

Measuring the entanglement of bipartite pure states

J. M. G. Sancho¹ and S. F. Huelga^{1,2}

¹ *Departamento de Física, Universidad de Oviedo, Calvo Sotelo s/n, 33007 Oviedo, Spain*

² *Optics Section, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*

(Received 11 October 1999; published 7 March 2000)

The problem of the experimental determination of the amount of entanglement of a bipartite pure state is addressed. We show that measuring a single observable does not suffice to determine the entanglement of a given unknown pure state of two particles. Possible minimal local measuring strategies are discussed, and a comparison is made on the basis of their best achievable precision.

PACS number(s): 03.67.-a, 03.65.Bz

I. INTRODUCTION

Quantum-mechanical states of multiparticle systems can be entangled, a fact well known since the early days of the formulation of the quantum theory [1]. However, the status of this property has changed substantially in recent years. Entanglement can also be viewed as a resource, and as such features in different processes of potential practical importance, for example quantum teleportation [2], quantum cryptography [3] or even high precision measurements [4]. From a theoretical point of view, entanglement of bipartite pure states, and its properties under local quantum operations, are reasonably well understood. There is a unique measure of entanglement for these systems, provided by the von Neumann entropy [5], and optimal methods of entanglement manipulation are known [6]. However, there is a remaining practical question which has not been addressed so far: How can one optimally measure the amount of entanglement of an unknown bipartite pure state?

At first sight, this question may seem obvious. Reconstructing the reduced density operator of any of the two subsystems will do the job. However, the essential point is that we require the determination to be optimal and the reconstruction of the reduced density matrix may provide redundant information, given that we are asking for just one feature of the composite state: its entanglement. This is a single number, and the first question to be answered is whether there exists a single operator whose experimental measure may provide us with just the amount of entanglement of the state. Note that further details of the state itself are not of interest in the problem we are posing here [7]. We will prove in the following that such an operator does not exist, a conclusion that confirms what could initially be regarded as an educated guess. Knowing the impossibility of a test using repeated measurements of a single observable, we will discuss possible strategies aimed at being minimal, in the sense of involving the smallest number of observables. In order to avoid any ambiguity when counting the number of observables involved in a given measurement protocol, we will define such a number as the different number of *meters* each observer has to read out. Among different minimal strategies, (that is, strategies involving the same number of meters), we will call ‘‘optimal’’ the one providing the best accuracy when supplied with the same resources. We will show that, in fact, measuring the reduced density matrix

turns out to be a minimal way to proceed. Moreover, the protocol can be made optimal when involving projections along mutually orthogonal directions. It must be emphasized that our analysis, and its following conclusions, hold for pure states. The problem of optimally determining the amount of entanglement of bipartite mixed states may require a completely different approach. We have organized this paper as follows. In Sec. II we formally state the problem. Section III shows the impossibility of finding a single observable whose measure may allow the experimental determination of the amount of entanglement. Minimal local strategies are discussed in Sec. IV. When supplied with the same number of identically prepared bipartite pure states, we discuss the performance of two classes of minimal measurements from the point of view of achievable precision in determining the amount of entanglement. Sec. V is devoted to conclusions.

II. EXPERIMENTAL SCENARIO

Let us imagine the following situation. We are provided with a state preparer which creates pairs of two-level particles (qubits) in an unknown entangled state. These entangled pairs are distributed to two remote locations where two observers, Alice and Bob, may perform local measurements as well as interchange classical communication. The internal dynamics of the device is not specified and the only thing Alice and Bob know is that, with high accuracy, the state they share is pure. Therefore, the two-qubit state can be written as

$$\rho = |\psi_{AB}\rangle\langle\psi_{AB}|, \quad (1)$$

where

$$|\psi\rangle_{AB} = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle. \quad (2)$$

In this expression ($|0\rangle, |1\rangle$) refer to eigenvectors of operators σ_z , the complex coefficients a_i , ($i=0, \dots, 3$), being completely unknown. In addition, we assume that the machine may supply a large number of identical pairs. The aim is to use the resulting pairs for a quantum information task and, therefore, the only property we are interested in is its amount of entanglement. Moreover, we require the measurement aimed at determining the amount of entanglement to be optimal in the following sense. First, the protocol should involve the smallest possible number of observables. Such

tests will be called minimal. Second, among minimal tests, we will define as optimal the class of protocols that yields the best resolution when supplied with the same resources, i.e., the same number of identically prepared two-level systems. The problem is still rather general and, for simplicity, three further assumptions will be made.

(1) The experimental situation is such that it only allows one to act on one pair at a time. In other words, we restrict ourselves to incoherent measurements. Alice and Bob are not allowed to store a given number of particles and perform a joint measurements on them [8].

(2) No ancillary systems are available, and the only allowed incoherent measurements are projective ones.

(3) The adopted protocol is rigid, in the sense that we will not accumulate information from a given set of initial measurements and readjust our strategy afterwards.

In these conditions, we will show that no single operator measurement allows one to determine the amount of entanglement of an unknown bipartite pure state.

III. IMPOSSIBILITY OF A SINGLE-OBSERVABLE MEASURING STRATEGY

The amount of entanglement of a bipartite pure state is given by its von Neumann entropy,

$$E(\psi_{AB}) = -\text{tr}(\rho_A \ln_2 \rho_A) = -\text{tr}(\rho_B \log_2 \rho_B), \quad (3)$$

where $\rho_{A(B)} = \text{tr}_{B(A)} \rho$ is the reduced density matrix of each subsystem, and ρ is given by Eq. (1). In terms of the concurrence C [9], defined as

$$\begin{aligned} C^2(\psi_{AB}) &= |\langle \psi | \sigma_y \otimes \sigma_y | \psi^* \rangle|^2 \\ &= |a_0 a_3 - a_1 a_2|^2 \\ &= 4 \det \rho_A = 4 \det \rho_B, \end{aligned} \quad (4)$$

the amount of entanglement can be expressed in a compact form as follows:

$$\begin{aligned} E(\psi_{AB}) &= -\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \ln_2 \left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \\ &\quad - \left(\frac{1 - \sqrt{1 - C^2}}{2}\right) \ln_2 \left(\frac{1 - \sqrt{1 - C^2}}{2}\right). \end{aligned} \quad (5)$$

It should be noted that if all coefficients a_i were real, the concurrence could be obtain via the repeated measurement a single observable $\sigma_y \otimes \sigma_y$. We will now prove that, in general, i.e., where no *a priori* information is provided about the state of the bipartite system, it is not possible to evaluate $C^2(\psi_{AB})$ by means of measuring a set of orthogonal projectors $\mathbb{P}_i = |O_i\rangle\langle O_i|$, $\sum_{i=0}^3 \mathbb{P}_i = 1$, where the $|O_i\rangle$'s form an orthonormal basis of a certain operator \hat{O} . This measurement would allow us to compute the four probabilities $p_i = |\langle O_i | \psi \rangle|^2$, and therefore it provides three independent real numbers. It is obvious that this will not be enough to fully reconstruct the pure state, $|\psi\rangle$, but one may still ask whether the resulting information may be enough to compute one

property of the state: its amount of entanglement. In order to check this, let us first rewrite the concurrence C in a more convenient form. For that, we will express the state $|\psi_{AB}\rangle$ in terms of the eigenbasis of the operator \hat{O} as

$$|\psi_{AB}\rangle = \sum_{i=0}^3 \langle O_i | \psi \rangle |O_i\rangle \equiv \sum_{i=0}^3 m_i e^{i\phi_i} |O_i\rangle, \quad (6)$$

where the coefficients m_i are purely real, and $\psi_i \in [0, 2\pi)$. Then C^2 can be written

$$\begin{aligned} C^2(\psi_{AB}) &= \left| \sum_{i=0}^3 \sum_{j=0}^3 \langle O_i | \psi \rangle^* \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \langle O_j | \psi \rangle \right|^2 \\ &= \left| \sum_{i=0}^3 \sum_{j=0}^3 m_i m_j e^{-i\phi_i} e^{-i\phi_j} \right. \\ &\quad \left. \times \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \right|^2. \end{aligned} \quad (7)$$

Let us define a new matrix K with elements given by

$$K_{ij} = \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle. \quad (8)$$

In terms of this quantity, the squared concurrence can be written as

$$C^2(\psi_{AB}) = \sum_{i,j,k,l=0}^3 m_i m_j m_k m_l e^{i(\phi_k + \phi_l - \phi_i - \phi_j)} K_{ij} K_{kl}^*. \quad (9)$$

Looking at this expression one can already formulate the guess that it will not be possible to obtain C^2 by just measuring the probabilities p_i , given that no information about the relative phases ψ_i will be unveiled by the measurement. In what follows we will prove explicitly that C^2 , and therefore the amount of entanglement of the bipartite pure state, cannot be an univaluated function of the probabilities p_i on (equivalently, of the coefficients m_i).

A. A useful lemma

Let us define two new auxiliary matrices S and σ with matrix elements given by

$$S_{ij} = \langle O_j^* | O_i \rangle \quad (10)$$

and

$$\sigma_{ij} = \langle O_i | \sigma_y \otimes \sigma_y | O_j \rangle. \quad (11)$$

It is easy to check that the following properties hold.

(1) The matrix K of Eq. (8) satisfies $K = K^T$, as follows immediately from the hermiticity of the operator $\sigma_y \otimes \sigma_y$.

(2) If the $|O_i\rangle$'s form an orthonormal basis, the corresponding conjugate vectors $|O_i^*\rangle$ also form an orthonormal basis. Then the matrix S defined above is just the change of basis matrix between the two representations, i.e.,

$$|O_i\rangle = \sum_j S_{ij} |O_j^*\rangle, \quad (12)$$

and therefore $S^\dagger S = \mathbb{1}$ and $|\det(S)| = 1$. In particular, this implies that $\det(S^\dagger) \neq 0$. (3) $\det(\sigma) = 1$.

We now have all the ingredients for proving the following lemma.

Lemma: $\det(K) \neq 0$.

Proof: The matrix elements of K can be written in terms of those of σ and S as follows:

$$\begin{aligned} K_{ij} &= \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \\ &= \sum_{l=0}^3 \langle O_i | \sigma_y \otimes \sigma_y | O_l \rangle \langle O_l | O_j^* \rangle \\ &= \sum_{l=0}^3 \sigma_{il} S_{jl}^* = \sum_{l=0}^3 \sigma_{il} S_{lj}^\dagger. \end{aligned}$$

Therefore, $K = \sigma S^\dagger$ and the lemma follows from properties (2) and (3).

B. Impossibility of a single-observable test

We will now prove that assuming that the measurement of a single observable allows us to determine the concurrence of the state, and therefore its amount of entanglement, yields a contradiction with the previous lemma. Given that the state $|\psi\rangle$ is unknown, the test we are seeking must be universal; that is, the hypothetical observable \hat{O} has to provide the amount of entanglement of whatever input state. The idea underlying our proof is to show that there will always be a particular case yielding to a contradiction. Therefore, if no *a priori* information is provided, the minimal test will necessarily require measuring more than one observable.

Consider the particular case where $m_0 = m_1 = 1/\sqrt{2}$ and $m_2 = m_3 = 0$. In this case, Eq. (9) takes the form

$$\begin{aligned} C^2 &= \frac{1}{4} (|K_{00}|^2 + |K_{11}|^2 + 4|K_{01}|^2 + 2e^{i\phi}(K_{00}K_{01}^* + K_{01}K_{11}^*) \\ &\quad + 2e^{-i\phi}(K_{01}K_{00}^* + K_{11}K_{01}^*) + e^{2i\phi}K_{00}K_{11}^* \\ &\quad + e^{-2i\phi}K_{11}K_{00}^*), \end{aligned}$$

where we have set $\phi = \phi_1 - \phi_0$, and made use of property (1). If we assume that C^2 is only a function of the real numbers m_i , i.e., independent of the relative phase ϕ , the fact that the functions $(1, e^{i\phi}, e^{-i\phi}, e^{2i\phi}, e^{-2i\phi})$ are linearly independent yields the set of equalities

$$C^2 = \frac{1}{4} (|K_{00}|^2 + |K_{11}|^2 + 4|K_{01}|^2), \quad (13)$$

$$K_{00}K_{01}^* + K_{01}K_{11}^* = 0, \quad (14)$$

$$K_{00}K_{11}^* = 0. \quad (15)$$

Equation (15) implies that either $K_{00} = 0$ and/or $K_{11} = 0$. Taking Eq. (14) into account, this corresponds to the cases where $K_{01} \neq 0$ or $K_{01} = 0$. In other words, we obtain that two out of the three complex numbers (K_{00}, K_{01}, K_{11}) must be zero.

If we repeat this argument for all the cases where any two of the coefficients m_i are equal to $1/\sqrt{2}$, and the remaining two equal to zero, we end up with the requirement that in all the sets of three complex numbers

$$\begin{aligned} &(K_{00}, K_{01}, K_{11}) \quad (K_{00}, K_{02}, K_{22}) \quad (K_{00}, K_{03}, K_{33}), \\ &(K_{11}, K_{12}, K_{22}) \quad (K_{11}, K_{13}, K_{33}), \\ &(K_{22}, K_{23}, K_{33}), \end{aligned}$$

there must be at least two of them equal to zero in any set. This fact imposes a certain symmetry for the allowed K matrices. Explicitly, K can only be either

$$K_1 = \begin{pmatrix} K_{00} & 0 & 0 & 0 \\ 0 & 0 & K_{12} & K_{13} \\ 0 & K_{12} & 0 & K_{23} \\ 0 & K_{13} & K_{23} & 0 \end{pmatrix} \quad (16)$$

or

$$K_2 = \begin{pmatrix} 0 & K_{01} & K_{02} & 0 \\ K_{01} & 0 & K_{12} & 0 \\ K_{02} & K_{12} & 0 & 0 \\ 0 & 0 & 0 & K_{33} \end{pmatrix}, \quad (17)$$

and analogous forms obtained when interchanging the roles of the indexes, or of the form

$$K_3 = \begin{pmatrix} 0 & K_{01} & K_{02} & K_{03} \\ K_{01} & 0 & K_{12} & K_{13} \\ K_{02} & K_{12} & 0 & K_{23} \\ K_{03} & K_{13} & K_{23} & 0 \end{pmatrix}. \quad (18)$$

It should be noted that many other cases could be obtained if any of the matrix elements written as nonzero were zero; however, these additional cases are not of interest here, as will become clear below.

Our proof ends by showing that, in any of the allowed forms for K , some of the possibly nonzero coefficients in K turn out to be zero. Therefore, all the allowed forms for K —i.e., all forms compatible with the requirement of the concurrence being a univaluated function of the real numbers p_i —will have a determinant equal to zero, which contradicts the lemma stated above

This can be easily shown for matrices of the form K_1 or K_2 , following an argument parallel to the one used above and choosing three of the coefficients m_i equal to $1/\sqrt{3}$ and the remaining one equal to zero. Here let us analyze the case of K matrices of the form K_3 . If we set the m_i coefficients to the values $m_0 = 0$ and $m_i = 1/\sqrt{3}$ for $(i=1, \dots, 3)$, the squared concurrence given by Eq. (9) reads

$$C^2 = \frac{4}{3\sqrt{3}} (|K_{12}|^2 + |K_{13}|^2 + |K_{23}|^2 + e^{i\alpha} K_{12} K_{13}^* + e^{-i\alpha} K_{13} K_{12}^* + e^{i\beta} K_{12} K_{23}^* + e^{-i\beta} K_{23} K_{12}^* + e^{i\gamma} K_{13} K_{23}^* + e^{-i\gamma} K_{23} K_{13}^*),$$

where we have introduced the relative phases $\alpha = \phi_3 - \phi_2$, $\beta = \phi_3 - \phi_1$, and $\gamma = \phi_2 - \phi_1$. Again using the argument invoked in proving the allowed forms for the matrix K , we obtain that the following equalities must hold:

$$K_{12} K_{13}^* = 0,$$

$$K_{12} K_{23}^* = 0,$$

$$K_{13} K_{23}^* = 0.$$

Therefore, within the three numbers (K_{12}, K_{13}, K_{23}) , two must be zero. If we repeat the argument, each time making one of the m_i 's zero and the other three equal to $\sqrt{3}$, we end up with allowed forms for K_3 , with either four or two matrix elements different from zero. For instance, an allowed form of K is given by

$$K_3^1 = \begin{pmatrix} 0 & K_{01} & 0 & 0 \\ K_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{23} \\ 0 & 0 & K_{23} & 0 \end{pmatrix}. \quad (19)$$

However, now choosing all the coefficients m_i equal to $1/2$ suffices to obtain the constraint $K_{01} K_{23}^* = 0$, which yields a zero determinant for K . The same reasoning applies to the other five possible cases. Therefore, assuming that C^2 is only a function of the probabilities p_i yields the condition $\det(K) = 0$, but we proved in Sec. II that the orthonormality of the vectors O_i demands the determinant of K to be non-zero. As a result, it is not possible to find a single operator \hat{O} whose measurement allows us to determine the amount of entanglement of the pure state $|\psi_{AB}\rangle$.

IV. MINIMAL TESTS

The previous analysis shows that a measuring strategy employing a single observable does not allow Alice and Bob to know the amount of entanglement of the state they are sharing. Knowing this fact, the natural question to ask is how to determine a minimal measuring strategy than may allow them to evaluate $E(\psi_{AB})$. It is clear that if they measure two different observables \hat{O}_{AB} of the form analyzed above, they will fully reconstruct the original pure state and can, therefore, compute its amount of entanglement. It is a remarkable fact that, acting on the whole Hilbert space of the two particles, we cannot isolate the information related to the amount of entanglement alone by means of measuring a single observable. If no information is known about the state, determining its amount of entanglement leads to a full reconstruction of the state. However, such a nonlocal implementation may not be the easiest strategy to implement experimentally, and local strategies are preferred. In this section we

will discuss possible ways to proceed if one is constrained to act locally, and we will determine the expected precision of the protocols.

A. Local actions without exchange of classical communication

When Alice and Bob are constrained to act locally and no classical communication can be exchanged among them, the minimal measuring strategy corresponds to the local reconstruction of the reduced density operators. For instance, Alice may reconstruct the operator $\rho_A = \text{tr}_B \rho_{AB}$, and at the end of the protocol send a final message to Bob whose length will depend on the required resolution.

To achieve this, Alice needs to perform three projective measurements along linearly independent directions. We will show in the following that measuring along three orthogonal directions is in fact optimal, in the sense that choosing this configuration yields the smallest associated uncertainty in the experimental determination of the determinant of the reduced density operator [see Eq. (4)].

Let us write the reduced density matrix in the general form

$$\rho_A = \frac{1}{2} (1 + \sigma \cdot S) = \frac{1}{2} \begin{pmatrix} 1 + S_z & S_x - iS_y \\ S_x + iS_y & 1 - S_z \end{pmatrix} \quad (20)$$

in terms of the corresponding Bloch's vector. With the above parametrization we have a one-to-one correspondence between directions in three-dimensional space and directions within the Bloch sphere. Note that the determinant of ρ_A only depends on the modulus of the Bloch's vector. In other words, it is rotationally invariant. Suppose now we are planning to measure the amount of entanglement projecting the reduced density matrix of a given state along three linearly independent directions. The uncertainty associated with this measurement will depend on (1) the modulus of the corresponding Bloch's vector, as the amount of entanglement does, and (2) the relative position of Bloch's vector with respect to the three projective directions.

Because of conditions (1) and (2), assuming the initial distribution of states to be isotropic, the average uncertainty after measuring sufficiently many states will only depend on the relative position of the three directions we project along. In particular, this implies we can choose a given direction to be the z axis. We will call the other two directions, \hat{n} and \hat{m} so that the angles they form with the z axis are θ_n and θ_m . Then we can write the average uncertainty as

$$\delta_{av} = f(\theta_n, \theta_m, \phi_{nm}).$$

Here ϕ_{nm} is the relative azimuth angle $\phi_m - \phi_n$. Moreover, because of condition (2), the following equalities must hold

$$f(\theta_n, \theta_m, \phi_{nm}) = f(\pi - \theta_n, \theta_m, \pi - \phi_{nm}),$$

$$f(\theta_n, \theta_m, \phi_{nm}) = f(\theta_n, \pi - \theta_m, \phi_{nm} - \pi),$$

$$f(\theta_n, \theta_m, \phi_{nm}) = f(\pi - \theta_n, \pi - \theta_m, \phi_{nm}).$$

This is equivalent to stating that we could redefine the three positive axes (simultaneously or not) without changing the average uncertainty. Finally, from the previous set of equations one obtains that the function f has to have an extremum at

$$\theta_m = \pi/2, \quad \theta_n = \pi/2, \quad \phi_{nm} = \pi/2.$$

$$\delta_{av} = \left\langle \sqrt{\left| \frac{\partial \det(\rho_a)}{\partial P_0} \right|^2 \delta P_0 + \left| \frac{\partial \det(\rho_a)}{\partial P_m} \right|^2 \delta P_m + \left| \frac{\partial \det(\rho_a)}{\partial P_n} \right|^2 \delta P_n} \right\rangle, \quad (21)$$

as a function of the relative difference ϕ_{nm} for a fixed value of $\theta_m = \theta_n$. In the definition above, P_0, P_m , and P_n are the probabilities to obtain a *spin-up* when measuring along directions \hat{z}, \hat{m} , and \hat{n} , respectively, and where each δP denotes the squared variance given by $\delta P = P(1-P)/N$. The bracket means the average over the isotropic distribution of states. It is clear that δ_{av} in fact reaches a minimum when $\phi_{nm} = \pi/2$. Similar figures can be plotted, all of them supporting that the minimum uncertainty is indeed achieved when the three directions of projection are chosen to be mutually orthogonal.

B. Local actions with exchange of classical communication

Let us now assume that Alice and Bob agree to cooperate. Then the amount of entanglement can be evaluated from the measurement of two Pauli operators on each side. If they agree to measure different operators in each round, they again fully reconstruct the state. However, if one of them always measures the same Pauli operator, (for instance, if they choose to compute the observables $\sigma_z \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_z$ and, in a subsequent round, the observables $\sigma_x \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_z$, for which they should read out three *meters*), they can obtain the amount of entanglement, but they will neither obtain full information about the state itself, nor about the whole reduced density matrix. Indeed, if we denote by P_i ($i=0, 1, 2$, and 3) the four probabilities associated with the outcomes $(+, +, +, -)$ when measuring $\sigma_z \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_z$, and by P_{++}, P_{+-}, P_{-+} , and P_{--} the corresponding probabilities when measuring $\sigma_x \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_z$, it can easily be shown that the probabilities P_{ij} ($i, j = +, -$) can be written in terms of the probabilities P_i as follows:

$$P_{++} = \frac{1}{2}(P_0 + P_1 + 2\sqrt{P_0 P_1} \cos \phi_{01}),$$

$$P_{+-} = \frac{1}{2}(P_0 + P_1 - 2\sqrt{P_0 P_1} \cos \phi_{01}),$$

$$P_{-+} = \frac{1}{2}(1 - P_0 - P_1 + 2\sqrt{P_3 P_2} \cos \phi_{23}),$$

We have numerically calculated the average uncertainty δ_{av} over 10.000 states of the composite system uniformly distributed over the complex four-dimensional joint Hilbert space, and calculated this uncertainty for a series of different measurements with $\theta_m = \pi/2$ and $\theta_n = \pi/2$, but ϕ_{nm} ranging from 0 to π . Results are shown in Fig. 1, where we show the mean uncertainty in determining the amount of entanglement δ_{av} , defined as

$$P_{--} = \frac{1}{2}(1 - P_0 - P_1 - 2\sqrt{P_3 P_2} \cos \phi_{23}), \quad (22)$$

where we have rewritten the amplitudes of the initial state as $a_i = m_i e^{i\phi_i}$ ($i=0, 1, 2$, and 3), and called $\phi_{ij} = \phi_i - \phi_j$. From the previous set of equations, we see that the functions $\cos(\phi_0 - \phi_1)$ and $\cos(\phi_2 - \phi_3)$ can be expressed in terms of measurable quantities in the forms

$$\cos(\phi_0 - \phi_1) = \frac{2P_{++} - P_0 - P_1}{2\sqrt{P_0 P_1}}, \quad (23)$$

$$\cos(\phi_2 - \phi_3) = \frac{2P_{-+} + P_0 + P_1}{2\sqrt{P_3 P_2}}.$$

Noting that

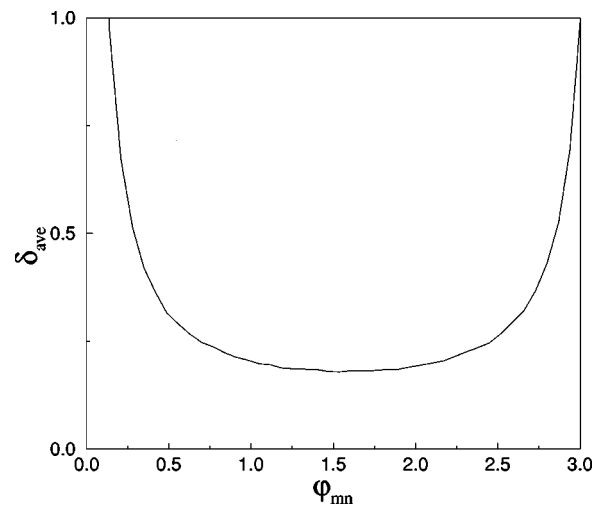


FIG. 1. Average uncertainty as a function of the relative azimuth angle ϕ_{mn} for a fixed value of $\theta_m = \theta_n$. The average uncertainty in determining the amount of entanglement is minimal when the three linearly independent directions of the projection are chosen to be orthogonal.

$$C^2 = 4[P_1P_2 + P_0P_3 - 2\sqrt{P_0P_1P_2P_3} \times \cos(\phi_0 - \phi_1 + \phi_3 - \phi_2)], \quad (24)$$

we see that this measuring strategy suffices to determine the squared concurrence, and therefore the amount of entanglement of the pure state but it does not allow a full reconstruction of the initial state. As we will show in Sec. IV C, this protocol is not optimal, in the sense of providing the best accuracy when locally measuring a minimal set of observables.

C. Which strategy yields the best resolution?

The measuring strategies described above are both minimal, in the sense of involving the smallest number of meters to be read out. However, it is not obvious whether the precision achieved following these two strategies will be the same. In fact, we will show in the following that it is not. When provided with the same resources, (that is, using the same number of identically prepared entangled pairs), we can obtain the amount of entanglement with higher precision by means of a local reconstruction of the reduced density operator. If we denote by N the number of entangled pairs, N being large in the statistical sense, a numerical simulation with 10^6 states from an isotropic initial distribution yields the following results.

(1) The measurement procedure by means of the local reconstruction of the reduced density operator has an associated uncertainty which scales with N as

$$\delta_{loc} = \frac{0.3}{\sqrt{N}}.$$

(2) The associated uncertainty with a local measurement of the form described in Sec. IV B is substantially much larger [10]. More precisely,

$$\delta_{loc+cc} = \frac{2.3}{\sqrt{N}}.$$

Note that, once N is given, the resulting number of measurements in each measuring protocol is different. While in the first case each single probability will scale as $P \approx 1/\sqrt{N/3}$, the larger number of measurements causes each probability in the second procedure to scale as $P \approx 1/\sqrt{N/2}$.

From these results one may be led to the conclusion that the best resolution will always be achieved by means of reconstructing the reduced density operator. However, this may not be true. Imagine that, in the context of the second protocol, Bob measures a different Pauli operator. If the direction of projection is orthogonal to the z axis, this procedure will also allow him to reconstruct the initial state. Will the associated uncertainty now be reduced with respect to the case analyzed above? In the light of the results obtained

when measuring the reduced density operator, this should be the case. It should be noted, however, that the number of observables required in this measurement protocol is no longer minimal, as it would require the observer Bob to read out an additional meter.

V. CONCLUSIONS

We have analyzed the problem of experimentally determining the amount of entanglement of bipartite pure states when one has a large supply of identically prepared systems on which one is restricted to act by means of projective measurements. We showed that, provided that the entangled state is totally unknown, no measuring strategy involving a single operator exists. Therefore, acting on the Hilbert space of the composite system does not allow one to single out the amount of entanglement without allowing one to determine the state completely. When local actions are considered, the minimal protocol for determining the amount of entanglement involves measuring three different observables. There we have analyzed two classes of minimal tests. In the first one, no exchange of classical communication is required, and entails a local reconstruction of the reduced density operator. The procedure is optimal, in the sense of it having the smallest associated uncertainty, when measuring along three mutually orthogonal directions. The second class of protocol requires the use of classical communication. Here we have analyzed a possible strategy, and showed that it suffices to determine the amount of entanglement of the pure state but not its full reconstruction. The associated resolution turns out to be worse than the one corresponding to the measurement of the reduced density operator. The analyzed protocol is not necessarily the most precise among the whole class of measuring strategies by means of local actions with the exchange of classical communication. However, an increase in the resolution would be made at the price of increasing the number of meters to be read out and the protocol would no longer be minimal. Establishing the best accuracy that a protocol involving the exchange of classical communication, and which allows the full reconstruction of the state, can achieve, and analyzing how it compares to the precision associated with the full reconstruction of the the reduced density operator, is an interesting open question.

ACKNOWLEDGMENTS

The authors thank M.B. Plenio, P. Hayden, D. Jonathan, G. Vidal, A.K. Ekert, D.P DiVincenzo, and J.I. Cirac for discussions on the subject of this paper. J.M.G.S. also acknowledges M. Ferrero for continuous encouragement and useful discussions. This work was supported by The Leverhulme Trust, the European Science Foundation, The Engineering and Physical Sciences Research Council (EPSRC), and DGICYT Project No. PB-95-0594 (Spain), and benefited from the participation in the ESF-QIT workshop in Cambridge.

- [1] E. Schrödinger, *Naturwissenschaften* **23**, 807 (1935); A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- [2] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
- [3] A.K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
- [4] D.J. Wineland, J.J. Bollinger, W.M. Itano, F.L. Moore, and D.J. Heinzen, *Phys. Rev. A* **46**, R6797 (1992); W.M. Itano, J.C. Bergquist, J.J. Bollinger, J.M. Gilligan, D.J. Heinzen, F.L. Moore, M.G. Raizen, and D.J. Wineland, *ibid.* **47**, 3554 (1993); S.F. Huelga, C. Macchiavello, T. Pellizzari, A.K. Ekert, M.B. Plenio, and J.I. Cirac, *Phys. Rev. Lett.* **79**, 3865 (1997); S.F. Huelga, P.L. Knight, C. Macchiavello, M.B. Plenio, and V. Vedral, *Appl. Phys. B* **67**, 723 (1998).
- [5] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [6] H-K. Lo and S. Popescu, LANL e-print, quant-ph/9707038; M.A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999); G. Vidal, *ibid.* **83**, 1046 (1999); D. Jonathan and M.B. Plenio, *ibid.* **83**, 1455 (1999); L. Hardy, *Phys. Rev. A* **60**, 1912 (1999).
- [7] The problem of the optimal reconstruction of pure states have been analyzed by different authors. See S. Massar and S. Popescu, *Phys. Rev. Lett.* **74**, 1259 (1995); R. Derka, V. Buzek, and A.K. Ekert, *ibid.* **80**, 1571 (1998); J.I. Latorre, P. Pascual, and R. Tarrach, *Phys. Rev. Lett.* **81**, 1351 (1998); Recently, the optimal and minimal measuring strategies for reconstructing unknown mixed states of two-level systems were analyzed by G. Vidal, J.I. Latorre, P. Pascual, and R. Tarrach, *Phys. Rev. A* **60**, 126 (1999).
- [8] The problem of optimally determining the amount of entanglement of bipartite pure states by means of coherent measurements was addressed by A. Acín, R. Tarrach, and G. Vidal, e-print quant-ph/9911008.
- [9] W.K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [10] The same resolution would be obtained if Alice projects along directions \hat{x} and \hat{y} or along \hat{z} and \hat{y} .