 3.

Level k of the characterization consists of the complete determination of the k-separability properties of the state ρ, i.e., considering all possible partitions into exactly k sets and determine whether the state is separable. At each level k, we have various classes, namely all possible combinations of k-separability and k-inseparability. If the number of k-partite splits is k_0, we have 2^{k_0} possible configurations at this level in principle.

However, the different levels of this structure are not independent of each other and thus some of the possible configurations are forbidden by the structure at higher/lower levels. We call each allowed configuration of the whole hierarchic classification a "class," since it corresponds to different physical properties. We have that permutations of the parties lead to different classes.

Note that all levels of this characterization are required to fully classify a state. It is not sufficient to give only the number of k-separable splits at each level and define classes in terms of this numbers, as done for $N=3$ in Ref. [13]. In this case, one obtains the remaining configurations by permuting the parties, while for $N>3$ this last property is no longer valid, i.e., one can have two physically different situations (not only up to permutations) corresponding to the same number of k-separable states at a certain level. This will be explained in more detail in Sec. V B.

In principle, it may turn out that some of the classes we give here are empty. In fact, for the family of states we are going to consider in the following, we have that k-separability is implied by the corresponding 2-separabilities, i.e., by the biseparability properties of all bipartite splits containing the k-partite split S_k in question. This means that these states are already fully classified by the structure at level 2 (biseparability properties). However, for $N=3$ examples for 3-inseparable states which are biseparable with respect to all bipartite splits are known [21], which makes it likely that similar examples (apart from the trivial generalization of those states ρ_B to N qubits by taking e.g., $|0\rangle\langle 0|^{\otimes N-3} \otimes \rho_B$) also exist for $N>3$.

2. *Partitions of N-qubits*

A partition of N, $\mathcal{L}$, is given by $\mathcal{L}_{\vec{r}} = \{1^{r_1} 2^{r_2} \ldots N^{r_N}\}$ with $\sum_{j=0}^{N} j r_j = N$, and the number of sets[1] $k = \sum r_i$. For example, $N=4$ and $\mathcal{L}_{21} = \{1^2, 2^1\}$ denotes all possible partitions of N into 3 sets such that one set consists of 2 parties, the other two sets consist of one party each. Note that we may have many partitions that correspond to the same number of sets, say k, which we called "k-partite splits."

[1]The number of sets is equivalent to the number of parties.

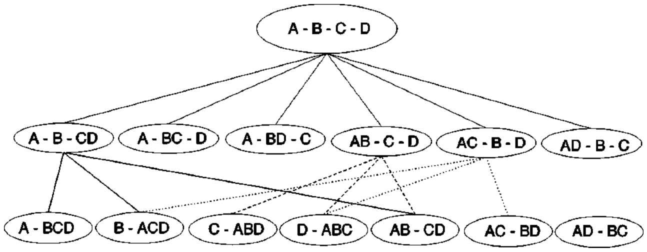

FIG. 2. Hierarchy structure for classification of 4-qubit systems. Note that not all connections at the lowest level are drawn.

Using the well-developed theory of partitions (see, e.g., Ref. [19]), one finds that the number of possible configurations (including all possible permutations among the parties) for a certain partition $\mathcal{L}_{\vec{r}}$ is given by

$$|\mathcal{L}_{\vec{r}}| = \frac{N!}{\prod r_j! \prod (j!)^{r_j}}. \tag{4}$$

The total number of partitions $\mathcal{L}_{\vec{r}}$ is given by the partition function $p(N)$, which grows rapidly with N, e.g., $p(10) = 42$, $p(50) = 204\,226$, $p(100) = 190\,569\,292$. A closed expression for $p(N)$ is known and can be found, e.g., in Ref. [19]. Using these relations, one can in principle obtain the number of possible k-partite splits of a N-qubit system.

3. *Contained splits and implications for classification*

Let $l<k$. We say a k-partite split S_k *belongs to* (equivalently *is contained in*) a l-partite split S_l if S_l can be obtained from S_k by joining some of the parties of S_k. For three parties ($N=3$) we have, for example, that the bipartite split $A_1-(A_2A_3)$ contains the 3-partite split $A_1-A_2-A_3$, since the split $A_1-(A_2A_3)$ can be obtained by joining the parties (A_2A_3). Note that in general we do not have a one-to-one correspondence in either direction. On one hand, each k-partite split is contained in various l-partite splits, while on the other hand a number of different k-partite splits may be contained in the same l-partite split (see also Fig. 2).

Furthermore, k-separability with respect to a certain k-partite split S_k implies l-separability with respect to all those l-partite splits which contain $S_k (l<k)$. However we learned in the case of three qubits [13] that 2-separability with respect to all possible 2-partite splits is not sufficient to guarantee the corresponding 3-separability in general. Thus, we have that l-separability with respect to all those l-partite splits, which contain a certain k-partite split S_k is a necessary, but not sufficient condition for k-separability with respect to S_k.

Let us apply this observation to our classification. We have that k-separability partly fixes the structure at lower levels $l<k$ (k-inseparability has no influence at lower levels), while l-inseparability fixes some properties at higher levels $k>l$ (l-separability only provides necessary conditions for separability at higher levels). In particular, k-separability with respect to a certain k-partite split S_k already implies l-separability of all l-partite splits containing

$S_k (l<k)$. On the other hand, l-inseparabilty with respect to a certain l-partite split S_l implies that all k-partite splits that belong to $S_l (k>l)$ are also k-inseparable. This means that once one finds a k-partite split S_k to be k-separable, one does not have to consider all l-partite splits containing S_k since they are automatically l-separable, which reduces the necessary effort to fully classify a state.

C. Distillability properties within a specific class

One can also consider the process of distillation (entanglement purification) and relate it to this classification. Let us consider a specific class, characterized by all their k-separability properties. A necessary condition for the distillation of a maximally entangled pair e.g., between A_i and A_j is that all those splits for which A_i and A_j belong to different parties are k-inseparable. In fact, it is sufficient to consider only the bipartite splits fulfilling this property, since this already implies the inseparability of all k-partite splits $(k>2)$ of this kind.

In a similar way, we find a necessary condition for the creation of a j-GHZ state, i.e., a GHZ state shared among j parties, e.g., $A_{\vec{i}} \equiv \{A_{i_0}, \ldots, A_{i_j}\}$: We consider all those bipartite splits where not all of the parties $A_{\vec{i}}$ are joint at one side. The inseparability of all those bipartite splits is a necessary condition for the creation of a j-GHZ state between the parties $A_{\vec{i}}$.

By investigating these necessary conditions for distillability, we immediately observe that there exists a huge number of classes that are inseparable at some level (and thus entangled), but cannot be distilled. Hence all these classes correspond to different kinds of bound entanglement. For example, we have that inseparability with respect to any k-partite split already implies that the state is entangled, but one can still have the 2-separability properties such that the necessary conditions for distillation of a pair between any two parties are not fulfilled. Sometimes this bound entanglement may be activated by allowing some additional entanglement between some subsystems. An example for this is given in Sec. V A.

IV. FAMILY OF STATES ρ_N

In the following, we show that any arbitrary N-qubit state ρ can be brought to a standard form ρ_N [13]. We also give a full classification of this family of states in terms of their separability and distillability properties.

A. Notation

We introduce the orthonormal GHZ-basis [20]

$$|\Psi_j^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}[|j\rangle|0\rangle \pm |(2^{N-1}-j-1)\rangle|1\rangle], \tag{5}$$

where $j=j_1 j_2 \ldots j_{N-1}$ is understood in binary notation. We have that $|j\rangle_{A_1 \ldots A_{N-1}}$ is the state of the first $(N-1)$ qubits. For example, for $N=5$ and $j=6$ this reads $|\Psi_6^{\pm}\rangle \equiv 1/\sqrt{2}(|0110\rangle_{A_1 \ldots A_4}|0\rangle_{A_5} \pm |1001\rangle_{A_1 \ldots A_4}|1\rangle_{A_5})$, since $(6 \equiv 0110)$ in binary notation. Each basis state is a GHZ state, and all basis elements are connected by N-local unitary operations. So $|\Psi_0^+\rangle = 1/\sqrt{2}(|0 \ldots 0\rangle + |1 \ldots 1\rangle)$ is only an arbitrary GHZ state, which can be selected by the choice of a local basis in $A_1 \ldots A_N$. We emphasis this here, since the states $|\Psi_0^{\pm}\rangle$ seem to play a special role in what follows. In the following, we consider the family of states

$$\rho_N = \sum_{\sigma=\pm} \lambda_0^{\sigma} |\Psi_0^{\sigma}\rangle\langle\Psi_0^{\sigma}| + \sum_{j=1}^{2^{(N-1)}-1} \lambda_j (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \tag{6}$$

which is the straightforward generalization of the family ρ_3 introduced in Ref. [13] to N qubits. Due to the normalization condition $\mathrm{tr}(\rho_N)=1$, we have that ρ_N is described by 2^{N-1} independent real parameters. The labeling is chosen such that $\Delta \equiv \lambda_0^+ - \lambda_0^- \geq 0$.

B. Depolarization to ρ_N

In this section, we are going to show that an arbitrary state ρ can be depolarized to the standard form (6) by a sequence of N-local operations while keeping the values of $\lambda_0^{\pm} \equiv \langle\Psi_0^{\pm}|\rho|\Psi_0^{\pm}\rangle$ and $2\lambda_j \equiv \langle\Psi_j^+|\rho|\Psi_j^+\rangle + \langle\Psi_j^-|\rho|\Psi_j^-\rangle$ unchanged. Similarly as in the three-qubit case, this implies that the necessary and sufficient conditions for distillability and nonseparability found for ρ_N automatically translate into sufficient conditions for arbitrary states.

We will now explicitly construct the required sequence of N-local operations to obtain the desired depolarization procedure. By mixing we understand in the following that a certain operation is (randomly) performed with $p=\frac{1}{2}$, while with $p=\frac{1}{2}$ no operation is performed. The following N rounds of mixing operations are sufficient to make ρ diagonal in the basis (5) without changing the diagonal coefficients: In the first round, we apply simultaneous spin flips at all N locations. The result of this mixing operation is that all off-diagonal elements of the form $|\Psi_k^+\rangle\langle\Psi_l^-|$ and $|\Psi_k^-\rangle\langle\Psi_l^+|$ are eliminated. The remaining $(N-1)$ rounds consist of applying σ_z to particles A_k and A_N (and the identity to all other particles), where k runs from 1 to $(N-1)$. The effect of the kth operation is the following: a state $|\Psi_j^{\pm}\rangle$ picks up a minus sign if j, written in binary notation, has a ''1'' at the kth position and remains unchanged if it has a ''0'' there. Since the corresponding j and i of two different basis states $|\Psi_j^{\pm}\rangle$ and $|\Psi_i^{\pm}\rangle$ differ in at least one digit, this implies that in at least one mixing round one state, say $|\Psi_j^{\pm}\rangle$ will pick up a minus sign while the other state, say $|\Psi_i^{\pm}\rangle$ will remain unchanged. This ensures that all off-diagonal elements of the form $|\Psi_j^{\pm}\rangle\langle\Psi_i^{\pm}|$ are eliminated in this mixing round. Finally, we have that after all N mixing rounds, ρ is diagonal in the basis (5).

It remains to depolarize the subspaces spanned by $\{|\Psi_j^{\pm}\rangle\}$ for each $j>0$. This can be accomplished by using random operations that change $|0\rangle_\alpha \rightarrow e^{i\phi_\alpha}|0\rangle_\alpha$ $(\alpha=A_1, \ldots, A_N)$ with $\Sigma_k \phi_{A_k} = 2\pi$ (this condition ensures that $\lambda_0^{\pm}$ remains unchanged). This implies that an arbitrary state ρ can be brought to the standard form ρ_N by a sequence of N-local operations.

One can readily check that the partial transpose of ρ_N with respect to the bipartite split $(A_1 \ldots A_{N-1})-A_N$ is positive iff $\Delta \equiv \lambda_0^+ - \lambda_0^- \leqslant 2\lambda_{2^{N-1}-1}$. Similar conditions hold for each possible bipartite split, i.e., ρ_N has PPT with respect to a certain bipartite split iff $\Delta \leqslant 2\lambda_k$ for a specific (unique) k corresponding to this bipartite split. To determine the corresponding k, let us consider a bipartite split where l qubits $A_{\vec{k}} \equiv \{A_{k_1}, \ldots A_{k_l}\}$ are jointly located at one side, while the remaining $N-l$ qubits are located at the other side. Without loss of generality we can assume that $A_N \notin A_{\vec{k}}$. In this case, the corresponding λ_k is given by k, which, written in binary notation, has ones at the positions $(k_1, \ldots, k_l)$ and zeros at all other positions and only the highest $(N-1)$ bits are considered (the lowest one, k_N, is allways zero because we assumed that $A_N \notin A_{\vec{k}}$). For instance we consider $N=6$ and the bipartite split $S_2=(A_1A_4A_5)-(A_2A_3A_6)$. We have that the corresponding k is given by $k=100\,110$, where we only have to consider the highest 5 bits. Thus, $k=10\,011 \equiv 19$ and we have that the state ρ_N has PPT with respect to the bipartite split S_2 iff $\Delta \leqslant 2\lambda_{19}$

C. Separability of ρ_N

Let us enumerate the separability properties of the states (6). Then we obtain

(i) We consider a specific k-partite split S_k of ρ_N. Iff all bipartite splits that contain S_k have PPT, then ρ_N is k-separable with respect to this specific k-partite split.

In order to prove (i), we find it useful to consider first two special cases of (i) in order to illustrate the underlying ideas. These special cases are the following:

(ii) We consider a specific bipartite split, where we have that l qubits $A_{\vec{k}} \equiv \{A_{k_1}, \ldots A_{k_l}\}$ are jointly located at one side, while the remaining $N-l$ qubits are located at the other side. Iff we have that the partial transpose corresponding to this bipartite split is positive, i.e., $\rho_N^{T_{A_{\vec{k}}}} \equiv \rho_N^{T_{A_{k_1}} \cdots T_{A_{k_l}}} \geqslant 0$ then ρ_N is separable with respect to this bipartite split, i.e., it can be written in the form

$$\rho_N = \sum_i |\chi_i\rangle_{A_{\vec{k}}}\langle\chi_i| \otimes |\varphi_i\rangle_{\mathrm{rest}}\langle\varphi_i|. \tag{7}$$

(iii) We consider all possible $2^{N-1}-1$ bipartite splits of a N-qubit system. Iff for each of those splits the corresponding partial transposition is positive, then ρ_N is fully separable, i.e., ρ_N is N-separable.

These statements are illustrated for the simplest cases of a 3 and 4-qubit system in Sec. V. From (i) follows that k-separability with respect to a certain k-partite split S_k of the states ρ_N is implied by the 2-separability properties of the bipartite splits containing S_k. Thus, the family ρ_N is completely characterized by its 2-separability properties, which already determine the hierarchic structure proposed in Sec. III B.

In the remainder of this section, we are going to prove the statements (i)–(iii). Let us start by proving (ii). The basic idea of the proof is to define a state $\hat{\rho}$, which we show to be 2-separable with respect to the bipartite split in question and which can be depolarized to ρ_N. Since a separable state is converted into a separable one by depolarization (which is a N-local process), this automatically implies the 2-separability of ρ_N. We have that ρ_N has positive partial transposition with respect to the bipartite split $A_{\vec{k}}$ if $\Delta \leqslant 2\lambda_k$ for k corresponding to $\vec{k}$. We define

$$\hat{\rho} = \rho_N + \frac{\Delta}{2}(|\Psi_k^+\rangle\langle\Psi_k^+| - |\Psi_k^-\rangle\langle\Psi_k^-|). \tag{8}$$

The state $\hat{\rho}$ is positive since $\Delta \leqslant 2\lambda_k$ and has the property $\hat{\rho} = \hat{\rho}^{T_{A_{\vec{k}}}}$. For a bipartite split of the form (one qubit)-rest, this already implies the separability of $\hat{\rho}$, since it has been shown in [18] that all states in $\mathbb{C}^2 \otimes \mathbb{C}^N$, which fulfill $\tilde{\rho}^{T_A} = \tilde{\rho}$ are separable. For all the other splits we show the separability directly. We rewrite $\hat{\rho}$ as follows:

$$\begin{aligned}\hat{\rho} = {} & \Delta(|\Psi_0^+\rangle\langle\Psi_0^+| + |\Psi_k^+\rangle\langle\Psi_k^+|) + \left(\lambda_k - \frac{\Delta}{2}\right)(|\Psi_k^-\rangle\langle\Psi_k^-| \\ & + |\Psi_k^+\rangle\langle\Psi_k^+|) + \lambda_0^-(|\Psi_0^-\rangle\langle\Psi_0^-| + |\Psi_0^+\rangle\langle\Psi_0^+|) \\ & + \sum_{j=1, j\neq k}^{2^{(N-1)}-1} \lambda_j(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|).\end{aligned} \tag{9}$$

We have that all prefactors are positive (since $\Delta \leqslant 2\lambda_k$). The terms in lines 2-4 are completely separable, which can be seen by using that $(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|) = |j0\rangle\langle j0| + |(2^{N-1}-j-1)1\rangle\langle(2^{N-1}-j-1)1|$. The term in line 1 is biseparable with respect to the bipartite split $\vec{k}$. To see this, let us rewrite the basis states as follows:

$$\begin{aligned}|\Psi_0^+\rangle &= \frac{1}{\sqrt{2}}(|0\ldots0\rangle_{A_{\vec{k}}}|0\ldots0\rangle_{\mathrm{rest}} \\ &\quad + |1\ldots1\rangle_{A_{\vec{k}}}|1\ldots1\rangle_{\mathrm{rest}}), \\ |\Psi_k^+\rangle &= \frac{1}{\sqrt{2}}(|1\ldots1\rangle_{A_{\vec{k}}}|0\ldots0\rangle_{\mathrm{rest}} \\ &\quad + |0\ldots0\rangle_{A_{\vec{k}}}|1\ldots1\rangle_{\mathrm{rest}}).\end{aligned} \tag{10}$$

We define $|\vec{\pm}\rangle = 1/\sqrt{2}(|0\ldots0\rangle \pm |1\ldots1\rangle)$. It is now straightforward to check that line one of Eq. (9) can be written as $\Delta(|\vec{+}\rangle_{A_{\vec{k}}}\langle\vec{+}| \otimes |\vec{+}\rangle_{\mathrm{rest}}\langle\vec{+}| + |\vec{-}\rangle_{A_{\vec{k}}}\langle\vec{-}| \otimes |\vec{-}\rangle_{\mathrm{rest}}\langle\vec{-}|)$, which is clearly biseparable with respect to the bipartite split $\vec{k}$ and concludes the proof in one direction. If we consider on the other hand that ρ_N is biseparable with respect to the bipartite split $\vec{k}$, it follows trivially that it also has PPT corresponding to this bipartite split, since positive partial transposition is a necessary condition for separability [7].

To prove the third statement (iii), we show that if ρ_N has PPT with respect to all possible bipartite splits, then ρ_N is

N-separable (note again that the opposite is trivially true). This condition implies that $\Delta/2 \leqslant \lambda_j$ for all j. Again, the idea is to define an operator $\tilde{\rho}$ which can be depolarized into the form ρ_N by using local operations and that is fully separable. Let $\tilde{\rho}$ be a state of the form (6) with coefficients $\tilde{\lambda}_0^{\pm} \equiv \lambda_0^{\pm}$, and $\tilde{\lambda}_k^{\pm} \equiv \lambda_k \pm \Delta/2$ $[k=1,\ldots,(2^{N-1}-1)]$. Clearly, $\tilde{\rho}$ can be depolarized into ρ_N. We now rewrite $\tilde{\rho}$ as follows:

$$\tilde{\rho} = \frac{1}{2} \sum_{k=0}^{2^{N-1}-1} (\tilde{\lambda}_k^+ + \tilde{\lambda}_k^- - \Delta)(|\Psi_k^+\rangle\langle\Psi_k^+| + |\Psi_k^-\rangle\langle\Psi_k^-|) + \Delta \sum_{k=0}^{2^{N-1}-1} |\Psi_k^+\rangle\langle\Psi_k^+|. \tag{11}$$

Since all possible partial transposes are positive, we have that all coefficients in (11) are positive. The first term in (11) can be written as $\Sigma_{k=0}^{2^{N}-1}(\tilde{\lambda}_k^+ + \tilde{\lambda}_k^- - \Delta)/2(|k,0\rangle\langle k,0| + |(2^{N-1}-k-1),0\rangle\langle(2^{N-1}-k-1),0|)$ and is thus fully separable. It remains to show that the second term in Eq. (11) is also N-separable. Let us first define the states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. To show the separability of the second term, we write it as $\Sigma_{j=0}^{2^{N}-1}|\phi_j\rangle\langle\phi_j|$ where $|\phi_j\rangle$ are all the states of the form $|\sigma_1\sigma_2,\ldots\sigma_N\rangle$ with $\sigma_i = \pm$, which have an even number of minuses. All the states $|\phi_j\rangle$ are fully separable, which concludes the proof.

Now we are ready to prove the first statement (i). The basic idea of the proof is very similar to the one used in the previous proofs: We define a state ρ' which can be depolarized to ρ_N and we show that ρ' is k-separable. We have that a number of bipartite splits have PPT. To a specific bipartite splits corresponds the relation $\Delta \leqslant 2\lambda_{i_0}$, which is the condition that this specific bipartite split has PPT. Thus, we have that $\Delta \leqslant 2\lambda_i$, where $i \in \{i_0,\ldots,i_l\} \equiv \vec{i}$, and each λ_i corresponds to a bipartite split that does not further divide systems that were joint for the k-partite split we consider. Let ρ' be a state of the form (6) with coefficients $\lambda'^{\pm}_0 \equiv \lambda_0^{\pm}$, $\lambda'^{\pm}_i \equiv \lambda_i \pm \Delta/2$ for $i \in \vec{i}$ and $\lambda'^{\pm}_k \equiv \lambda_k$ for $k \notin \vec{i}$. Clearly, ρ' can be depolarized to ρ_N. Similarly as in the previous proofs, we rewrite ρ' as follows:

$$\begin{aligned}\rho' = {} & \Delta\Big(|\Psi_0^+\rangle\langle\Psi_0^+| + \sum_{i\in\vec{i}} |\Psi_i^+\rangle\langle\Psi_i^+|\Big) \\ & + \sum_{i\in\vec{i}} \left(\lambda_i - \frac{\Delta}{2}\right)(|\Psi_i^-\rangle\langle\Psi_i^-| + |\Psi_i^+\rangle\langle\Psi_i^+|) \\ & + \sum_{k\notin\vec{i}} \lambda_k(|\Psi_k^+\rangle\langle\Psi_k^+| + |\Psi_k^-\rangle\langle\Psi_k^-|) \\ & + \lambda_0^-(|\Psi_0^-\rangle\langle\Psi_0^-| + |\Psi_0^+\rangle\langle\Psi_0^+|). \end{aligned} \tag{12}$$

All coefficients in Eq. (12) are positive, and lines (2–4) are fully separable. It remains to show the k-separability of line 1 of Eq. (12). To see this, we define the states $|\vec{\pm}\rangle_{\text{party}l} = 1/\sqrt{2}(|0\ldots0\rangle_{\text{party}l} \pm |1\ldots1\rangle_{\text{party}l})$, and the states $|\Phi_j\rangle \equiv |\sigma_1\rangle_{\text{party}1} \otimes \ldots \otimes |\sigma_k\rangle_{\text{party}k}$ with $\sigma_i = \pm$ and the number of minuses is even. It is now straightforward to check that line 1 of Eq. (12) can be written as $\Delta\Sigma_j|\Phi_j\rangle\langle\Phi_j|$ and is thus k-separable with respect to the k-partite split we consider, and concludes the proof in one direction.

If we consider on the other hand that ρ_N is k-separable with respect to a specific k-partite split S_k, it follows that it is also biseparable with respect to all bipartite splits that contain S_k, since any of those bipartite splits corresponds to joining systems which were divided for the k-partite split. But the positivity of the partial transposition is a necessary condition for biseparability corresponding to a certain bipartite split [9], from which follows that positivity of all bipartite splits we consider is also a necessary condition for k-separability. So again the conditions we found are necessary and sufficient.

D. Distillability of ρ_N

We will turn now to analyzing the distillability properties of ρ_N:

(i) We consider all possible bipartite splits of the N qubits where the particles A_i and A_k belong to different parties. Iff all such splits have negative partial transposition, then a maximally entangled pair between particle A_i and A_k can be distilled.

(ii) We consider the k parties $A_{\vec{i}} \equiv \{A_{i_0},\ldots,A_{i_k}\}$ ($k \leqslant N$) and consider all those bipartite splits where not all of the parties $A_{\vec{i}}$ are joint at one side. Iff all those splits have negative partial transposition, then a k-GHZ state (i.e., a GHZ state shared between k parties) between the parties $A_{\vec{i}}$ can be distilled.

To show (i) and without loss of generality we take $i=N$ and $k=N-1$. In that case, the condition we impose on the partial transpositions is equivalent to require that $\Delta/2 > \lambda_j$ with j odd [note that with the notation we are using, the state of the $N-1$ qubit determines the parity of the states $|j\rangle$ in Eq. (5)]. In order to show the distillability of a maximally entangled state between A_{N-1} and A_N, it is sufficient to show that a pair with fidelity $F > 0.5$ (overlap with the maximally entangled state $|\Phi^+\rangle$) between those two parties can be created [1]. If we project all the qubits except the ones at A_N and A_{N-1} onto the state $|+\rangle$ we see that the resulting state obtained from ρ_N has $F > 0.5$ and can thus be distilled to a maximally entangled state between A_N and A_{N-1} iff

$$\Delta/2 > \sum_{j\,\text{odd}} \lambda_j. \tag{13}$$

Even though we have that $\Delta/2 > \lambda_j$ for all j odd, the condition (13) might not be fulfilled. In this case we use the following purification procedure: The idea is to combine M systems in the same state ρ_N, perform a measurement and obtain one system with the same form (6) but in which the new Δ is exponentially amplified with respect to λ_k, k odd. In order to do that, let us define the operator

$$P = |00\ldots00\rangle\langle00\ldots00| + |10\ldots00\rangle\langle11\ldots11|, \tag{14}$$

which acts on M qubits. Now we proceed as follows: We take M systems, and apply the operator P in all N locations. This corresponds to measuring a POVM that contains P obtaining the outcome associated to P. The resulting state $P^{\otimes N}\rho_N^{\otimes M}(P^\dagger)^{\otimes N}$ has the first system in an (unnormalized) density operator of the form (6) but with new coefficients $\tilde{\Delta}$ and $\tilde{\lambda}_k$. In order to calculate these new coefficients, we need the following observations: First, the operator $\rho_N^{\otimes M}$ is diagonal in the basis $\{|\chi_{k_0 \dots k_M}^{\sigma_1 \dots \sigma_M}\rangle\}$ with coefficients $\lambda_{k_1}^{\sigma_1} \dots \lambda_{k_M}^{\sigma_M}$ where

$$|\chi_{k_0 \dots k_M}^{\sigma_1 \dots \sigma_M}\rangle = |\Psi_{k_0}^{\sigma_0}\rangle \otimes \dots \otimes |\Psi_{k_M}^{\sigma_M}\rangle, \tag{15}$$

and $k_j \in \{0, \dots 2^{N-1}-1\}, \sigma_j = \pm$. Second, we need the action of the operator $P^{\otimes N}$ on the basis states (15). We find

$$P^{\otimes N}|\chi_{k_0 \dots k_M}^{\sigma_1 \dots \sigma_M}\rangle = \delta_{k_0 \dots k_M}|\Psi_{k_0}^{\sigma}\rangle_{\text{system 1}}|0 \dots 0\rangle_{\text{rest}}, \tag{16}$$

where $\sigma = +$ if the number of minuses in $\{\sigma_1 \dots \sigma_M\}$ is even, and $\sigma = -$ otherwise. Note also that we only have a contribution if $k_0 = k_1 = \dots = k_M$. Using these results, it is now straightforward to check that the first system of $P^{\otimes N}\rho_N^{\otimes M}(P^\dagger)^{\otimes N}$ is an (unnormalized) density operator of the form (6) with new coefficients

$$\tilde{\Delta}/2 = (\Delta/2)^M; \tilde{\lambda}_k = \lambda_k^M. \tag{17}$$

Given that $\Delta/2 > \lambda_k$, k odd, for sufficiently large M we have that condition (13) is fulfilled, i.e., that after the projection of all systems except A_{N-1} and A_N on the state $|+\rangle$, the resulting state has $F > 0.5$ and is thus distillable, which concludes the proof in one direction. On the other hand, the condition we impose for distillability is also necessary. Having a maximally entangled pair between the parties A_{N-1} and A_N implies that all bipartite splits in question have NPT. Since local operations keep the positivity of the partial transposition [10], we must start with NPT of all bipartite splits in question.

To prove (ii), we just have to recognize that the condition we impose guarantees that maximally entangled pairs between any two parties within $A_{\vec{i}}$ can be distilled. This is clearly sufficient to create a GHZ state among those parties, e.g., by means of teleportation [creating the GHZ state locally at A_{i_0} and teleporting the $(k-1)$ qubits to the parties $\{A_{i_1}, \dots, A_{i_k}\}$ using maximally entangled pairs created among A_{i_0} and A_j with $j \in \{i_1, \dots i_k\}$]. Note also the condition we impose is also necessary, since a GHZ state allows us to create maximally entangled pairs among all parties involved, which implies that all bipartite splits in question have NPT and thus have to have NPT at the beginning, since one cannot convert a state from PPT to NPT by means of local operations [10].

V. EXAMPLES

A. Three-qubit systems

In this section, we investigate the simplest case of three qubits, each hold by one of the parties A, B, or C.

1. Classification

In order to perform the classification proposed in Sec. III B, we consider the 3-partite split of the system as well as all possible bipartite splits, where all together three such splits exist. Each corresponds to having one system (e.g., A) on one side and the two other systems (e.g., B and C) on the other side. In other words, for this specific bipartite split of our three-qubit system we allow the parties B and C to act together, i.e., $\mathcal{H} = \mathbb{C}_A^2 \otimes \mathbb{C}_{BC}^4$. Thus each of these bipartite splits will give us an upper limit of what can be done by three-local operations on the system. To perform the classification, we have to consider the separability properties of the 3-partite and bipartite splits. In particular, whether they can be written in one or more of the following forms:

$$\rho = \sum_i |a_i\rangle_A\langle a_i| \otimes |b_i\rangle_B\langle b_i| \otimes |c_i\rangle_C\langle c_i| \tag{18a}$$

$$\rho = \sum_i |a_i\rangle_A\langle a_i| \otimes |\varphi_i\rangle_{BC}\langle \varphi_i| \tag{18b}$$

$$\rho = \sum_i |b_i\rangle_B\langle b_i| \otimes |\varphi_i\rangle_{AC}\langle \varphi_i| \tag{18c}$$

$$\rho = \sum_i |c_i\rangle_C\langle c_i| \otimes |\varphi_i\rangle_{AB}\langle \varphi_i|. \tag{18d}$$

Here, $|a_i\rangle$, $|b_i\rangle$, and $|c_i\rangle$ are (unnormalized) states of systems A, B, and C, respectively, and $|\varphi_i\rangle$ are states of two systems. We call a state biseparable with respect to a certain bipartite split if it is separable with respect to this split, e.g., a state is biseparable with respect to the bipartite split A-(BC) if it can be written in the form (18b). Similary, a state is called triseparable (3-separable) if it is separable with respect to the split A-B-C, i.e., can be written in the form (18a).

At the top level of the classification (level 3), we consider the tripartite split A-B-C and determine whether the state ρ is 3-separable or not. At the second level (level 2), one considers all possible bipartite splits $[A$-(BC),B-(AC), C-$(AB)]$ and determines whether the state ρ can be written in one or more of the forms (18b),(18c),(18d). At this level of the classification, one has $2^3 = 8$ different possibilities, each corresponding to a physically different situation. For arbitrary three-qubit systes one thus finds the following complete set of 9 disjoint classes.

Class 1—Fully inseparable states. States that cannot be written in any of the above forms (18). An example is the GHZ state [20] $|\Psi_0^+\rangle$.

Classes 2.1, 2.2, 2.3—1-qubit biseparable states. Class 2.1: biseparable states with respect to qubit A are states that are separable in A-(BC), but nonseparable otherwise. That is, states that can be written in the form (18b) but not as Eqs. (18c) or (18d). An example is the state $|0\rangle_A \otimes |\Phi^+\rangle_{BC}$, where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a maximally entangled state of two qubits. Similarly, class 2.2 and 2.3 correspond to biseparable states with respect to qubit B and C respectively.

Classes 3.1, 3.2, 3.3—2-qubit biseparable states: Class 3.1: biseparable states with respect to qubits A and B are states that are separable in A-(BC) and B-(AC), but nonseparable in C-(AB). That is, states that can be written in the forms (18b) and (18c) but not as Eq. (18d). For examples, see below. Similary, classes 3.2 (3.3) are biseparable with respect to the qubits A and $C(B$ and $C)$.

Class 4—3-qubit biseparable states: Those are states that are separable in A-(BC), B-(AC), and C-(AB) (i.e., separable with respect to each bipartite split), but which are not completely separable, i.e., cannot be written as Eq. (18a). For an example, see Ref. [21].

Class 5—Fully separable states: These are states that can be written in the form (18a) and are thus also separable with respect to each bipartite split. A trivial example is a product state $|1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C$.

Note that the classes 2.1, 2.2, 2.3 (respectively 3.1, 3.2, 3.3) were identified in Ref. [13], since they can be obtained from each other by permuting the parties. In this case, 5 distinct classes remain.

2. Family ρ_3

Let us now concentrate on the family of three-qubit states ρ_3 (6). This family is characterized by 4 parameters, $\{\Delta \equiv \lambda_0^+ - \lambda_0^-, \lambda_1, \lambda_2, \lambda_3\}$, and we have that any state can be depolarized to this standard form. The GHZ basis (5) reads in this case

$$|\Psi_j^\pm\rangle \equiv \frac{1}{\sqrt{2}}[|j\rangle_{AB}|0\rangle_C \pm |(3-j)\rangle_{AB}|1\rangle_C], \tag{19}$$

where $|j\rangle_{AB} \equiv |j_1\rangle_A |j_2\rangle_B$ with $j = j_1 j_2$ in binary notation. For example, $|\Psi_0^\pm\rangle = 1/\sqrt{2}(|000\rangle \pm |111\rangle)$ are standard GHZ states, as well as $|\Psi_3^\pm\rangle = 1/\sqrt{2}(|110\rangle \pm |001\rangle)$ $(3 \equiv 11$ in binary notation). Note that all 8 basis states are connected by 3-local unitary operations, i.e., each basis state is a maximally entangled GHZ state. Due to the fact that the local bases in A, B, and C can be chosen arbitrarily, none of the basis states has any preferences.

We will now investigate the separability and distillability properties of ρ_3 and give a full classification in terms of the classes introduced above. It turns out that using the partial transpose criterion for each bipartite split characterizes the state ρ_3 completely. The conditions under which the operator ρ_3 has positive partial transpose with respect to each qubit are as follows

$$\begin{aligned} \rho_3^{T_A} \geq 0 &\quad \text{iff} \quad \Delta \leq 2\lambda_2, \\ \rho_3^{T_B} \geq 0 &\quad \text{iff} \quad \Delta \leq 2\lambda_1, \\ \rho_3^{T_C} \geq 0 &\quad \text{iff} \quad \Delta \leq 2\lambda_3. \end{aligned} \tag{20}$$

TABLE I. Separability and distillability classification of ρ_3.

Positive Operators	Class	Distillability
None	1	(GHZ) $|\Psi_0^+\rangle_{ABC}$
$\rho_3^{T_A}$	2.1	(Pair) $|\Phi^+\rangle_{BC}$
$\rho_3^{T_A}, \rho_3^{T_B}$	3.1	Activate with $|\Phi^+\rangle_{AB}$
All	5	

Recall that each of these conditions correspond to a (virtually) bipartite split of the system. Investigating, e.g., $\rho_3^{T_A}$, we actually have in mind a bipartite split of the system into A on one side and BC on the other side. From these conditions we also see that in general no further depolarization that keeps the form of ρ_3 is possible (except the depolarization toward the completely depolarized state, which can always be done trivially). We show this by giving a counterexample. Imagine we would like to depolarize the subspaces spanned by $\{|\Psi_1^\pm\rangle, |\Psi_2^\pm\rangle\}$ and thereby equalize the coefficients λ_1 and λ_2. For certain values of the parameters, this would imply that the state after this depolarization has negative partial transposition with respect to one party, while it started with positive partial transposition. Since one cannot change the positivity of the partial transpose by local operations [22], this further depolarization is impossible in general. [e.g., ρ_3 with $\lambda_0^+ = \frac{2}{3}, \lambda_2 = \frac{1}{3}$, all other parameter 0 has $\rho_3^{T_A} \geq 0$, but would have negative partial transposition with respect to all bipartite splits after the (imaginary) further depolarization in question].

3. Separability of ρ_3

Let us specialize the theorems about separability obtained in Sec. IV C to $N=3$. In this case, we do not need the general theorem (i), but only give examples for the statements (ii) and (iii):

(ii) ρ_3 is separable with respect to the bipartite split A-(BC), i.e., it can be written in the form (18b) iff $\rho_3^{T_A} \geq 0$ [and analogously for Eqs. (18c) and (18d) with $\rho_3^{T_B} \geq 0$ and $\rho_3^{T_C} \geq 0$, respectively].

(iii) ρ_3 is completely separable, i.e., it can be written as Eq. (18a) iff $\rho_3^{T_A}, \rho_3^{T_B}, \rho_3^{T_C} \geq 0$. Note that these are *iff* statements and thus provide a full characterization of ρ_3 in terms of the separability properties. The resulting classification is summarized in Table I. Here, we have that 3-qubit biseparability implies fully separability (tri-separability), while in general this is not necessarily the case (otherwise class 4 would be empty). We have the the family ρ_3 is completely characterized by its biseparability properties, so only the structure at level 2 is necessary for classification.

Furthermore, this also provides us with sufficient conditions for non-separability for arbitrary states ρ. Namely if there exist a basis (19) such that the corresponding state ρ_3 after depolarization has, e.g., the property that $\rho_3^{T_A}$ is negative, this implies that ρ is nonseparable in A-(BC). Note also that no conclusion can be drawn about the separability properties of ρ given PPT of the depolarized state ρ_3, since the depolarization process might convert a nonseparable state ρ into a separable state ρ_3.

4. Distillability of ρ_3

We now state the distillability properties of ρ_3:

(i) One can distill a maximally entangled state $|\Phi^+\rangle_{\alpha\beta}$ between α and β iff both $\rho_3^{T_\alpha},\rho_3^{T_\beta}$ have negative partial transposition.

(ii) Iff all three partial transposes are negative, we can distill a GHZ state (since we can distill an entangled state between A and B and another between A and C and then connect them to produce a GHZ state [23]).

(iii) If we have that $\rho_3^{T_C}$ is negative but $\rho_3^{T_A},\rho_3^{T_B}\geqslant 0$ (i.e., have PPT) and we have maximally entangled states between A and B at our disposal, then we can *activate* the entanglement between ABC and create a GHZ state.

Note that (iii) is an example for the activation of bound entanglement. In this example, the state is inseparable with respect to the bipartite split C-(AB) and thus entangled. However, no entanglement between any two subsystems can be created, since we have that the PPT of A and B implies the separability of A-(BC) and B-(AC) [see (ii) in Sec. V A 3). However entanglement between A and B is sufficient to allow the creation of a GHZ state. One can show this by noting that the singlets allow to teleport states between locations A and B. Thus, for all practical purposes we can consider a pair of qubits AB as a four-level system in which we can perform arbitrary operations. The situation is equivalent to that in which one has 2-level systems entangled to 4-level systems such that the density operator describing one pair acts on $\mathbb{C}^2\otimes\mathbb{C}^4$ and has a negative partial transpose. It can be easily shown [16] that in systems $2\times N$ negative partial transpose is a necessary and sufficient condition for distillation, and one can thus distill arbitrary states. Using again teleportation, one can end up with a GHZ state shared by A, B, and C.

Again, these results also provide us with sufficient conditions for distillability of arbitrary states ρ. From (i) follows: If there exist a basis (19) such that the corresponding state ρ_3 after depolarization has, e.g., the property that $\rho_3^{T_A},\rho_3^{T_B}$ are negative, this implies that a maximally entangled pair between A and B can be distilled from ρ. Note here that to make this condition for distillability necessary *and* sufficient for arbitrary states ρ, one should find a depolarization procedure (may be assisted by some appropriate local filtering operations) such that the NPT property is maintained. That is, the corresponding state ρ_3 after depolarization should still have $\rho_3^{T_A},\rho_3^{T_B}$ negative.

Note also that a sufficient condition to distill a GHZ-state from ρ is that there exist two different bases (19) such the corresponding states ρ_3 after depolarization only have two of the partial transposes kept negative, where for the second basis one has to differ from the first one. For example, basis 1 allows us to keep A and B each having NPT (and thus to distill a pair between A and B), while basis 2 keeps B and C having NPT. These together ensure that a GHZ state can be created. Clearly, one has to start with ρ which has all three partial transposes negative. Again, one may try to make this condition necessary and sufficient by introducing a (filtering assisted) depolarization procedure such that at least two of the partial transposes are kept negative for a certain choice of basis.

B. Four-qubit system

Here, we consider the special case of a 4-partite system in order to illustrate the theorems about separability and distillability obtained in the previous sections. For convenience, let us call the parties *A*, *B*, *C*, and *D* instead of $A_1, \ldots, A_4$.

1. Classification

We start by illustrating the classification for general 4-qubit systems. At the top level of the structure is the 4-separability, that is the question whether the state ρ is separable with respect to the 4-partite split $(4)A$-B-C-D, i.e., fully separable. At the second level, we have to consider 6 different 3-partite splits of the system: (3a) A-B-(CD), (3b) A-(BC)-D, (3c) A-(BD)-C, (3d) (AB)-C-D, (3e) (AC)-B-D and (3f) (AD)-B-C. At the third level, all together 7 different 2-partite splits exist, namely (2a) A-(BCD), (2b) B-(ACD), (2c) C-(ABD), (2d) D-(ABC), (2e) (AB)-(CD), (2f) (AC)-(BD) and (2g) (AD)-(BC).

We have for instance that the 3-partite split (3a) is contained in the 2-partite splits (2a), (2b), and (2e). All other bipartite splits cannot be obtained from (3a) by joining some of the parties, since they would divide the system (CD) along different parties, and thus the tripartite split (3a) does not belong to them.

The classification thus takes place as follows (see also Fig. 2): At the top level (level 4), one has to decide whether the state is 4-separable or not. In the case it is 4-separable, it automatically follows that it is also 3- and 2-separable with respect to all possible 3- or 2-partite splits. If it is 4-inseparable, one has to investigate the various kinds of 3-separability at level 3, where one can have all possible combinations of the 6 kinds (3a)–(3f) of 3-separability and and 3-inseparability. We have 2^6 different configurations at this level. At the next level of the classification (level 2), one investigates all possible bipartite splits (2a)–(2g) closer. One finds 2^7 different configurations at this level. Without taking the connections between different levels into account (which reduces the total number of allowed configurations), we have all together $(2)(2^6)(2^7)=16\,384$ possible combinations of 4-, 3-, and 2-separability and inseparability with respect to all possible partitions.

The structure at level 2 is partly determined by the structure at level 3 and vice versa. If, e.g., the state is 3-separable with respect to the 3-partite split (3a), it follows that the bipartite splits (2a), (2b), and (2e) at level 2 which contain (3a) are also 2-separable. In the case where a state is 3-separable with respect to the splits (3a), (3b), and (3c), it even follows that the state is 2-separable with respect to all possible bipartite splits. Although it still can be 3-inseparable with respect to the 3-partite splits (3d), (3e), or (3f) in principle, the underlying structure at level 2 is already completely determined by the structure at level 3. 3-inseparability with respect to a specific 3-partite split S_3, on the other hand, still allows all combinations of 2-separability and 2-inseparability within the bipartite splits at level 2 which contain S_3. Conversely, 2-inseparabilty with respect to the bipartite split (2a) implies 3-inseparability with respect to the 3-partite splits (3a), (3b), and (3c), while 2-separability still leaves all possibilities at level 3 open. From this one also sees that it is neither sufficient to consider only the 4- and 2-separability to classify the state completely, nor to consider only 4- and 3-separability. Taking now all these connections between different levels into account, we find that many of the 16 384 possible combinations are forbidden. In fact, one can check that only 346 different, allowed configurations remain [24]. For $N=4$, we thus have 346 different classes (including permutations among the parties), which should be compared to the 9 classes we found for $N=3$.

Furthermore, it is not sufficient to classify the states by the number of k-separable states at level k of the hierarchic classification. For example, we have that 3-separability with respect to the 3-partite splits (3a), (3b), and (3c) already implies 2-separability with respect to all bipartite splits. On the other hand, 3-separability with respect to the 3-partite splits (3a), (3b), and (3d) does not determine the biseparability properties of the bipartite split (2f), which may thus still be 2-inseparable. The two kinds of 3 times 3-separability correspond to different physical situations, and one cannot obtain one configuration from the other one by permuting the parties. Thus, it is not sufficient to give only the number of 3-separable states, one also needs the information which of the splits is separable and which is not.

2. Separability and distillability properties of ρ_4

Let us now turn to the family of states ρ_4 and illustrate its separability properties. We will give an example for each theorem.

(i) Let us consider the 3-partite split A-B-(CD) (3a). Iff we have that the 2-partite splits (2a), (2b), and (2e) [which contain (3a)] have PPT, then ρ_4 is 3-separable with respect to this 3-partite split.

(ii) Iff the partial transposition with respect to the bipartite split (AB)-(CD) (2e) is positive, then ρ_4 is 2-separable in (AB)-(CD).

(iii) Iff for all possible 2-partite splits (2a)-(2g) we have that the corresponding partial transposition is positive, then ρ_4 is 4-separable.

Note that this family of states is completely characterized by its 2-separability properties, since from 2-separability follows the corresponding 3-separability as well as the corresponding 4-separability. So in this case only one level of the hierarchic structure, namely level 2, is required to fully classify the states ρ_4.

Finally we consider the distillability properties of ρ_4.

(i) Iff the partial transposition with respect to the bipartite splits (2c), (2d), (2f), and (2g) is negative, then a maximally entangled pair between C and D can be distilled. Note that these four splits are the only ones of relevance, since the parties C and D are not joint there.

(ii) Iff the partial transpositions with respect to the bipartite splits (2a), (2b), (2c), (2e), (2f), and (2g) are negative, then a GHZ state between the parties A-B-C can be distilled. Note that the split (2d) is the only one that is not of relevance here, since the parties ABC are joint in this case. If in addition also the split (2d) has NPT, then a GHZ state between all four parties can be created.

If it turns out that there exist nondistillable states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ with NPT as conjectured in Refs. [16] and [17], this automatically implies that the conditions (i) for distillability obtained for the states ρ_4 are *not* sufficient for arbitrary states ρ. To see this, let us consider the question of distillability of a maximally entangled pair between C and D. Assume that the partial transposition with respect to the bipartite splits (2c), (2d), (2f), and (2g) is negative. Let us concentrate on the bipartite split (2f). According to the conjecture in [16,17], the negativity of the partial transposition with respect to this split is not sufficient to ensure distillability. So we have that there exist states which are not distillable even if we allow for joint actions at (AC) and (BD) and are thus also not distillable when allowing only local operations in A, B, C, and D (as mentioned earlier, each bipartite split provides us with an upper limit of what can be done by local operations).

VI. MIXTURES OF GHZ-STATE WITH IDENTITY

We will apply now our results to the case in which we have a maximally entangled state of N particles mixed with the completely depolarized state

$$\rho(x) = x|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1-x}{2^N}\mathbb{1}. \tag{21}$$

This is clearly a special case of the state ρ_N with $\lambda_0^- = \lambda_j = (1-x)/2^N$, $\lambda_0^+ = x + (1-x)/2^N$ and thus $\Delta = x$. These states have been analyzed in the context of robustness of entanglement [15], NMR computation [14], and multiparticle purification [11]. In all these contexts bounds are given regarding the values of x for which $\rho(x)$ is separable or purificable. For example, in Refs. [14] and [15] they show that in the case $N=3$ if $x \leq 1/(3+6\sqrt{2}),1/25$ then the state is separable, respectively. In Ref. [11] it is shown that for $N=3$ if $x > 0.32\,263$ then $\rho(x)$ is distillable. Using our results we can state that $\rho(x)$ is fully nonseparable and distillable to a maximally entangled state iff $x > 1/(1+2^{N-1})$, and fully separable otherwise. Specializing this for the case $N=3$ we obtain that for $x > 1/5$ it is non-separable and distillable [25].

Note also that the purification procedure proposed in this work is pretty unefficient compared to the two-step procedure proposed in Ref. [11], although it allows us to determine stronger bounds for the value of x. However, one can slightly modify the procedure proposed in Ref. [11] such that the protocols P1 and P2 are no longer performed alternately as in the original version, but rather in a specific (state dependent) order, e.g., P1-P1-P1-P2-P1, etc. When doing so, we found by numerical investigation (for $N=3$) that all states ρ_3 which are purificable to a GHZ state using our procedure are also purificable using the modified procedure of Ref. [11], which thus provides an efficient purification protocol for states of the form ρ_3.

VII. SUMMARY

In summary, we have proposed a classification of arbitrary multiqubit systems. For a family of states, we gave a full characterization of the separability and distillability properties. These states play the role of Werner states in these systems since any state can be reduced to such a form by depolarization. Thus, our results provide sufficient conditions for non-separability and distillability for general states.

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[24] For example, we have that the following combination is in principle allowed: The state ρ is 4-inseparable, 3-inseparable with respect to all possible 3-partite splits (3a)–(3f), but 2-separable with respect to all possible bipartite splits (2a)–(2g). On the other hand, we have that the following combination is not allowed: ρ is 4-inseparable, 3-separable with respect to all possible 3-partite splits (3a)–(3f) and 2-separable with respect to the bipartite splits (2b)–(2g), but 2-inseparable with respect to the bipartite split (2a). Since, e.g., the 3-separability with respect to the 3-partite split (3a) already implies the 2-separability with respect to the bipartite split (2a) (see Fig. 2), it follows that this configuration is impossible and thus forbidden.

[25] In R. Schack and C. M. Caves, e-print quant-ph/9904109, it was independently shown using different methods that for $N=3$, the state $\rho(x)$ is separable for $x\leqslant 1/5$. For larger N however, only weaker bounds were obtained.