Dynamical symmetry, integrability of quantum systems, and general character of quantum regular motion

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(Received 1 June 1998; revised manuscript received 1 February 1999; published 17 March 2000)

The notion of quantum-classical correspondence is carefully investigated in order to prepare firm grounds for studying the spatiotemporal evolution of quantum states in the same spirit as for corresponding classical cases. Three relevant problems, (1) the integrability of dynamical equations of quantum systems, (2) the initial minimum uncertainty states one-to-one correspondent to classical phase points, and (3) the effective Planck constants for systems having analogous dynamical properties but exhibiting different quantum effects, have been successfully resolved. Then the solution $\rho_\gamma(t)$ of the dynamical equation of a quantum integrable system is shown to be expressed as an analytical functional of the initial minimum uncertainty state ρ_γ^0 varying smoothly with γ and *t*. Such a general character of the quantum regular motion serves as a reference for the study of quantum irregular motion under the action of perturbed Hamiltonian.

PACS number(s): 03.65.−w, 05.45.−a

I. INTRODUCTION

The state of motion of a classical system described by the solution $\underline{x}(t)$ of Hamilton canonical equations at any instant t corresponding to a given initial state $\underline{x}_0$ can be generally expressed as the map $\underline{x}_0 \rightarrow \underline{x}(t) = f(\underline{x}_0, t)$. If the map is a symplectic diffeomorphic one, the motion is stable with respect to slight alterations of the initial state and is thus regular. But in case the symplectic diffeomorphism of the map is strongly violated, the motion is exponentially instable with slight alterations of the initial state and is thus chaotic.

In principle, the state of motion of a quantum system could be studied in a corresponding way. But due to the principle of uncertainty, both the initial state $\rho_\gamma^0 = |\Psi_\gamma^0\rangle\langle\Psi_\gamma^0|$ and the state $\rho_\gamma(t) = |\Psi_\gamma(t)\rangle\langle\Psi_\gamma(t)|$ at any instant t are of statistical nature, $|\Psi_\gamma(t)\rangle$ should have the wave nature and satisfy the principle of superposition. The Schrödinger equation governing the evolution of the state $|\Psi_\gamma(t)\rangle$ should be linear in nature. Thus it seems impossible for a quantum system to have its chaotic motion with exponential instability as in the corresponding classical case [1–3].

However, Arnold has pointed out in his monograph [4]: ''The basic concepts and theorems of Hamiltonian mechanics are invariant under the group of symplectic diffeomorphisms acting on the phase space,'' and that ''The Hamiltonian point of view allows us to solve completely a series of mechanical problems which do not yield solutions by other means. It has even greater value for the approximate methods of perturbation theory (celestial mechanics), for understanding the general character of motion in complicated mechanical systems (ergodic theory, statistical mechanics) and in connection with other areas of mathematical physics (optics, quantum mechanics, etc.)'' As the framework of quantum mechanics is mainly formulated on the basis of Hamiltonian mechanics, in principle it is not impossible to study the spatio-temporal evolution of quantum states for given initial states to see whether or not the state $|\Psi_\gamma(t)\rangle$ can be expressed as an analytical functional of $|\Psi_\gamma^0\rangle$ varying smoothly with γ and t.

But due to historical reasons, quantum-mechanical studies have mainly concentrated on stationary problems as well as transition rates per unit time. As to the study of spatio-temporal evolution of quantum states in parallelism to corresponding classical cases, there still exist problems to be resolved.

(1) The integrability of dynamical equations of quantum systems.

(2) The initial minimum uncertainty states one-to-one correspondent to classical phase points.

(3) The effective Planck constants for systems having analogous dynamical properties but exhibiting different quantum effects.

We shall first make efforts to resolve these problems. The integrability of dynamical equations of classical and quantum systems can be shown explicitly if the Hamiltonian of the system has already been expressed in actions alone free from angle variables. But it is difficult to ascertain whether or not one can find an appropriate canonical transformation to reduce the Hamiltonian of a general system to the required form. In quantum mechanics, it is even more difficult because basic canonically conjugate dynamical variables are represented by incommutable operators. Even if the classical canonical transformation between $(\underline{p}, \underline{q})$ and $(\underline{I}, \underline{\theta})$ has already been found, it is still unable to obtain the corresponding quantum canonical transformation straightforwardly [5]. If one tries to study the problems from the topological property of quantum state space as in the Liouville theorem [4], one will immediately face the fundamental difficulty that the general property of quantum state space known to us is just the mathematical property of Hilbert space. For a few special systems, it has been known since the early days that the

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quantum state space can be spanned by a set of orthonormal eigenstates of the corresponding stationary Schrödinger equation either obtained by solving the differential equation as a boundary condition problem or obtained by solving the problem algebraically [6]. Later, more quantum systems of physical interest have been studied in this way [7] and such special functions have been studied systematically from the viewpoint of Lie algebra [8–11]. Since the interacting boson model proposed by Arima and Iachello [12] enables one to carry out comparative spectroscopic studies for a series of nuclei, several authors have tried to define the quantum integrability with dynamical groups having several subgroup chains [13–16]. But the correspondence between the classical integrability defined by the Liouville theorem and the quantum integrability temporarily defined in such a way has not yet been established.

After analyzing these previous works, we see that in order to resolve the problem of quantum integrability one has to start from known quantum integrable systems and characterize their relevant properties algebraically, such that the condition for existence of the required quantum canonical transformation for operators and the corresponding classical canonical transformation for dynamical variables can be expressed with analogous analytical relations. Moreover, the obtained results for a special integrable system can be readily extended to a class of systems with a group of Lie transformations [17] keeping the relevant algebraic property invariant. Preliminary results have already been reported [18]. We shall study the problem in detail in this article by starting our discussions from a three-dimensional isotropic harmonic oscillator in Sec. II and extend the discussions to more general three-dimensional oscillators having the same kind of boundary conditions in Sec. III.

Owing to the uncertainty principle in quantum mechanics, it is not possible to consider generally the correspondence between quantum state space and the classical phase space. But we have pointed out previously [19] for integrable systems, $\underline{I}$ are first integrals of motion, classical invariant tori with definite $\underline{I}$ and any value of $\underline{\theta}$ just correspond to simultaneous orthonormal eigenstates of $\underline{I}$ with perfectly undefined $\underline{\theta}$. At the same time, the ground state of the system has the minimum uncertainty, while the displaced ground states having the minimum uncertainty too can thus be taken as the one-to-one correspondence of classical phase points. Using such states as initial states for solutions of quantum dynamical equations, the spatiotemporal evolution of quantum states can be studied in the same spirit as for the corresponding classical case. This problem is studied in detail in Sec. IV.

In the previous paragraph, we have taken the displaced ground state of an integrable system as the one-to-one correspondence of the classical phase points. As classical limit cannot be attained straightforwardly with the universal Planck constant, it implies that the minimum uncertainty state here must be characterized with effective Planck constants for systems having analogous dynamical properties but exhibiting different quantum effects. By making appropriate scaling transformations, dynamical equations for three-dimensional isotropic harmonic oscillators with different inertia masses and potential strengths can be expressed indeed in the same form but the commutation relations for basic canonically conjugate variables must be expressed with different effective Planck constants. This problem is explained in detail in Sec. II.

With these three problems resolved, the quantum dynamics can be studied in precisely the same way as in classical dynamics. However, we shall restrict ourselves to discussions on quantum regular dynamics in this article. General discussions are given in Sec. IV, numerical illustrations are given in Sec. V. Finally, a brief summary is given in the last section.

II. DYNAMICAL SYMMETRY ALGEBRA AND LIOUVILLE'S THEOREM ON INTEGRABLE SYSTEMS

We shall study the problem of quantum integrability in this section by taking the special integrable system, the three-dimensional isotropic oscillator, as the starting point. The discussions consist of three steps: (1) to obtain independent sets of raising and lowering operators as simultaneous solutions of eigenequations of first integrals of motion such that a complete basis set of orthonormal states for the quantum state space can be found [20], (2) to express the condition for completeness of the basis set generally with a closed Lie algebra, and (3) to find the quantum canonical transformation between the conjugate pair of raising and lowering operators and action-angle variables according to the obtained closed Lie algebra. Moreover, in order to show the quantum-classical correspondence explicitly, the scaling transformations for oscillator systems with different parametric quantities are first performed such that the fundamental Poisson brackets for canonical conjugate variables are expressed with different effective Planck constants.

The Hamiltonian of the three-dimensional isotropic harmonic oscillator is of the form

$$H^0=\frac{1}{2m}\sum_{j=1,2,3}p_j^2+\frac{m\omega^2}{2}\sum_{j=1,2,3}q_j^2, \tag{2.1a}$$

the commutation relations between q_j and p_j are

$$[q_j,p_j]=i\hbar, \quad (j=1,2,3). \tag{2.1b}$$

The dynamical equations are

$$i\hbar\frac{dq_j}{dt}=[q_j,H^0], \quad i\hbar\frac{dp_j}{dt}=[p_j,H^0], \quad (j=1,2,3). \tag{2.1c}$$

If we carry out the scaling transformations by referring to a certain oscillator system with finite frequency ω_0 and taking $\sqrt{m\omega_0\hbar}$, $(\omega_0/\omega)\sqrt{\hbar/m\omega_0}$, and $(\omega_0/\omega)1/\omega_0$ as units of p_j, q_j, and t, respectively, Eqs. (2.1a) and (2.1b) will be expressed in dimensionless form as

$$\frac{H^0}{\hbar\omega_0}=\frac{1}{2}\sum_{j=1,2,3}\left[\left(\frac{p_j}{\sqrt{m\omega_0\hbar}}\right)^2+\left(\frac{q_j}{\left(\frac{\omega_0}{\omega}\right)\sqrt{\frac{\hbar}{m\omega_0}}}\right)^2\right] \tag{2.2a}$$

and

$$\left[\frac{q_j}{\left(\frac{\omega_0}{\omega}\right)\sqrt{\frac{\hbar}{m\omega_0}}},\frac{p_j}{\sqrt{m\omega_0\hbar}}\right]=i\left(\frac{\omega}{\omega_0}\right),\quad (j=1,2,3). \tag{2.2b}$$

While the dynamical equations become

$$i\left(\frac{\omega}{\omega_0}\right)\frac{d\left(\frac{q_j}{\left(\frac{\omega_0}{\omega}\right)\sqrt{\frac{\hbar}{m\omega_0}}}\right)}{d(\omega t)}=\left[\frac{q_j}{\left(\frac{\omega_0}{\omega}\right)\sqrt{\frac{\hbar}{m\omega_0}}},\frac{H^0}{\hbar\omega_0}\right],$$

$$i\left(\frac{\omega}{\omega_0}\right)\frac{d\left(\frac{p_j}{\sqrt{\hbar m\omega_0}}\right)}{d(\omega t)}=\left[\frac{p_j}{\sqrt{\hbar m\omega_0}},\frac{H^0}{\hbar\omega_0}\right],\quad (j=1,2,3), \tag{2.2c}$$

which remain of the original form. In fact, the scaling transformations here are just special canonical transformations which keep the form of dynamical equations unaltered.

Now we see that with the help of an arbitrary chosen oscillator system with finite frequency ω_0, quantum effects of oscillator systems with different frequencies ω can be compared with each other while the quantity ω/ω_0 plays the role of effective Planck constant. For convenience, we shall denote $H_0/\hbar\omega_0$, $p_j/\sqrt{m\hbar\omega_0}$, $q_j/(\omega_0/\omega\sqrt{\hbar/m\omega_0})$, ωt, and ω/ω_0 simply as H^0, p_j, q_j, t, and $\hbar$ afterwards, such that

$$H^0=\frac{1}{2}\sum_{j=1,2,3}(q_j^2+p_j^2), \tag{2.3a}$$

$$[q_j,p_j]=i\hbar,\quad (j=1,2,3) \tag{2.3b}$$

$$\frac{dq_j}{dt}=[q_j,H^0],\quad \frac{dp_j}{dt}=[p_j,H^0],\quad (j=1,2,3). \tag{2.3c}$$

This system has at the same time three different sets of first integrals of motion. In the first case, the system can be regarded as composed of three uncoupled simple harmonical oscillators. The first integrals of motion in involution are

$$I_j=\frac{1}{2}(q_j^2+p_j^2),\quad (j=1,2,3). \tag{2.4}$$

We can find immediately the raising and lowering operators

$$b_j^\dagger=\sqrt{\frac{1}{2\hbar}}(q_j-ip_j),\quad b_j=\sqrt{\frac{1}{2\hbar}}(q_j+ip_j), \tag{2.5}$$

as simultaneously eigensolutions of eigenequations of $I_{k=1,2,3}$ with different eigenvalues [20]. The pairs of conjugate operators $b_j^\dagger,b_j$ together with I_j form a Heisenberg Weyl algebra,

$$\frac{1}{\hbar}[I_j,b_j^\dagger]=b_j^\dagger,\quad \frac{1}{\hbar}[I_j,b_j]=-b_j,\quad [b_j^\dagger,b_j]=1,\quad (j=1,2,3). \tag{2.6}$$

Then the general simultaneous orthonormal eigenstates of $I_{k=1,2,3}$ are

$$|\phi^{(I)}_{m_1m_2m_3}\rangle=C^{(I)}_{m_1m_2m_3}(b_1^\dagger)^{m_1}(b_2^\dagger)^{m_2}(b_3^\dagger)^{m_3}|0\rangle,$$

$$b_1|0\rangle=b_2|0\rangle=b_3|0\rangle=0. \tag{2.7}$$

The closed Lie algebra for this case is just the direct sum of these three subalgebras.

In the second case, the system can be regarded as composed of two uncoupled subsystems, the first one consisting of the third degree of freedom is just a simple harmonic oscillator while the second one consisting of other two degrees of freedom just behaves as an axially isotropic harmonic oscillator. The closed Lie algebra for this case is the direct sum of subalgebras of these two uncoupled subsystems. The subalgebra of the first subsystem has already been considered. For the second subsystem, the first integrals of motion in involution are

$$L_3=q_1p_2-q_2p_1,\quad J_{II}=\frac{1}{2}\sum_{j=1,2}(q_j^2+p_j^2). \tag{2.8}$$

The first integral L_3 characterizes the axial symmetry of the Hamiltonian of the subsystem, while the first integral J_{II} which commutes with L_3 just characterizes the motion of such a subsystem with axial symmetry. We can find raising and lowering operators

$$\frac{1}{\sqrt{2}}(b_1^\dagger\pm ib_2^\dagger),\quad \frac{1}{\sqrt{2}}(b_1\mp ib_2);$$
$$\frac{1}{2\sqrt{2}}\sum_{j=1,2}b_j^\dagger b_j^\dagger,\quad \frac{1}{2\sqrt{2}}\sum_{j=1,2}b_jb_j, \tag{2.9}$$

as simultaneous solution of eigenequations of L_3 and J_{II} with different eigenvalues. These raising and lowering operators together with corresponding integrals of motion form Lie subalgebras

$$\left\{h(4):\frac{1}{2\hbar}(J_{II}\pm L_3),\frac{1}{\sqrt{2}}(b_1^\dagger\pm ib_2^\dagger),\frac{1}{\sqrt{2}}(b_1\mp ib_2),1\right\}, \tag{2.10a}$$

$$\left\{ so(2,1): \frac{1}{2} \sum_{j=1,2} \left(b_j^\dagger b_j + \frac{1}{2} \right), \frac{1}{2\sqrt{2}} \times \sum_{j=1,2} (b_j^\dagger b_j^\dagger), \frac{1}{2\sqrt{2}} \sum_{j=1,2} (b_j b_j) \right\}. \quad (2.10b)$$

Noticing that L_3 commutes with $1/2\sqrt{2}\Sigma_{j=1,2}(b_j^\dagger b_j^\dagger)$, $1/2\sqrt{2}\Sigma_{j=1,2}(b_j b_j)$ but J_{II} does not commute with $1/\sqrt{2}(b_1^\dagger \pm i b_2^\dagger)$, $1/\sqrt{2}(b_1 \mp i b_2)$, the first integral of motion L_3 corresponding to the axial symmetry constraining should be taken as an intrinsic property and fixed first before considering the motion of this subsystem with axial symmetry. The closed Lie subalgebra of this subsystem should be taken as the semidirect sum of the Heisenberg algebra $\{b_1^\dagger, b_1, b_2^\dagger, b_2^\dagger, 1\}$ and

$$\{so(2):L_3\} \oplus \left\{ so(2,1): \frac{1}{2} \sum_{j=1,2} \left(b_j^\dagger b_j + \frac{1}{2} \right), \frac{1}{2\sqrt{2}} \times \sum_{j=1,2} (b_j^\dagger b_j^\dagger), \frac{1}{2\sqrt{2}} \sum_{j=1,2} (b_j b_j) \right\}.$$

The corresponding simultaneous orthonormal eigenstates of the independent integrals of motion are

$$|\phi_{nln_3}^{(II)}\rangle = C_{nln_3}^{(II)} \left(\frac{1}{2\sqrt{2}} \sum_{j=1,2} b_j^\dagger b_j^\dagger \right)^x \times \left[\frac{1}{\sqrt{2}} (b_1^\dagger \pm i b_2^\dagger) \right]^y (b_3^\dagger)^{n_3} |0\rangle,$$

$$(n = 2x + y, l = \pm y, x = 0,1,2, \ldots, y = 0,1,2, \ldots, n_3 = 0,1,2, \ldots). \quad (2.11)$$

In the third case, the system can be regarded as a spherically isotropic harmonic oscillator. The Hamiltonian of the system is invariant to the rotation about any axis, and thus can commute with the Casimir operator L^2 of the SO(3) group. The three operators

$$L_3 = (q_1 p_2 - q_2 p_1), L^2 = L_1^2 + L_2^2 + L_3^2, \quad J_{III} = \frac{1}{2} \sum_{j=1,2,3} (p_j^2 + q_j^2) \quad (2.12)$$

form a complete set of commutable operators, the raising and lowering operators

$$\frac{1}{\sqrt{2}}(b_1^\dagger \pm i b_2^\dagger), \frac{1}{\sqrt{2}}(b_1 \mp i b_2); \frac{1}{\sqrt{2}}(L_1 + iL_2), \frac{1}{\sqrt{2}}(L_1 - iL_2); \quad (2.13a)$$

$$\frac{1}{2\sqrt{2}} \sum_{j=1,2,3} (b_j^\dagger b_j^\dagger), \frac{1}{2\sqrt{2}} \sum_{j=1,2,3} (b_j b_j)$$

can be obtained as simultaneous solutions of eigenequations of L_3, L^2, and J_{III} with different eigenvalues. Noticing that L_3 and L^2 commute with $1/2\sqrt{2}\Sigma_{j=1,2,3}(b_j^\dagger b_j^\dagger)$, $1/2\sqrt{2}\Sigma_{j=1,2,3}(b_j b_j)$ but L^2 and J_{III} do not commute with $1/\sqrt{2}(b_1^\dagger \pm i b_2^\dagger)$, $1/\sqrt{2}(b_1 \mp i b_2)$, the first integrals of motion L_3 and L^2 should be taken as intrinsic properties and fixed first before considering the motion of other two degrees of freedom. The closed Lie algebra for this case should be taken as the semidirect sum of the Heisenberg algebra $\{b_1^\dagger, b_1, b_2^\dagger, b_2, b_3^\dagger, b_3, 1\}$ and

$$\left\{ so(3): L_3, \frac{1}{\sqrt{2}}(L_1 + iL_2), \frac{1}{\sqrt{2}}(L_1 - iL_2) \right\} \oplus \left\{ so(2,1): \frac{1}{2} \sum_{j=1,2,3} \left(b_j^\dagger b_j + \frac{1}{2} \right), \frac{1}{2\sqrt{2}} \times \sum_{j=1,2,3} (b_j^\dagger b_j^\dagger), \frac{1}{2\sqrt{2}} \sum_{j=1,2,3} (b_j b_j) \right\}. \quad (2.13b)$$

We can use the property of the lowest-weight state to find first the simultaneous orthonormal eigenstates of J_{III}, L^2, and L_3 with eigenvalues of L^2 and L_3 equal to $l(l+1)\hbar^2$ and $-l\hbar$, respectively, and corresponding eigenvalue of J_{III} being $n\hbar = (2x + l)\hbar$

$$|\phi_{nl-l}^{(III)}\rangle = C_{nl-l}^{(III)} \left[\frac{1}{2\sqrt{2}} \sum_{j=1,2,3} (b_j^\dagger b_j^\dagger) \right]^x \left[\frac{1}{\sqrt{2}} (b_1^\dagger - i b_2^\dagger) \right]^l |0\rangle,$$

$$(n = 2x + l, x, l = 0,1,2, \ldots). \quad (2.14)$$

Having obtained the lowest weight state $|\phi_{nl-l}^{(III)}\rangle$ we can obtain the general simultaneous orthonormal eigenstates

$$|\phi_{nlm}^{(III)}\rangle = C_{nlm}^{(III)} \left[\frac{1}{2\sqrt{2}} \sum_{j=1,2,3} (b_j^\dagger b_j^\dagger) \right]^x \times \left[\frac{1}{\sqrt{2}} (L_1 + iL_2) \right]^y \left[\frac{1}{\sqrt{2}} (b_1^\dagger - i b_2^\dagger) \right]^l |0\rangle,$$

$$(n = 2x + l, x = 0,1,2, \ldots, l = 0,1,2, \ldots, m = y - l, y = 0,1,2, \ldots 2l). \quad (2.15)$$

We have given in the above three complete sets of orthonormal eigenstates of the integrable system H^0 corresponding to three different cases. Though these are well-known results, we shall extract general characters from them in order to characterize the integrability of the system generally.

First, the three pairs of raising and lowering operators are obtained as simultaneous solutions of eigenequations of commutable dynamical variables with different eigenvalues, these three pairs of raising and lowering operators are independent of each other. Thus they can be used as three pairs of conjugate dynamical variables in the place of the original pairs of basic canonically conjugate dynamical variables.

Second, any pair of raising and lowering operators together with corresponding first integrals of motion form a closed Lie algebra. This fact shows us that the orthonormal eigenstates of H^0 thus obtained form indeed a complete basic set for the state space of the system H^0.

Due to the Jacobi identity between three operators A_1,A_2,A_3,

$$[[A_1,A_2],A_3]+[[A_2,A_3],A_1]+[[A_3,A_1],A_2]=0, \tag{2.16}$$

any pair of the raising and lowering operators $\Gamma^\dagger,\Gamma$ and the commutator $[\Gamma^\dagger,\Gamma]$ form a closed Lie algebra if and only if

$$[\Gamma,\Gamma^\dagger]=1 \quad \text{or} \quad \pm\frac{J}{\hbar}. \tag{2.17}$$

In the first case,

$$\frac{1}{\hbar}J=\Gamma^\dagger\Gamma+\frac{1}{2},\ \ \frac{1}{\hbar}[J,\Gamma^\dagger]=\Gamma^\dagger, \qquad \frac{1}{\hbar}[J,\Gamma]=-\Gamma,\ \ [\Gamma,\Gamma^\dagger]=1, \tag{2.18}$$

the three operators $J,\Gamma^\dagger,\Gamma$ form the Heisenberg-Wyle algebra. In the latter two cases,

$$\frac{1}{\hbar}[J,\Gamma^\dagger]=\Gamma^\dagger,\ \ \frac{1}{\hbar}[J,\Gamma]=-\Gamma,\ \ [\Gamma,\Gamma^\dagger]=\pm\frac{J}{\hbar}, \tag{2.19}$$

the three operators $J,\Gamma^\dagger,\Gamma$ form so(2,1) and so(3) algebras, respectively. Therefore the closed Lie algebras involved can be only of these three types.

Third, all these three types of closed Lie algebras connect the pair of conjugate dynamical variables $\Gamma^\dagger,\Gamma$ directly to the third Hermitian dynamical variable $J/\hbar$. Therefore it is possible to take J as action variable and introduce in addition the corresponding angle variable θ,

$$\frac{1}{i\hbar}[\theta,J]=1, \tag{2.20}$$

and express the transformation relation between $\Gamma^\dagger$, Γ and J,θ generally as

$$\Gamma=e^{-i\theta}f\left(\frac{J}{\hbar}\right), \quad \Gamma^\dagger=f\left(\frac{J}{\hbar}\right)e^{i\theta}, \tag{2.21}$$

where $f(J/\hbar)$ is a function to be determined. Inserting these formulas to Eq. (2.17), we have

$$\left[\exp\left(\hbar\frac{\partial}{\partial J}\right)-1\right]f^2\left(\frac{J}{\hbar}\right)=\begin{cases}1,\\ \pm\dfrac{J}{\hbar}.\end{cases} \tag{2.22}$$

Solving this equation we find

$$f\left(\frac{J_j}{\hbar}\right)=\begin{cases}\sqrt{\dfrac{J}{\hbar}-\dfrac{1}{2}},\\ \sqrt{\mp\dfrac{J}{\hbar}\left(1-\dfrac{J}{\hbar}\right)}.\end{cases} \tag{2.23}$$

Inserting these results to Eq. (2.21) we have

$$\theta=\begin{cases}\dfrac{1}{2}\left\{(-i)\ln\left[\left(\dfrac{J}{\hbar}-1\right)^{-1/2}\right]\Gamma^\dagger+\text{H.c.}\right]\right\},\\ \dfrac{1}{2}\left\{(-i)\ln\left[\mp\dfrac{J}{\hbar}\left(1-\dfrac{J}{\hbar}\right)^{-1/2}\right]\Gamma^\dagger+\text{H.c.}\right]\right\}\end{cases}. \tag{2.24}$$

Together with the relations,

$$\frac{J}{\hbar}=\begin{cases}\Gamma^\dagger\Gamma+\dfrac{1}{2}\\ \pm[\Gamma,\Gamma^\dagger],\end{cases} \tag{2.25}$$

we have obtained the transformation between $(\Gamma,\Gamma^\dagger)$ and (θ,J). As pointed out before, $\Gamma,\Gamma^\dagger$ are used as a pair of conjugate dynamical variables and are expressed explicitly with basic dynamical variables, the transformation between $(\Gamma,\Gamma^\dagger)$ and (θ,J) can lead to the canonical transformation between (q,p) and (θ,J).

With such a transformation between (q,p) and (θ,J), the Hamiltonian of the system can be expressed with action variables alone and independent of angle variables θ, thus the first integrals of motion can be expressed only in terms of action variables. In the same spirit as the space-time symmetry, the integrable system is said to have a dynamical symmetry in the state space and the corresponding closed Lie algebra will be designated as the dynamical symmetry algebra and regarded as the characteristic of an integrable system.

Fourth, as discussions given before are based fully on algebraic relations expressed with quantum Poisson brackets, the obtained results can be readily transferred into classical mechanics by expressing the algebraic relations with classical Poisson brackets. While the quantum-classical correspondence can be explicitly exhibited with steadily decreasing effective Planck constants.

Fifth, in classical mechanics, the torus in the phase space of an integrable system invariant to the phase flow is designated with values of the first integrals of motion, while in quantum mechanics the subspace

$$\rho_{\underline{m}}=|\phi_{\underline{m}}\rangle\langle\phi_{\underline{m}}|, \tag{2.26}$$

of the state space of an integrable system invariant to the Schrödinger flow must be designated by discrete eigenvalues $\underline{m}$ of first integrals of motion $\underline{J}$.

The previous discussions for the three-dimensional isotropic harmonic oscillator in principle can be extended to systems with more than three degrees of freedom or to more complicated systems. The general concept is still valid there. But the admitted dynamical symmetry algebra is usually not given with the model system, it is essential to carry out the technical work of finding out the dynamical symmetry algebra admitted by the model system. For example, the Hamiltonian of the electron in the hydrogen atom moving under the Coulomb potential is invariant to SO(4) group transformations. Hence we must consider the correspondent SO(4)-invariant subalgebra. By enlarging the SO(4) group to SO(4,2) group, we have the required SO(2,1) subgroup with generators L_{46}, L_{45}, and L_{56} [21] (notations same as in Ref. [21] have been used here). The Schrödinger equation for the hydrogen atom has an energy-dependent boundary condition, hence we must consider the Schrödinger equation itself instead of the Hamiltonian of the hydrogen atom. Then, by first left-multiplying the Schrodinger equation by $(L_{56}-L_{46})$, we obtain an equation expressed linearly in group generators L_{56} and L_{46} of the SO(2,1) subgroup. In order to obtain bound states of the hydrogen atom we can eliminate the term involving L_{46} leaving the equation expressed with L_{56} alone with a nonunitary transformation. In this way, we have obtained the required SO(4)-invariant subalgebra in which L_{56} plays the role of the action variable.

III. DYNAMICAL-SYMMETRY-PRESERVING (DSP) LIE TRANSFORMATIONS AND INTEGRABLE CLASS OF SYSTEMS HAVING THE SAME DYNAMICAL SYMMETRY

It has been discussed in Sec. II that the integrability of a known integrable system can be profoundly characterized by its dynamical symmetry algebra, which enables one to express the Hamiltonian in terms of actions alone free from angle variables. Thus the existence of first integrals of motion is closely related to the fact that the Hamiltonian of the system is invariant to the rotation with respect to angle variables. The problem whether or not a class of systems have the same dynamical symmetry as this system can thus be studied with the continuous transformation group which keeps the dynamical symmetry as its invariant property. For this purpose, we introduce an auxiliary Hamiltonian

$$H(\lambda)=(1-\lambda)H^0+\lambda H', \tag{3.1}$$

where H^0 denotes the known integrable system as given by Eq. (2.3a), the commutation relation between q_j and p_j remains the same, while H' denotes the three-dimensional quartic oscillator given as follows:

$$H'=\sum_{j=1,2,3}\frac{1}{2}(p_j^2+q_j^2)+\sum_{j=1,2,3}\frac{\alpha_j}{2}q_j^4+\sum_{j,k=1,2,3}\gamma_{jk}q_j^2q_k^2. \tag{3.2}$$

Such a system has the same boundary condition as the system H^0, the state space can be spanned by same basis sets as for the known integrable system H^0. In special case (I) $\gamma_{jk}=0$, $(j,k=1,2,3)$, there does not exist any coupling between the three degrees of freedom. In special case (II), $\alpha_1=\alpha_2=\gamma_{12}$, $\gamma_{23}=\gamma_{31}=0$, the subsystem composed of the first and second degrees of freedom has the isotropic property and moves independently from another subsystem composed of the third degree of freedom. In special case (III) $\alpha_1=\alpha_2=\alpha_3=\gamma_{12}=\gamma_{23}=\gamma_{31}$, the whole system has the spherical isotropy. In such cases, the Hamiltonian can be expressed with action-angle variables determined from H^0 for the corresponding dynamical symmetry. However, due to the presence of quartic terms, the Hamiltonian of the whole system or independent subsystems is not free from angle variables. Now the problem is just to eliminate the angle variables there.

Since the dynamical symmetry algebra is expressed as direct and semidirect sum of corresponding subalgebras, the elimination of angle variables can be carried out for each degree of freedom independently. The Hamiltonian of the subsystem with a single degree of freedom originally realized as $\{A_\nu^0:J^0,\Gamma^{0\dagger},\Gamma^0\}$ is now realized as $\{A_\nu(\lambda):J(\lambda),\Gamma^\dagger(\lambda),\Gamma(\lambda)\}$ where λ denotes a parameter varying continuously in a certain range of values. They can be transformed with each other as

$$A_\mu(\lambda)=U(\{A_\nu^0\},\lambda)A_\mu^0U^\dagger(\{A_\nu^0\},\lambda), \tag{3.3}$$

$$A_\mu^0=U^\dagger(\{A_\nu(\lambda)\},\lambda)A_\mu(\lambda)U(\{A_\nu(\lambda)\},\lambda). \tag{3.4}$$

Here $U(\{A_\nu^0\},\lambda)$ is a unitary transformation

$$U(\{A_\nu^0\},\lambda)=\exp[S(\{A_\nu^0\},\lambda)], \tag{3.5}$$

$S(\{A_\nu^0\},\lambda)$ is an anti-Hermitian function expressed as power series of $\{A_\nu^0\}$ and λ, Eq. (3.4) is just the inverse transformation of Eq. (3.3).

Substituting the relation (3.4) to the Hamiltonian $H(\{A_\nu^0\},\lambda)$ we have

$$\begin{aligned}H(\{A_\nu^0\},\lambda)&=U^\dagger(\{A_\nu(\lambda)\},\lambda)H(\{A_\nu(\lambda)\},\lambda)U(\{A_\nu(\lambda)\},\lambda)\\&=\mathcal{H}(\{A_\nu(\lambda)\},\lambda).\end{aligned} \tag{3.6}$$

The Hamiltonian expressed in $\{A_\nu(\lambda)\}$ has a new form $\mathcal{H}(\{A_\nu(\lambda)\},\lambda)$. With an appropriately chosen Lie transformation, $\mathcal{H}(\{A_\nu(\lambda)\},\lambda)$ can be expressed with $J(\lambda)$ alone free from $\Gamma^\dagger(\lambda),\Gamma(\lambda)$ or $\theta(\lambda)$. Though the hierarchy of equations for determining the series expansion of the anti-Hermitian generating function can be written down [15], but the explicit solution can hardly be found. We can only ascertain its existence implicitly by using Kolmogorov's superconvergent perturbation [22]. Recently, Scherer [23] has applied the Kolmogorov's superconvergent perturbation theory to the Hamiltonian operator in quantum mechanics. While for our purpose we must apply the Kolmogorov's superconvergent series to the continuous dynamical-symmetry-preserving (DSP) Lie transformation [17].

According to the algebraic relations of the subalgebra given as Eqs. (2.15)–(2.17), the Hamiltonian $H(\{A_\nu^0\},\lambda)$ can always be expressed as a Fourier series of $e^{im\theta}$. Hence $H(\{A_\nu^0\},\lambda)$ can be written generally as

$$H(\{A_\nu^0\},\lambda)=H_0(J^0,\lambda)+\sum_{p=1,2,\ldots}\lambda V_p(\{A_\nu^0\}), \tag{3.7a}$$

$$V_p(\{A_\nu^0\})=\sum_{m=1,2,\ldots}[(\Gamma^{0\dagger})^m v_{pm}(J^0)+\text{H.c.}]. \tag{3.7b}$$

The elimination of terms involving $\Gamma^{0\dagger}$ and Γ^0 can be carried out in successive steps. In the first step, the Hamiltonian originally expressed with $\{A_\nu^0\}$ is expressed by $\{A_\nu^{(1)}(\lambda)\}$, terms involving $\Gamma^{(1)\dagger}$ and $\Gamma^{(1)}$ retained are reduced to the order of magnitude $O(\lambda^2)$. In the second step the Hamiltonian originally expressed with $\{A_\nu^{(1)}(\lambda)\}$ is expressed by $\{A_\nu^{(2)}(\lambda)\}$, terms involving $\Gamma^{(2)\dagger}$ and $\Gamma^{(2)}$ retained are reduced to the order of magnitude $O(\lambda^4)$. In order to eliminate terms involving $\Gamma^{(n)\dagger}$ and $\Gamma^{(n)}$ associated with $\lambda^p(p=2^{(n-1)},2^{(n-1)}+1,\ldots,2^n-1)$ in the nth step, we take

$$U(\{A_\nu^{(n)}(\lambda)\},\lambda)=\exp\left[\sum_{p=2^{(n-1)}}^{2^n-1}\lambda^p S_p(\{A_\nu^{(n)}(\lambda)\},\lambda)\right], \tag{3.8}$$

$$S_p(\{A_\nu^{(n)}(\lambda)\},\lambda)=\sum_{m=1,2,\ldots}[\{\Gamma^{(n)\dagger}(\lambda)\}^m u_{pm}(J^{(n)}(\lambda))-\text{H.c.}]. \tag{3.9}$$

Substituting Eqs. (3.8) and (3.9) into (3.6) and carrying out the transformation explicitly, we have

$$U^{(n)}H^{(n-1)}U^{(n)}=H_0^{(n-1)}+\sum_{p=2^{(n-1)}}^{2^n-1}\lambda^p\{[S_p^{(n)},H_0^{(n-1)}]+V_p^{(n-1)}\}+O(\lambda^{2n}). \tag{3.10}$$

Because the anti-Hermitian generating function given by Eq. (3.9) is of the similar form as the Hermitian perturbing term $V^{(n-1)}$ given by Eq. (3.7b). The yet unknown functions $u_{pm}^{(n)}(J^{(n)}(\lambda))$ can be determined individually from Eqs. (3.10)

$$[S_p^{(n)},H_0^{(n-1)}]+V_p^{(n-1)}=0,\quad (p=2^{(n-1)},\ 2^{(n-1)}+1,\ldots 2^n-1). \tag{3.11}$$

After a straightforward calculation we have

$$u_{pm}^{(n)}(J^{(n)}(\lambda))=v_{pm}^{(n-1)}(J^{(n)}(\lambda))[\hbar\omega_{pm}^{(n-1)}(J^{(n)}(\lambda))]^{-1},\quad (p=2^{(n-1)},2^{(n-1)}+1,\ldots 2^n-1), \tag{3.12}$$

with

$$\hbar\omega_{pm}^{(n-1)}(J^{(n)}(\lambda))=\left\{\left[\exp\left(2m\hbar\frac{\partial}{\partial J}\right)-1\right]H_0^{(n-1)}(J)\right\}_{J=J^{(n)}(\lambda)}. \tag{3.13}$$

Since $H_0^{(n-1)}$ is the Hamiltonian of the subsystem with a single degree of freedom or the whole system under definite symmetry constrainings, $\hbar\omega_{pm}^{(n-1)}(J^{(n)}(\lambda))$ cannot vanish. We have thus obtained $H^{(n)}(\{A_\nu^{(n)}(\lambda)\},\lambda)$ which can be expressed again in the form of Eqs. (3.7a) and (3.7b), terms involving $\Gamma^{(n)\dagger}$ and $\Gamma^{(n)}$ retained are of the order of magnitude $O(\lambda^{2n})$. For a sufficiently small λ_1, the required Lie transformation can always be found. With the notion of analytical extension, we can take the system $H(\lambda=\lambda')$ instead of $H(\lambda=0)$ as reference to carry out the same kind of study. By repeating such studies again and again, we can eventually ascertain that $H(\lambda=1)$ is also an integrable system of the same dynamical symmetry as $H(\lambda=0)$ and is expressed only with action variable $J(\lambda)$ in the new realization

$$J(\lambda)=U(\{A_\nu^0\},\lambda)J^0U^\dagger(\{A_\nu^0\},\lambda). \tag{3.14}$$

But we should notice that the Hamiltonian is generally a nonlinear function of $J(\lambda)$.

Since the eigenfunctions of the integrable Hamiltonian $H(\lambda)$ are simultaneously eigenfunctions of action variables in the new realization, the orthonormal eigenstates of $H(\lambda)$ are thus related to the corresponding orthonormal eigenstates of $H(\lambda=0)$ by the Lie transformation $U(\{A_\nu^0\},\lambda)$

$$|\phi_{\underline{m}}(\lambda)\rangle=U(\{A_\nu^0\},\lambda)|\phi_{\underline{m}}^0\rangle, \tag{3.15}$$

$$U(\{A_\nu^0\},\lambda)=\sum_{\underline{m}}|\phi_{\underline{m}}(\lambda)\rangle\langle\phi_{\underline{m}}^0|. \tag{3.16}$$

From Eq. (3.15) we see that the orthonormal eigenstates $|\phi_{\underline{m}}(\lambda)\rangle$ of the system $H(\lambda)$ in one-to-one correspondence to orthonormal eigenstates $|\phi_{\underline{m}}^0\rangle$ of H^0 have the following properties: (i) $\lim_{\lambda\to 0}|\phi_{\underline{m}}(\lambda)\rangle=|\phi_{\underline{m}}^0\rangle$, (ii) $|\phi_{\underline{m}}(\lambda)\rangle$ vary continuously with λ adiabatically in the same manner as $H(\lambda)$ and (iii) $\langle\phi_{\underline{m}}(\lambda)|\phi_{\underline{m}}(\lambda+\varepsilon)\rangle=1+O(\varepsilon^2)$. Therefore Eq. (2.16) implies that if $H(\lambda)$ belongs to the same integrable class as H^0, orthonormal eigenstates of $H(\lambda)$ in one-to-one correspondence to those of H^0 should be obtainable with the method of iterative perturbation [24,25].

On the contrary, the breaking of the dynamical symmetry algebra implies that there exist couplings between at least two degrees of freedom. We shall particularly examine behaviors of the subsystem $H'(J_i,\Gamma_i^\dagger,\Gamma_i)$ $(i=1,2)$ composed of the two degrees of freedom. Let us consider an auxiliary Hamiltonian

$$H'(J_i,\Gamma_i^\dagger,\Gamma_i)=(1-\lambda)H_0(J_i)+\lambda H'(J_i,\Gamma_i^\dagger,\Gamma_i), \tag{3.17}$$

and write it generally in the following form:

$$H(J_i,\Gamma_i^\dagger,\Gamma_i,\lambda)=H_0(J_i,\lambda)+\sum_{p=1,2,\ldots}\lambda^pV_p(J_i,\Gamma_i^\dagger,\Gamma_i), \tag{3.18a}$$

$$V_p(J_i,\Gamma_i^\dagger,\Gamma_i)=\sum_{m_1m_2\ldots}[(\Gamma_1^\dagger)^{m_1}(\Gamma_2^\dagger)^{m_2}u_{pm_1m_2}(J_i)+\text{H.c.}], \tag{3.18b}$$

arguments $(J_i,\Gamma_i^\dagger,\Gamma_i)$ should be read as $(J_i,\Gamma_i^\dagger,\Gamma_i,i=1,2)$, $i=1,2$ is omitted here. Corresponding to Eqs. (3.8) and (3.9) we should have

$$U^{(n)}(J_i^{(n)}(\lambda),\Gamma_i^{(n)\dagger}(\lambda),\Gamma_i^{(n)}(\lambda)) =\exp\left[\sum_{p=2^{(n-1)}}^{2^n-1}\lambda^p S_p(J_i^{(n)}(\lambda),\Gamma_i^{(n)\dagger}(\lambda),\Gamma_i^{(n)}(\lambda))\right] \quad (3.19)$$

$$S_p(J_i^{(n)}(\lambda),\Gamma_i^{(n)\dagger}(\lambda),\Gamma_i^{(n)}(\lambda)) =\sum_{m_1m_2=1,2,\ldots}[(\Gamma_1^{(n)\dagger})^{m_1}(\Gamma_2^{(n)\dagger})^{m_1}u_{pm_1m_2}(J_i^{(n)})-\text{H.c.}]. \quad (3.20)$$

In order to eliminate terms involving $\Gamma_1^{(n)\dagger}(\lambda),\Gamma_1^{(n)}(\lambda)$, and $\Gamma_2^{(n)\dagger}(\lambda),\Gamma_2^{(n)}(\lambda)$ associated with $\lambda^p,(p=2^{(n-1)},2^{(n-1)}+1,\ldots,2^n-1)$ in the nth step, Eqs. (3.12) and (3.13) should be expressed as

$$u_{pm_1m_2}(J^{(n)}(\lambda))=v_{pm_1m_2}^{(n-1)}(J_i^{(n)}(\lambda))[\hbar\omega_{pm_1m_2}^{(n-1)}(J_i^{(n)}(\lambda))]^{-1}, \quad (p=2^{(n-1)},2^{(n-1)}+1,\ldots,2^n-1) \quad (3.21)$$

with

$$\hbar\omega_{pm_1m_2}^{(n-1)}(J_i^{(n)}(\lambda)) =\left\{\left[\exp\left(\sum_{i=1,2}\hbar m_i\frac{\partial}{\partial J_i}\right)-1\right]H_0^{(n-1)}(J_i)\right\}_{J_i=J_i^{(n)}(\lambda)}. \quad (3.22)$$

Since $H_0^{(n-1)}(J_i(\lambda))$ is a nonlinear function of $J_i^{(n)}(\lambda)$, as λ increases there will eventually occur the case

$$H_0(J_i^{(n)}(\lambda)+m_i\hbar)-H_0(J_i^{(n)}(\lambda))=0, \quad (3.23)$$

corresponding to commensurable frequencies. It is generally impossible to obtain for sufficiently large values of λ the implicit solution of the required Lie transformation to reduce the Hamiltonian to the form $\mathcal{H}(J_i(\lambda))$. As the level spacing tends to zero in the classical limit $\hbar\to 0$, the condition for the commensurability of frequencies may even occur at infinitesimally small value of λ in the classical limit.

Due to the dynamical symmetry breaking, there will not exist the analytical relation of the form (3.15) for one-to-one correspondent orthonormal eigenstates of integrable system H^0 and nonintegrable system $H(\lambda)$. Therefore, we conclude that if and only if $H(\lambda)$ belongs to the same integrable class as H^0, orthonormal eigenstates of $H(\lambda)$ in one-to-one correspondence to those of H^0 can be obtained with the method of iterative perturbation. Though the explicit form of the required DSP Lie transformation cannot be explicitly found, but it can be expressed in the form (3.16) and numerically obtained with one-to-one correspondent orthonormal eigenstates by using the method of iterative perturbation.

IV. QUANTUM DYNAMICS WITH COMPLETE CLASSICAL CORRESPONDENCE

In Secs. II and III, we have demonstrated the outstanding property of the dynamical symmetry with which the integrability of a quantum system can be clearly exhibited by the existence of subspaces invariant to the Schrödinger flow and thus able to be characterized by definite values of action variables. However, in addition to this kind of stationary problem, we have to consider the general dynamical problem which deals with the solution of the dynamical equation corresponding to a given initial state. In classical mechanics, the initial state is naturally chosen as the phase point described with pairs of conjugate dynamical variables corresponding to a certain dynamical symmetry. While in quantum mechanics, the initial state must be chosen as the minimum-uncertainty state expressed in an analytical form and characterized by expectation values of same pairs of conjugate dynamical variables. Such an initial quantum state tends to the corresponding phase point in classical limit $\hbar\to 0$. The quantum dynamics formulated in this way has evidently complete classical correspondence.

The ground state of an integrable quantum system is almost localized to the classical phase point with minimum uncertainty but still has a zero-point fluctuation. Such a state can be found with the dynamical symmetry algebra and can be sent to any desired place with an element of the corresponding dynamical symmetry group. We shall use such states as initial states to obtain corresponding solutions of quantum dynamical equations.

The ground state of the integrable system corresponding to a certain dynamical symmetry should be determined from the subalgebra related to the Hamiltonian of the whole system or subalgebras related to independent subsystems. These subalgebras must be of the three types discussed in Sec. II. We shall discuss them individually as follows. In order to avoid confusion, corresponding to a dynamical variable A, the quantum operator and its expectation value are denoted by $\hat{A}$ and $\overline{A}$, respectively, in this section.

In the Heisenberg-Weyl algebra, the ground state $|\phi_0\rangle$ has expectation values of $\hat{q}$ and $\hat{p}$ both equal to zero,

$$\langle\phi_0|\hat{q}|\phi_0\rangle=\langle\phi_0|\hat{p}|\phi_0\rangle=0. \quad (4.1)$$

Thus we have

$$\langle\phi_0|[(\hat{q}-\langle\phi_0|\hat{q}|\phi_0\rangle)^2+(\hat{p}-\langle\phi_0|\hat{p}|\phi_0\rangle)^2]|\phi_0\rangle =\langle\phi_0|(\hat{q}^2+\hat{p}^2)|\phi_0\rangle=\hbar. \quad (4.2)$$

The displaced ground state is given as

$$|\Phi_\gamma^0\rangle=\exp\left[\frac{i}{\hbar}(\overline{p}\hat{q}-\overline{q}\hat{p})\right]|\phi_0\rangle, \quad \gamma=\overline{q}+i\overline{p}, \quad (4.3)$$

which has expectation values

$$\langle\Phi_\gamma^0|\hat{q}|\Phi_\gamma^0\rangle=\langle\phi_0|(\hat{q}+\overline{q})|\phi_0\rangle=\overline{q}, \quad (4.4)$$

$$\langle\Phi_\gamma^0|\hat{p}|\Phi_\gamma^0\rangle=\langle\phi_0|(\hat{p}+\overline{p})|\phi_0\rangle=\overline{p}. \tag{4.5}$$

The state $|\Phi_\gamma^0\rangle$ is characterized by the expectation values $\overline{q},\overline{p}$ in correspondence to the coordinates of a classical phase point. Corresponding to Eq. (4.2) we have

$$\langle\Phi_\gamma^0|[(\hat{q}-\overline{q})^2+(\hat{p}-\overline{p})^2]|\Phi_\gamma^0\rangle=\langle\phi_0|(\hat{q}^2+\hat{p}^2)|\phi_0\rangle=\hbar. \tag{4.6}$$

Evidently the uncertainty measure defined above is invariant to the displacement.

The above expression for uncertainty is symmetrical with respect to $\Delta\hat{q}$ and $\Delta\hat{p}$, it is usually defined as $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle\geqslant\hbar^2/4$. As the principle of uncertainty has been generally embodied in the fundamental Poisson bracket for basic canonically conjugate variables, while the results of Eq. (4.6) are obtained only for the case of the Heisenberg-Weyl subgroup which is particularly related to Euclidean phase space, we prefer to use Eq. (4.6) to characterize the minimum-uncertainty state and then generalize it to cases corresponding to SO(3) and SO(2,1) subgroups [26].

In the case of SO(3) and SO(2,1) with discrete infinite-dimensional representation $D^+(\Phi)$ bounded below, the ground state is the lowest-weight state $|\phi_-^0\rangle$ which is the simultaneous eigenstates of the Casimir operator $\hat{C}=\hat{J}_1^2+\hat{J}_2^2+g_{33}\hat{J}_3^2$ and $\hat{J}_3$ with eigenvalues $g_{33}\Phi(\Phi+1)\hbar^2$ and $(-\Phi\hbar)$, respectively. Here $g_{33}=+1,\Phi$ denotes a positive integer or half integer for the case SO(3), $g_{33}=-1$, $-1<\Phi<0$ for the case SO(2,1), the same notations as in Ref. [21] have been used here. Utilizing the properties of the lowest-weight state $|\phi_-^0\rangle$

$$\hat{J}_3|\phi_-^0\rangle=-\Phi\hbar|\phi_-^0\rangle,\quad \hat{J}_-|\phi_-^0\rangle=\langle\phi_-^0|\hat{J}_+=0 \tag{4.7}$$

and the algebraic relations

$$[\hat{J}_+\hat{J}_--\hat{J}_-\hat{J}_+]=2\hbar g_{33}\hat{J}_3 \tag{4.8}$$

we have

$$\begin{aligned}&\langle\phi_-^0|[(\Delta\hat{J}_1)^2+(\Delta\hat{J}_2)^2+g_{33}(\Delta\hat{J}_3)^2|\phi_-^0\rangle\\&=\langle\phi_-^0|\hat{J}_+\hat{J}_--\frac{1}{2}(\hat{J}_+\hat{J}_--\hat{J}_-\hat{J}_+)]|\phi_-^0\rangle\\&=\begin{cases}\Phi\hbar, & \mathrm{SO}(3)\\ -\Phi\hbar, & \mathrm{SO}(2,1).\end{cases}\end{aligned} \tag{4.9}$$

The lowest-weight state $|\phi_-^0\rangle$ among the whole set of orthonormal eigenstates of $\hat{J}_3$ gives minimum uncertainty.

For the rotated lowest-weight state

$$|\Phi^0(\alpha,\beta)\rangle=\hat{g}(\alpha,\beta)|\phi_-^0\rangle, \tag{4.10}$$

$$\hat{g}(\alpha,\beta)=\exp\left[\frac{i}{\hbar}(\alpha\hat{J}_1+\beta\hat{J}_2)\right], \tag{4.11}$$

the expectation values $\langle\Phi^0(\alpha,\beta)|\hat{J}_{\nu=1,2,3}|\Phi^0(\alpha,\beta)\rangle$ are

$$\begin{aligned}&\langle\Phi^0(\alpha,\beta)|\hat{J}_{\nu=1,2,3}|\Phi^0(\alpha,\beta)\rangle\\&=\langle\phi_-^0|\hat{g}^\dagger(\alpha,\beta)\hat{J}_{\nu=1,2,3}\hat{g}(\alpha,\beta)|\phi^0\rangle\end{aligned} \tag{4.12}$$

in which

$$\begin{aligned}\hat{g}^\dagger(\alpha,\beta)\hat{J}_\nu\hat{g}(\alpha,\beta)&=\Sigma\nu' C_{\nu\nu''}(\alpha,\beta)\hat{J}\nu\\&\quad\times\Sigma\nu C_{\nu\nu'}(\alpha,\beta)C_{\nu\nu''}(\alpha,\beta)\rangle\\&=\delta_{\nu'\nu''}.\end{aligned} \tag{4.13}$$

Hence the uncertainty measure

$$\begin{aligned}&\langle\Phi^0(\alpha,\beta)|[(\Delta\hat{J}_1)^2+(\Delta\hat{J}_2)^2+g_{33}(\Delta\hat{J}_3)^2]|\Phi^0(\alpha,\beta)\rangle\\&=\langle\phi_-^0|[(\Delta\hat{J}_1)^2+(\Delta\hat{J}_2)^2+g_{33}(\Delta\hat{J}_3)^2]|\phi_-^0\rangle\\&=\begin{cases}\Phi\hbar & \mathrm{SO}(3)\\ -\Phi\hbar & \mathrm{SO}(2,1)\end{cases}\end{aligned} \tag{4.14}$$

remains the same.

The lowest-weight state displaced with an element of the dynamical symmetry group has kept the required property of minimum uncertainty and tends to the classical phase point in the limiting case, thus it can be readily used as the initial state in formulating the quantum dynamics with complete classical correspondence. In this respect, it differs from the coherent state discussed by Zhang and co-workers [13,27].

Taking the minimum-uncertainty state as the initial state and denoting it generally as $|\psi^0\rangle$, we have the expectation value for the dynamical variable $\hat{A}^0$ at the initial instant $t=0$,

$$\overline{A^0}=\langle\psi^0|\hat{A}^0|\psi^0\rangle=\mathrm{Tr}(\hat{A}^0\hat{\rho}^0), \tag{4.15}$$

where

$$\hat{\rho}^0=|\psi^0\rangle\langle\psi^0|,\quad(\hat{\rho}^0)^2=\hat{\rho}^0 \tag{4.16}$$

is the density operator first introduced by von Neumann [28]. As $\overline{A(t)}$ should be uniquely determined at a definite instant, the variation of the state and operators with respect to t cannot be independent of each other. In the Heisenberg picture, states are independent of t, while operators for canonically conjugate variables vary with t,

$$\hat{\rho}_H=\hat{\rho}^0,\quad \hat{A}_H(t=0)=\hat{A}^0, \tag{4.17}$$

such that

$$\overline{A(t)}=\mathrm{Tr}(\hat{A}_H\hat{\rho}_H) \tag{4.18}$$

vary with t in correspondence to results in classical mechanics. We have thus

$$\frac{d\overline{A(t)}}{dt}=\mathrm{Tr}\left(\frac{d\hat{A}_H}{dt}\hat{\rho}^0\right). \tag{4.19}$$

In order to keep the quantum-classical correspondence we must have

$$\frac{d\hat{A}_H}{dt}=\frac{1}{i\hbar}[\hat{A}_H,\hat{H}]=\{\hat{A}_H,\hat{H}\}_{qu}. \tag{4.20}$$

For autonomous systems,

$$\hat{A}_H(t)=\exp\left[\frac{i}{\hbar}\hat{H}t\right]\hat{A}^0\exp\left[-\frac{i}{\hbar}\hat{H}t\right]. \tag{4.21}$$

With $\hat{A}_H(t)$ and $\hat{\rho}_H$ given by Eqs. (4.21) and (4.17), $\overline{A(t)}$ in (4.18) can also be expressed as

$$\overline{A(t)}=\mathrm{Tr}\left(\hat{A}^0\exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\rho}^0\exp\left[\frac{i}{\hbar}\hat{H}t\right]\right). \tag{4.22}$$

Therefore we have alternatively the Schrödinger picture in which operators for basic dynamical variables are independent of t while states of autonomous systems vary with t,

$$\hat{A}_s=\hat{A}^0, \quad \hat{\rho}_s(t=0)=\hat{\rho}^0, \tag{4.23}$$

such that

$$\overline{A(t)}=\mathrm{Tr}(\hat{A}_s\hat{\rho}_s). \tag{4.24}$$

Comparing it with Eq. (4.22), we have

$$\hat{\rho}_s(t)=\exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\rho}^0\exp\left[\frac{i}{\hbar}\hat{H}t\right] \tag{4.25}$$

or, equivalently,

$$\frac{\partial\hat{\rho}_s}{\partial t}=\frac{1}{i\hbar}[\hat{H},\hat{\rho}_s]=\{\hat{H},\hat{\rho}_s\}_{qu}, \quad \hat{\rho}_s(t=0)=\hat{\rho}^0, \quad (\hat{\rho}^0)^2=\hat{\rho}^0. \tag{4.26}$$

As the initial state is required to be a pure state, this equation of the form of Neumann equation has in fact the nature of Schrödinger equation. In transition to classical mechanics, we have correspondingly

$$\frac{\partial\rho_{cl}}{\partial t}=\{H_{cl},\rho_{cl}\}_{cl}, \quad \rho_{cl}(t=0)=\rho_{cl}^0, \quad (\rho_{cl}^0)^2=\rho_{cl}^0, \tag{4.27}$$

where ρ_{cl}^0 is just the classical counterpart of the minimum-uncertainty state $\hat{\rho}^0$. As the discussions about the minimum-uncertainty states are just based on algebraic relations expressed with quantum Poisson brackets, there exist certainly classical counterparts ρ_{cl}^0 which behave as δ functions in the phase space corresponding to different dynamical symmetries. Hence Eq. (4.27) of the form of Liouville equation can be used to describe a bundle of phase trajectories originating from an infinitesimally small cloud of phase points. In this way, we have indeed quantum dynamics with complete classical correspondence. As the correspondence principle has already been emphasized, dynamical equations in Heisenberg picture [Eqs. (4.17) and (4.20)] and in Schrödinger picture [Eqs. (4.23) and (4.26)] come out naturally.

The present form of quantum dynamics differs from the conventional one in considering solutions of the quantum dynamical equation for respective initial minimum uncertainty states. Since the linear superposition of two minimum uncertainty states is no longer a minimum uncertainty state, solutions of the quantum dynamical equation corresponding to different initial minimum uncertainty states will not obey the principle of superposition. Thus it is not a linear problem.

As the integrability and initial state are both discussed on the basis of dynamical symmetry, we have to consider two quite different situations: the evolution of an initial minimum-uncertainty state corresponding to a certain dynamical symmetry of H^0 under the action of (i) an integrable system belonging to the same integrable class as H^0 and (ii) a nonintegrable system without any kind of dynamical symmetry.

In the first case, the obtained solutions $\rho_\gamma(t)$ should be expressed as an analytical functional of ρ_γ^0 varying smoothly with γ and t. Various components of the initial state evolve essentially in the same way such that the coherence property of the initial state will be preserved during its evolution. While in the second case, due to the dynamical symmetry breaking, the obtained solutions $\rho_\gamma(t)$ can no longer be expressed as an analytical functional of ρ_γ^0. Various components of the initial state may evolve in quite different ways due to successive occurrences of nonlinear resonance. Thus the coherence property of the initial state may even be strongly violated.

In order to understand the violation of the general character of quantum regular motion more comprehensively, we shall give in the next section a detailed study of the general character of quantum regular motion together with numerical illustrations.

V. GENERAL CHARACTER OF QUANTUM REGULAR MOTION ILLUSTRATED WITH A LIPKIN MODEL

As discussed in the previous section, if we take the displaced ground state ρ_γ^0 of the known integrable system $H(\kappa=0)$ as an initial state to study its spatiotemporal evolution under the action of an integrable system $H(\kappa)$ belonging to the same integrable class as $H(\kappa=0)$, the obtained solution $\rho_\gamma(t)$ of the quantum dynamical equation is just an analytical functional of ρ_γ^0 varying smoothly with respect to γ and t. The solution $\rho_\gamma(t)$ is generally distorted but still behaves as a phase trajectory in the classical limit.

Let us consider as an example the two-level Lipkin [29] model which has been introduced to simulate collective monopole excitations of nuclei. The Hamiltonian of the two-level model is of the form

$$H(\kappa)=\frac{\varepsilon}{2}\sum_m\left(\frac{a_{m+}^\dagger a_{m+}}{\Omega}-\frac{a_{m-}^\dagger a_{m-}}{\Omega}+\frac{1}{\Omega}\right)-\frac{\kappa}{2}\left\{\sum_m\left(\frac{a_{m+}^\dagger a_{m-}}{\Omega}+\frac{a_{m-}^\dagger a_{m+}}{\Omega}\right)\right\}^2, \tag{5.1}$$

where $a_{m\pm}^\dagger, a_{m\pm}$ denote, respectively, creation and annihilation operators of fermions situated in upper and lower levels with the same angular momentum j, while $\Omega=2j+1$ denotes the degeneracy of these two levels. If the total number of fermions of the system is equal to Ω, the ground state of the system with $\kappa=0$ is just the state with the lower level fully occupied. Then the action of collective operator $\Sigma_m(a_{m+}^\dagger a_{m-})/\Omega$ or $\Sigma_m(a_{m-}^\dagger a_{m+})/\Omega$ just denotes the creation or annihilation of a monopole particle-hole pair and the resulting collectively excited states are just the eigenstates of the Hermitian collective operator $\frac{1}{2}[\Sigma_m(a^\dagger_{m+}a_{m-}-a^\dagger_{m-}a_{m+}+1]/\Omega$. These three collective operators behave as J_+,J_-, and J_0 of the so(3) algebra. Thus according to discussions given in Sec. II, $H(\kappa)$ represents an integrable system with a single collective degree of freedom. Comparing the commutation relations for these three collective operators with Eq. (2.17), we see that $1/\Omega$ plays the effective Planck constant here.

The state space of the Hamiltonian $H(\kappa\neq 0)$ is just spanned by the same basis set as $H(\kappa=0)$. It has been shown in Sec. III that with the application of a Lie transformation $U(\kappa,0)$, $H(\kappa)$ can be expressed generally as a nonlinear function of $J_0(\kappa)$ free from raising and lowering operators $J_\pm(\kappa)$. Hence systems $H(\kappa)$ belong to the same integrable class as $H(\kappa=0)$, the complete set of orthonormal eigenstates $|\phi_n(\kappa)\rangle$ of $H(\kappa)$ are just simultaneous orthonormal eigenstates of $J_0(\kappa)$, $|\phi_n(\kappa)\rangle=U(\kappa,0)|\phi_n^0\rangle$. Thus we have $U(\kappa,0)=\Sigma_n|\phi_n(\kappa)\rangle\langle\phi_n^0|$. Though it is not possible to obtain the explicit form of the required Lie transformation, $U(\kappa,0)$ can be obtained numerically through the one-to-one correspondent orthonormal eigenstates $|\phi_n(\kappa)\rangle$ of $H(\kappa)$ and $|\phi_n^0\rangle$ of $H(\kappa=0)$ with the method of iterative perturbation [24,25].

If the initial state is taken as the rotated ground state of $H(\kappa=0)$

$$\rho_\gamma^0=|\Phi_\gamma^0\rangle\langle\Phi_\gamma^0|,\quad |\Phi_\gamma^0\rangle=\exp[\gamma J_+-\gamma^* J_-]|\phi_-^0\rangle, \tag{5.2}$$

and evolves under the action of $H(\kappa\neq 0)$, we have

$$\begin{aligned}\rho_\gamma(t)&=\exp\left[-\frac{i}{\hbar}H(\kappa)t\right]\rho_\gamma^0\exp\left[\frac{i}{\hbar}H(\kappa)t\right]\\ &=\sum_{mm'}|\phi_m(\kappa)\rangle e^{-(i/\hbar)E_m(\kappa)t}\langle\phi_m(\kappa)|\rho_\gamma^0|\phi_{m'}(\kappa)\rangle\\ &\quad\times e^{(i/\hbar)E_{m'}(\kappa)t}\langle\phi_{m'}(\kappa)|.\end{aligned} \tag{5.3}$$

Here

$$\langle\phi_m(\kappa)|\rho_\gamma^0|\phi_{m'}(\kappa)\rangle=\sum_{nn'}\langle\phi_m(\kappa)|\phi_n^0\rangle\langle\phi_n^0|\rho_\gamma^0|\phi_{n'}^0\rangle\times\langle\phi_{n'}^0|\phi_{m'}(\kappa)\rangle \tag{5.4}$$

$$E_m(\kappa)=\langle\phi_m(\kappa)|H(\kappa)|\phi_m(\kappa)\rangle. \tag{5.5}$$

As both $\exp[\gamma J_+-\gamma^* J_-]$ and $\langle\phi_m(\kappa)|\phi_n^0\rangle=\langle\phi_m^0|U^\dagger(\kappa,0)|\phi_n^0\rangle$ here are dynamical symmetry preserving transformations, $\rho_\gamma(t)$ is an analytical functional of $\rho_\gamma^0(t)$ varying smoothly with γ and t. However, due to the fact that $H(\kappa)$ is a nonlinear function of $J(\kappa)$, $E_m(\kappa)$ is a nonlinear function of m, different matrix elements of $\rho_\gamma^0(t)$ will evolve with different frequencies. There will appear dispersion and regular interference patterns during a sufficiently long period of evolution.

These features are just quantum effects, the regularities exhibited in these plentiful features imply that the state $\rho_\gamma(t)$ has its analyticity still preserved. These quantum effects do not contradict the property of dynamical symmetry preservation. On the contrary, the absence of any evidence of decoherence even confirms indirectly the conclusion about dynamical symmetry preservation.

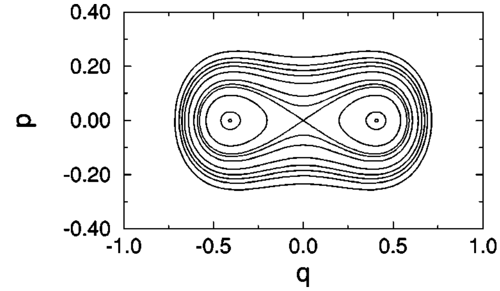


FIG. 1. Energy contour of $\mathcal{H}_{cl}(q,p)=E.\varepsilon=1.0,\ \kappa=0.8$.

In order to show these consequences more conveniently, the collective Hamiltonian will be described with a SO(3) model in q,p realization [30] with $J_0=\frac{1}{2}(q-ip)(q+ip)-\frac{1}{2}$, $J_+=\frac{1}{2}(q-ip)\,[1-\frac{1}{2}(q-ip)\,(q+ip)]^{1/2}$, $J_-=\frac{1}{2}[1-\frac{1}{2}(q-ip)(q+ip)]^{1/2}(q-ip)$, we have then

$$\mathcal{H}=\frac{\varepsilon}{2}(q-ip)(q+ip)-\frac{\kappa}{4}\left\{(q-ip)\left[1-\frac{1}{2}(q-ip)(q+ip)\right]^{1/2}+\text{H.c.}\right\}^2, \tag{5.6}$$

$$[q,p]=i\left(\frac{1}{\Omega}\right). \tag{5.7}$$

All systems with definite ε,κ but different Ω have the same classical counterpart corresponding to $1/\Omega=0$. Hence quantum effects in systems with different Ω can be readily shown.

In the classical limit $1/\Omega\to 0$, the classical orbits can be readily shown with the energy contours

$$\mathcal{H}_{cl}(p,q)=E \tag{5.8}$$

as in Fig. 1, parameters are taken as $\varepsilon=1.0,\kappa=0.8$. The uncertainty measure of $\rho_\gamma(t)$

$$[\Delta(\gamma,t)]^{1/2}=(\text{Tr}\{[(\Delta J_x)^2+(\Delta J_y)^2+(\Delta J_z)^2]\rho_\gamma(t)\})^{1/2} \tag{5.9}$$

calculated for solutions corresponding to a given initial state but with $1/\Omega=\frac{1}{100},\frac{1}{300}$ is given in Fig. 2. The time-averaged values are 0.24 and 0.14, respectively, the ratio between them agree roughly with the ratio between $\sqrt{1/100}$ and $\sqrt{1/300}$. As $1/\Omega$ decreases steadily, $[\Delta(\gamma,t)]^{1/2}$ decreases as $\sqrt{1/\Omega}$ and becomes much smaller than the range of variation of $\overline{q}(\gamma,t)$ and $\overline{p}(\gamma,t)$, quantum effects will then be negligible, the evolution of the state $\rho_\gamma(t)$ can be approximately represented with the classical phase trajectory.

However, the effective Planck constant $1/\Omega$ is a finitely small quantity. There should always exist quantum effects. In order to show them explicitly we consider the expectation value

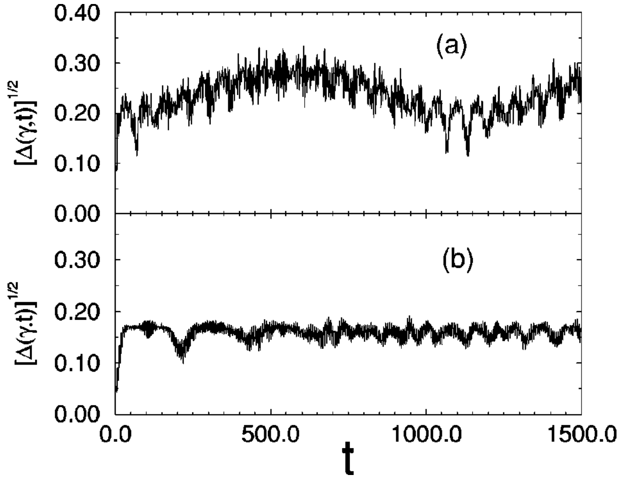


FIG. 2. Uncertainty measure $[\Delta(\gamma,t)]^{1/2}$ for a given initial state, curve (a) corresponds to $1/\Omega=\frac{1}{100}$ and (b) corresponds to $1/\Omega=\frac{1}{300}$.

$$\overline{q}(\gamma,t)=\mathrm{Tr}[q\rho_\gamma(t)]=\sum_{nn'}\langle\phi_n(\kappa)|q|\phi_{n'}(\kappa)\rangle \times\langle\phi_{n'}(\kappa)|\rho_\gamma(t)|\phi_n(\kappa)\rangle \tag{5.10}$$

and the probability lying to the left of the energy barrier

$$\begin{aligned}P_L(\gamma,t)&=\sum_{q_i<0}\mathrm{Tr}[|q_i\rangle\langle q_i|\rho_\gamma(t)]\\&=\sum_{q_i<0}\sum_{nn'}\langle\phi_n(\kappa)|q_i\rangle\langle q_i|\phi_{n'}(\kappa)\rangle\\&\quad\times\langle\phi_{n'}(\kappa)|\rho_\gamma(t)|\phi_n(\kappa)\rangle\end{aligned} \tag{5.11}$$

where $|q_i\rangle$ is the orthonormal eigenstate of q

$$q|q_i\rangle=q_i|q_i\rangle. \tag{5.12}$$

The corresponding numerical results are given in Figs. 3 and 4, respectively, for the case $1/\Omega=\frac{1}{100}$ with the other parameters the same as in Fig. 2.

For the initial minimum uncertainty state studied here, almost all the components lie below the top of the energy barrier, the eigenenergy of the odd-parity states are much lowered so that they become almost degenerated with the preceding even-parity ones. The almost-degenerated pair of levels are in phase with each other at the right side of the energy barrier but out of phase at the left side of the energy barrier. Hence $\overline{q}(\gamma,t)$ mainly oscillates around the well center at the right side of the energy barrier with a frequency $\omega\approx1$ which is determined from the mean level spacing between two neighboring almost-degenerated pairs of levels. Similarly $\overline{p}(\gamma,t)$ mainly oscillates around the mean value zero with the same frequency $\omega\approx1$. Hence the ''quantum trajectory'' mainly revolves around the well center at the right side of the energy barrier.

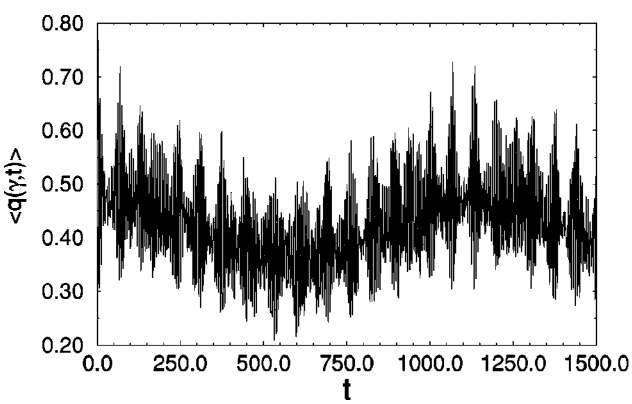


FIG. 3. Expectation value $\overline{q}(\gamma,t)$ for the same initial state. $1/\Omega=\frac{1}{100}$.

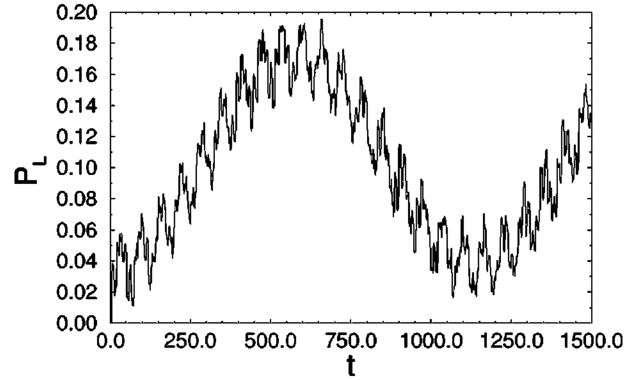


FIG. 4. Probability lying to the left of the barrier $P_L(\gamma,t)$ for the same initial state. $1/\Omega=\frac{1}{100}$.

However, due to the spreaded probability distribution of the state and the energy-dependent frequency of different components, the probability distribution of the state will be distorted. Consequently, the details of the ''quantum trajectory'' will vary with *t*. In other words $\overline{q}(\gamma,t)$ and $\overline{p}(\gamma,t)$ will have an amplitude modulation around their mean values. As the amplitude modulation is correlated with the distortion of the probability distribution, $[\Delta(\gamma,t)]^{1/2}$ will also have an amplitude modulation around its mean value. The largest amplitudes of $\overline{q}(\gamma,t)$ oscillation generally correspond to the smallest amplitudes of $[\Delta(\gamma,t)]^{1/2}$ oscillation and vice versa. Practically, $\rho_\gamma(t=0)$ is completely localized at the right side of the energy barrier. As $\rho_\gamma(t)$ evolves with *t*, the almost-degenerated pair of levels will not be completely out of phase at the left side of the energy barrier. The probability distribution at the left side of the energy barrier $P_L(\gamma,t)$ will not always be negligibly small. $P_L(\gamma,t)$ will have also an amplitude modulated oscillation as shown in Fig. 4.

As these interference patterns can exist only if the dynamical symmetry and analyticity of the initial state are preserved, the disappearance of such a kind of regular interference patterns can be regarded as an indication of decoherence and also an indirect indication of dynamical symmetry breaking.

As a whole, the essential point of quantum regular dynamics is the preservation of the dynamical symmetry and analyticity of the initial state during its evolution. Thus the essential point of quantum chaotic dynamics should be just the violent destruction of the general character of quantum regular motion as in classical chaotic dynamics.

VI. SUMMARY

In this paper, we have first investigated the quantum-classical correspondence by concentrating our attention to peculiar properties arising from the uncertainty principle in quantum mechanics. Though with the replacement of basic canonically conjugate dynamical variables by corresponding incommutable operators, quantum dynamical equations have been formulated correspondingly. But due to the uncertainty principle, we have to introduce quantum states of statistical nature to give expectation values for dynamical variables. Thus quantum state space is quite different from classical phase space.

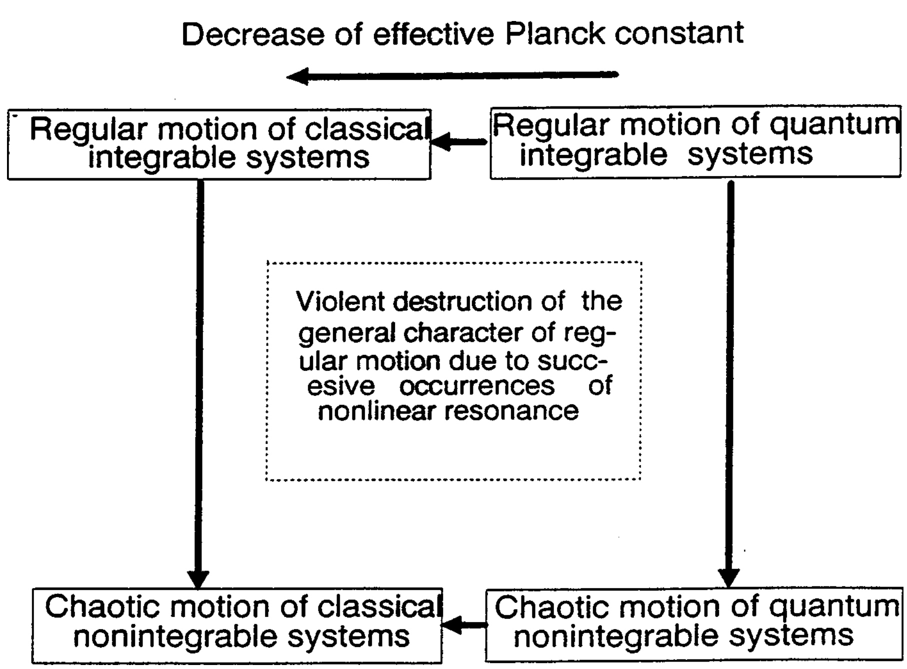
But instead of a completely defined single state, one can consider subspaces of an integrable system defined by definite values of action variables alone. In classical mechanics, such subspaces are just the tori invariant to the phase flow, while in quantum mechanics, the corresponding subspaces invariant to the Schrödinger flow will be the simultaneous orthonormal eigenstates of a complete set of commutable operators. In order to span the quantum state space completely by such a set of orthonormal states, pairs of conjugate operators together with accompanying Hermitian operators must form closed Lie algebras. Due to this fact, we can find the quantum canonical transformation between basic canonically conjugated variables and action-angle variables such that the Hamiltonian of this integrable system can be expressed in terms of action variables alone free from angle variables. Analogous to space-time symmetry, such a property is designated as dynamical symmetry and the associated Lie algebra as dynamical symmetry algebra.

Though such discussions are made for a special integrable system, but the integrability of this system is now characterized algebraically by its dynamical symmetry, the discussions can readily be extended to a class of integrable systems having the same dynamical symmetry. In this way, we have rendered the Liouville theorem applicable to both classical and quantum integrable systems.

Alternatively, the one-to-one correspondence between completely defined single quantum state with minimum uncertainty and classical phase points can be seen through classical limit. Having known the dynamical symmetry algebra of an integrable system, such a minimum uncertainty state can readily be found as a displaced ground state. Taking such a state as the initial state for quantum dynamical equations, its spatiotemporal evolution can be studied in the same spirit as for the corresponding classical case.

Though quantum dynamical equations for operators are formally analogous to classical canonical equations for dynamical variables, the manipulations in solving quantum dynamical equations must involve the commutation relations for basically conjugate operators. Due to the finiteness of the universal Planck constant, it is not possible to consider the classical limit straightforwardly. In fact, a microscopic system would be characterized by the small inertia mass and/or strong interaction strength such that its state varies rapidly. Hence it is possible to consider scaling transformations to express dynamical equations for systems with different parametric quantities in the same form but commutation relations with different effective Planck constants. Then the classical limit is attained by steadily decreasing the magnitude of the effective Planck constant for systems with different parametric quantities.

Having resolved these relevant problems, not only the general character of quantum regular motion can be studied in parallel to that of the classical regular motion, but also the



destruction of the general character of quantum regular motion can be studied in the same spirit as shown in the line diagram.

The general character of quantum regular motion has been clearly shown to be the preservation of the dynamical symmetry and analyticity of the initial minimum uncertainty state. It is expected that very complicated behaviors will arise as a consequence of the violent destruction of the general character of quantum regular motion under perturbation.

ACKNOWLEDGMENTS

The authors would like to thank Professor Y. S. Duan, Professor Y. Gu, Professor D. J. Fu, Professor Z. Q. Ma, Professor S. J. Wang, Professor M. L. Ge, and Professor Z. Y. Zhu for helpful discussions. The work was supported by the National Basis Research Project, ''Nonlinear Science'' of China, the National Natural Science Foundation of China, and the Natural Science Foundation of Fujian Province of China.

[1] G. Casati and B.V. Chirikov, in *Quantum Chaos, Between Order and Disorder*, edited by G. Casati and B.V. Chirikov (Cambridge University Press, Cambridge, England, 1995).
[2] E.A. Jackson, *Perspections of Nonlinear Dynamics* (Cambridge University Press, Cambridge, UK, 1989) Vol. 1, p. 3.
[3] L.E. Reichle, *The Transition to Chaos in Classical Conservative Systems: Quantum Manifestations* (Springer-Verlag, New York, 1992), p. 7.
[4] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978), pp. 162, 271.
[5] H.A. Kramers, *Quantum Mechanics* (North-Holland, Amsterdam, 1957), Sec. 42.
[6] P.A.M. Dirac, *Principle of Quantum Mechanics* (Oxford University Press, Oxford, UK, 1958).
[7] L. Infeld and T.E. Hull, Rev. Mod. Phys. **23**, 21 (1951).
[8] B. Kaufman, J. Math. Phys. **7**, 447 (1966).
[9] N.I. Vilemkin, *Special Functions and the Theory of Group Representations*, Transl. Math. Monogr. Vol. 22 (American Mathematical Society, Providence, RI, 1968).
[10] J.D. Talman, *Special Functions and Group Theoretical Approach* (Benjamin, New York, 1968).
[11] W. Miller, *Lie Theory and Special Functions* (Academic Press, New York, 1968).
[12] A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976); **111**, 201 (1978); **115**, 325 (1978); **123**, 468 (1979).
[13] W.M. Zhang, D.H. Feng, J.M. Yuan, and S.J. Wang, Phys. Rev. A **40**, 438 (1989); W.M. Zhang, D.H. Feng, and J.M. Yuan, *ibid.* **42**, 7125 (1990).
[14] S. Weigert and G. Muller, Chaos, Solitons and Fractals **5**, 1419 (1995).
[15] T. Gramespacher and S. Weigert, Phys. Rev. A **53**, 2971 (1996).
[16] D. Kuznezov, Phys. Rev. Lett. **79**, 532 (1997).
[17] A. Lichtenberg and M. Liebermann, *Regular and Chaotic Dynamics* (Springer, Berlin, 1992).
[18] Xu Gong-ou and Yang Ya-tian, Chin. Phys. Lett. **16**, 6 (1999).
[19] Xu Gong-ou, Yang Ya-tian, and Xing Yong-zhong, Chin. Phys. Lett. **16**, 318 (1999).
[20] J.Q. Chen, *Group Representation Theory for Physics* (World Scientific, Singapore, 1988).
[21] B.G. Wybourne, *Classical Groups for Physics* (John Wiley and Sons, New York, 1974).
[22] A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **98**, 527 (1954).
[23] W. Scherer, Phys. Rev. Lett. **74**, 1495 (1995).
[24] Xu Gong-ou, Wang Wen-ge, and Yang Yia-tian, Phys. Rev. A **45**, 5401 (1992).
[25] Xu Gong-ou, Xing Yong-zhong, and Yang Ya-tian, Chin. Phys. Lett. **16**, 86 (1999).
[26] R. Delbourgo and Le Fox, J. Phys. A **10**, L233 (1977).
[27] W.M. Zhang, D.H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[28] J. von Neumann, *Mathematical Foundation of Quantum Mechanics*, translated by R.T. Beger (Princeton University Press, Princeton, 1955).
[29] H.J. Lipkin, N. Meshkov, and A.S. Glick, Nucl. Phys. **62**, 188 (1965).
[30] Xu Gong-ou, Gong Jiang-bin, Wang Wen-ge, Yang Ya-tian, and Fu De-ji, Phys. Rev. E **51**, 1770 (1995).