Probabilistic teleportation and entanglement matching

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Teleportation may be interpreted as sending and extracting quantum information through quantum channels. In this report, it is shown that to get the maximal probability of exact teleportation through a partially entangled quantum channel, the sender (Alice) need only operate a measurement that satisfies an ''entanglement matching'' to this channel. An optimal strategy is also provided for the receiver (Bob) to extract the quantum information by adopting a general evolution.

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Quantum teleportation, the process that transmits an unknown qubit from a sender (Alice) to a receiver (Bob) via a quantum channel with the help of some classical information, was originally concerned by Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters (BBCJPW) [1]. In their scheme, such a quantum channel is represented by a maximally entangled pair (any of the Bell states) and the original state can be deterministically transmitted to Bob.

The process of teleportation may be regarded as sending and extracting quantum information via the quantum channel. We will apply this picture to investigate a partially entangled quantum channel. Because a mixed state can be purified to a Bell state with zero probability [2–4], a quantum channel of mixed states can never provide teleportation with fidelity 1. Therefore only pure entangled pairs should be considered if we want exact teleportation (even with some probability). Because of Schmidt disposition [5], a partially entangled pair may be expressed as

$$|\Phi\rangle_{2,3}=a|00\rangle_{2,3}+b|11\rangle_{2,3}(|a|^2+|b|^2=1,|a|>|b|). \quad (1)$$

(Hereafter, we assume particle 2 is at Alice's site and particle 3 at Bob's site.) The absolute value of the Schmidt coefficient $|b|$ is an invariant under local operations, and it corresponds to the entanglement entropy E of the state [6]. Such a state can be concentrated to a Bell state [6,7] with a probability of $2b^2$ and the concentrated pair may be used as a new quantum channel to carry out a teleportation.

In this report, Alice performs a Von Neumann measurement on her side while Bob performs a corresponding general evolution to reestablish the initial state with a certain probability. We will give a measure of the entanglement degree for Alice's measurement and show that the optimal probability of an exact teleportation is determined by the smaller of the entanglement degrees of Alice's measurement and the quantum channel. Thus the matching of these entanglement degrees should be considered and the entanglement degree of the measurement attains the meaning of Alice's ability to send quantum information.

First, we consider the case in which Alice operates a Bell measurement and give Bob's proper general evolution to reestablish the initial state with optimal probability. Considering the previously shared pair shown in Eq. (1) and the unknown state (which is to be sent) of particle 1 $|\phi\rangle_1=\alpha|0\rangle_1+\beta|1\rangle_1$, the total state can be written as $|\Psi\rangle_{1,2,3}=|\phi\rangle_1|\Phi\rangle_{2,3}=\alpha a|000\rangle_{1,2,3}+\alpha b|011\rangle_{1,2,3}+\beta a|100\rangle_{1,2,3}+\beta b|111\rangle_{1,2,3}$. If Alice operates a Bell measurement, Bob will get the corresponding unnormalized states as follows:

$$\begin{aligned}
\langle\Phi^+_{1,2}|\Psi\rangle_{1,2,3}&=\frac{\sqrt{2}}{2}(\alpha a|0\rangle_3+\beta b|1\rangle_3),\\
\langle\Phi^-_{1,2}|\Psi\rangle_{1,2,3}&=\frac{\sqrt{2}}{2}(\alpha a|0\rangle_3-\beta b|1\rangle_3),\\
\langle\Psi^+_{1,2}|\Psi\rangle_{1,2,3}&=\frac{\sqrt{2}}{2}(\beta a|0\rangle_3+\alpha b|1\rangle_3),\\
\langle\Psi^-_{1,2}|\Psi\rangle_{1,2,3}&=\frac{\sqrt{2}}{2}(\beta a|0\rangle_3-\alpha b|1\rangle_3),
\end{aligned} \quad (2)$$

where $\{|\Phi^{\pm}_{1,2}\rangle=\sqrt{2}/2(|00\rangle_{1,2}\pm|11\rangle_{1,2}),\ |\Psi^{\pm}_{1,2}\rangle=\sqrt{2}/2(|01\rangle_{1,2}\pm|10\rangle_{1,2})\}$ are Bell states of particle 1 and particle 2. Alice informs Bob of her measurement result, for example, $|\Phi^+_{1,2}\rangle$ [with the corresponding collapsed state of particle 3 as $\langle\Phi^+_{1,2}|\Psi\rangle_{1,2,3}=\sqrt{2}/2(\alpha a|0\rangle_3+\beta b|1\rangle_3)$, which is unnormalized], and Bob gives a corresponding general evolution. To carry out a general evolution, an auxiliary qubit with the original state $|0\rangle_{aux}$ is introduced. Under the basis $\{|0\rangle_3|0\rangle_{aux},|1\rangle_3|0\rangle_{aux},|0\rangle_3|1\rangle_{aux},|1\rangle_3|1\rangle_{aux}\}$, a collective unitary transformation

$$U_{sim}=\begin{pmatrix} \frac{b}{a} & 0 & \sqrt{1-\frac{b^2}{a^2}} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ \sqrt{1-\frac{b^2}{a^2}} & 0 & -\frac{b}{a} & 0 \end{pmatrix} \quad (3)$$

transforms the unnormalized product state $\sqrt{2}/2(\alpha a|0\rangle_3|0\rangle_{aux}+\beta b|1\rangle_3|0\rangle_{aux})$ to the result

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$$|\Phi\rangle_{3,aux}=\frac{\sqrt{2}}{2}\left[b(\alpha|0\rangle_3+\beta|1\rangle_3)|0\rangle_{aux}\right.$$
$$\left.+a\sqrt{1-\frac{b^2}{a^2}}\alpha|1\rangle_3|1\rangle_{aux}\right], \tag{4}$$

which is also unnormalized. Then a measurement to the auxiliary particle follows. If the measurement result is $|0\rangle_{aux}$, the teleportation is successfully accessed, while if the result is $|1\rangle_{aux}$, teleportation fails with the state of qubit 3 transformed to a blank state $|1\rangle_3$ and no information about the initial qubit 1 is left (thus an optimal probability of teleportation is accessed). The contribution of this unnormalized state to the probability of successful teleportation may be expressed by the probabilistic amplitude of $\alpha|0\rangle_3+\beta|1\rangle_3$ in Eq. (4) as $|(\sqrt{2}/2b)|^2=\frac{1}{2}|b|^2$.

Other states in Eq. (2) can be discussed in the same way, and their contributions to the probability of successful teleportation may be calculated directly by using a general method: if the unnormalized state in Eq. (2) is written as $\alpha x|0\rangle_3+\beta y|1\rangle_3$ or $\alpha x|1\rangle_3+\beta y|0\rangle_3$, after Bob's optimal operation, it gives a contribution to the whole successful probability of

$$p=(\min\{|x|,|y|\})^2. \tag{5}$$

Adding up all the contributions, the optimal probability of successful teleportation is obtained as $P=\frac{1}{2}|b|^2\times 4=2|b|^2$.

Then consider more general cases: Alice operates a measurement with the eigenstates

$$\begin{aligned}|\Phi\rangle_{1,2}^1&=a'|00\rangle_{1,2}+b'|11\rangle_{1,2},\\|\Phi\rangle_{1,2}^2&=b'|00\rangle_{1,2}-a'|11\rangle_{1,2},\\|\Phi\rangle_{1,2}^3&=a'|10\rangle_{1,2}+b'|01\rangle_{1,2},\\|\Phi\rangle_{1,2}^4&=b'|10\rangle_{1,2}-a'|01\rangle_{1,2}.\end{aligned} \tag{6}$$

where $(|a'|^2+|b'|^2=1,|a'|\geqslant|b'|)$. Because of Schmidt disposition, this basis represents all possible Von Neumann measurements of two particles when (a',b') varies. The four states above are orthogonal and have the same entanglement entropy, so the measurement's entanglement degree E can be defined as that of any of the four states. Collapsed states of particle 3 corresponding to the four measurement results can be written as

$$\begin{aligned}\langle\Phi_{1,2}^1|\Psi\rangle_{1,2,3}&=\alpha aa'|0\rangle_3+\beta bb'|1\rangle_3,\\\langle\Phi_{1,2}^2|\Psi\rangle_{1,2,3}&=\alpha ab'|0\rangle_3-\beta ba'|1\rangle_3,\\\langle\Phi_{1,2}^3|\Psi\rangle_{1,2,3}&=\beta aa'|0\rangle_3+\alpha bb'|1\rangle_3,\\\langle\Phi_{1,2}^4|\Psi\rangle_{1,2,3}&=\beta ab'|0\rangle_3-\alpha ba'|1\rangle_3,\end{aligned} \tag{7}$$

which is unnormalized. The general evolution to particle 3 is similar to what is shown in Eq. (3). From the result of Eq. (6), the probability of successful teleportation can be considered directly in the following two cases:

(1) $|a|\geqslant|a'|\geqslant|b'|\geqslant|b|$. In this case, because $|(ab')|^2=|a|^2\,(1-|a'|^2)$ and $|(ba')|^2=|a'|^2(1-|a|^2)$, the inequality $|ab'|\geqslant|ba'|$ is established, and $|aa'|\geqslant|bb'|$ is obvious, so the whole probability of successful teleportation may be written as

$$P=|(bb')|^2+|(ba')|^2+|(bb')|^2+|(ba')|^2=2|b|^2, \tag{8}$$

which is just the same as the case when Alice operates a Bell measurement.

(2) $|a'|\geqslant|a|\geqslant|b|\geqslant|b'|$. In this case, $|ba'|\geqslant|ab'|$, and the probability of successful teleportation is

$$P=|(bb')|^2+|(ab')|^2+|(bb')|^2+|(ab')|^2=2|b'|^2. \tag{9}$$

From the analysis above, the probability of successful teleportation is determined by the smaller of $|b|$ and $|b'|$, and may be regarded as being determined by the entanglement degree of Alice's measurement or the quantum channel, whichever is less.

Just as discussed above, teleportation may be regarded as preparation of the quantum channel and as sending and extraction of quantum information. The result above may be explained clearly with this picture. The entanglement degree of Alice's measurement can be considered as Alice's sending ability and the entanglement degree of the quantum channel can be taken as the width of it. Then the amount of transmitted quantum information is determined by the lower one of these two bounds: the width of the quantum channel $2|b|^2$ and the sending ability of Alice $2|b'|^2$. If they are just the same, a condition of ''entanglement matching'' is satisfied. If Bob always reestablishes the state to be sent with an optimal probability (which means he always extracts all the quantum information he received), an exact teleportation will be performed with probability equal to the amount of the quantum information transmitted, just as is shown in Eqs. (8) and (9).

Though Bell measurement is an essential task of quantum teleportation, it is very difficult to fully access it. It has been shown that Bell states cannot be distinguished completely by using linear devices [8,9], and this difficulty can be seen in some teleportation experiments [10]. Von Neumann measurements with less entangled eigenstates may be more efficient. From the result above, if a partial entanglement state $|\Phi\rangle_{2,3}=a|00\rangle_{2,3}+b|11\rangle_{2,3}$ is adopted as the quantum channel, the same optimal probability of successful teleportation can be accessed if only Alice's measurement satisfies the ''entanglement matching,'' while a Bell measurement or a POVM is not necessary. The matching here is essential to get the optimal probability, and it can also be regarded as the matching between the quantum channel's width and Alice's sending ability. Without such a matching, a waste of quantum information either at Alice's site or through the quantum channel will occur.

The result of entanglement matching can be generalized to the teleportation of a multiparticle system. Consider a k-particle system P at Alice's site with the state

$$|\Psi\rangle_P=\alpha_0|00\cdots 00\rangle_{P_1,\ldots,P_k}+\alpha_1|00\cdots 01\rangle_{P_1,\ldots,P_k}+\cdots+\alpha_{2^k-1}|11\cdots 11\rangle_{P_1,\ldots,P_k}.$$

Without loss of generality, the quantum channel between Alice and Bob is k independent entangled pairs with the state $\Pi_{i=1}^{k}(a_i|00\rangle_{A_i,B_i}+b_i|11\rangle_{A_i,B_i})$ (any other pure quantum channel can be transformed to this by local operations). Alice makes k collective measurements, each of which is a Von Neumann measurement with the following eigenstates:

$$|\Phi\rangle^{i,1}=a_i'|00\rangle_{P_i,A_i}+b_i'|11\rangle_{P_i,A_i},$$

$$|\Phi\rangle^{i,2}=b_i'|00\rangle_{P_i,A_i}-a_i'|11\rangle_{P_i,A_i},$$

$$|\Phi\rangle^{i,3}=a_i'|10\rangle_{P_i,A_i}+b_i'|01\rangle_{P_i,A_i},$$

$$|\Phi\rangle^{i,4}=b_i'|10\rangle_{P_i,A_i}-a_i'|01\rangle_{P_i,A_i}. \tag{10}$$

where $|a_i'|^2+|b_i'|^2=1$, $|a_i'|\geqslant|b_i'|$, and $i=1,2,\ldots,k$. Then Bob reestablishes the original state as $|\Psi\rangle^B$ with a certain probability by adopting a proper general evolution. Using similar methods as in the case of monoqubit teleportation, we may show that there also exists an entanglement matching in multiqubit teleportation: If c_i is defined as $\min\{|b_i'|,|b_i|\}$, the optimal probability of successful teleportation can be expressed as $2^k\Pi_{i=1}^{k}c_i^2$.

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