Genericity property of comoving potentials

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Comoving potentials or refractive indices acting on the propagation of waves of various nature are shown to exhibit a resonant effect giving rise to a general property we call genericity in the sense that they provide a *phase shift with any desired dependence on the momentum.* Application in atom optics to balance the natural spreading of matter waves in a vacuum is given. Optical analogies are also examined in view of applications in coherent control of short pulses.

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I. INTRODUCTION

Evolution of wave packets in real space and time is involved in many different domains, with applications ranging from acoustics to light or matter-wave optics [1]. This evolution is basically determined by the nature of the wave and is governed by the dispersion properties of the propagation medium. The coherent control of such an evolution naturally arises both from a fundamental and a practical point of view. In particular, controlled one- or two-color short light pulses have been recently used to drive the dynamics of (i) atoms and molecules [2], (ii) wave packets in quantum well structures and excitons in solids [3].

Coherent control means the ability to shift by a proper amount the individual phase of each spectral component of a given wave packet, that is to create media with customized dependence of the phase on the wave number or momentum. In light optics, this has been already realized, more or less approximately, by use of special media and optical devices such as prisms and gratings [4] but up to now the complete control and characterization of the phase in the pulse remains a difficult problem.

In this article we propose a new and, in principle, universal method to achieve any coherent control of wave packets based on the use of ''comoving'' perturbation fields [5]. By ''comoving'' we mean that (i) the perturbation field moves in the same direction (x) as the incoming wave and (ii) its velocity u is close to the relevant wave velocity (this point will be explained below). The perturbation is thus proportional to a dimensionless generic field

$$I(x,t)=I(x,x-ut)=I_v i(x)\cos\left[\frac{2\pi}{\lambda}(x-ut)\right], \qquad (1)$$

where $u=u(\nu)=\lambda(\nu)\nu$ and $i(x)$ is a normalized envelope modelizing the finite extension of the device. Note that one could choose as well a temporal, e.g., a rectangular, profile $i(t)$ limiting the interaction to a given finite time interval.

For instance, in light optics, the refractive index n can be properly perturbed by acoustic or electromagnetic waves, like in traveling-wave electro-optic modulators, according to

$$n-n_0=\eta I(x,x-ut), \qquad (2)$$

where η is characteristic of the medium [6]. Similarly, in matter-wave optics, potentials play the role of the refraction index and one might use a comoving potential

$$V=\vartheta I(x,x-ut), \qquad (3)$$

where ϑ characterizes the particle-field coupling strength. For example, we constructed a comoving magnetic field for which ϑ involves the particle magnetic moment. This field, produced by a set of helicoidal wires supplied with alternating currents of frequency ν, has been used in a Stern-Gerlach atom interferometer [5,7,8]. Minor changes in the experimental arrangement would produce an electric instead of a magnetic comoving field and ϑ would then involve either the induced electric dipole moment of the particle, just like in Sokolov's experiment [9], or the permanent dipole moment of a molecule. Comoving potentials would also appear in moving optical lattices provided that one uses a configuration combining those already used in [10] and [11], to make it simultaneously comoving and off-resonant. In such a case ϑ would be the magnitude of the optical potential. Many other configurations that fit the comoving scheme presented above can be imagined.

The organization of the paper will be as follows. In Sec. II, a general semiclassical expression of the phase shift produced by a single-frequency comoving potential is given. As a function of the velocity this phase shift exhibits a resonant behavior, upon which the so-called genericity property is based. In Sec. III the specific case of a matter-wave packet moving in a frequency-distributed comoving potential is examined. Section IV is devoted to an example of application of the genericity property, namely the possibility to balance the vacuum dispersion and to periodically recover the original matter-wave packet. Conclusions and possible applications in the field of light optics are given in Sec. V.

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II. THE SEMICLASSICAL PHASE SHIFT

For a light wave the phase shift is readily calculated in the semiclassical limit by use of the Fermat theorem. In the case of a one-dimensional configuration, one gets

$$\delta\varphi = k_0 \int dx[n(x,t(x)) - n_0], \tag{4}$$

where

$$t(x) = t_0 + x/v_\varphi$$

is the parametrization of the ray, v_φ being the *phase* velocity of the light. Similarly for a matter wave submitted to a perturbation V (depending on a single spatial coordinate), the magnitude of which is small compared to the initial kinetic energy E_0, the phase shift is given by [12,13]

$$\delta\varphi = k_0 \int dx \frac{-V(x,t(x))}{2E_0}, \tag{5}$$

where

$$t(x) = t_0 + x/v_g$$

is the parametrization of the classical unperturbed path and $v_g = \hbar k_0/m$ is the *group* velocity of the matter-wave packet. The replacement of v_φ by v_g is a manifestation of the fact that matter waves do not obey the same propagation equation as light waves, which gives rise to fundamental differences as soon as nonmonochromatic waves are considered. Nevertheless there remains a formal analogy between the two situations if one changes the parameter η into ϑ, $n - n_0$ into $-V/2E_0$, and the phase velocity into the group velocity. We will then write ρ for the relevant parameter (η or ϑ) and v for the relevant velocity (v_φ or v_g). We assume for the moment that (i) the incoming wave and the perturbation are synchronized and (ii) the magnitude of the perturbation is a constant for $-L \leqslant x \leqslant +L$ and zero elsewhere. Under such conditions the phase shift is

$$\delta\varphi = \int dx\, \rho i(x) I_\nu \cos\left[\frac{2\pi}{\lambda}\left(x - u(\nu)\frac{x}{v}\right)\right] \propto I_\nu L\, \mathrm{sinc}(K_- L), \tag{6}$$

where $\mathrm{sinc}(x)$ denotes the function $\sin(x)/x$ and $K_- = (2\pi/\lambda)\{1 - u(\nu)/v\}$ is the Doppler-shifted wave number of the perturbation. For $L \gg \lambda$, the extra phase is proportional to I_v times a very narrow function centered at $v = u(\nu)$. Moreover it is clear from Eqs. (4) and (5) that $\delta\varphi$ depends linearly on the potential. Therefore, by adding perturbations of different frequencies with appropriate intensities and phases, one is able to construct any desired dependence of $\delta\varphi$ on v, i.e., on the momentum k. This ability constitutes what we call the *genericity property of comoving potentials*. Let us just mention that comoving potentials offer other interesting possibilities, especially those dealing with frequency modulation. For example, at a fixed value of λ, $u(\nu) = \lambda\nu$, and a linear variation of ν in time (chirping) generates an accelerated moving field [14], etc.

III. EVOLUTION OF A MATTER-WAVE PACKET

Let us examine in some detail the behavior of a matter wave $\psi(x,t)$ subjected to a frequency- (or velocity-) distributed comoving potential

$$V(x,t) = \int d\nu\, H(\nu) \cos\left[2\pi\left(\nu(t - t_0) - \frac{x}{\lambda}\right)\right]. \tag{7a}$$

From now on, the spatial extension of V is assumed to be infinite [$i(x) \equiv 1$]. A limited extension would lead to unessential complications and a less strict univocal correspondence between the phase shift $\delta\varphi$ and the spectrum H. In the general case the velocity is $u = \lambda(\nu)\nu$. For the sake of simplicity we shall assume here that λ is independent of ν, as it is in Refs. [5] and [7], where λ is a fixed geometrical parameter, namely, the period of the helices that produce the magnetic field; t_0 is a shift in time between the perturbation and the incident wave, this latter one being taken as the time reference. $H(\nu)$ is the *cosine*-Fourier transform [$\nu \leftrightarrow (t - t_0)$] of the real potential $V(0,t)$. As a consequence only positive values of ν have to be considered and $H(\nu)$ is real. It is worth noting that V does not propagate as a whole. For instance, if $H(\nu)$ has a finite range and is symmetric around $\nu = \nu_0$, then V takes the form of a nonpropagating envelope modulating a propagating carrier wave

$$V(x,t) = h(t - t_0) \cos\left[2\pi\left(\nu_0(t - t_0) - \frac{x}{\lambda}\right)\right], \tag{7b}$$

where $h(t) = \int_{-\nu_0}^{\infty} dy\, H(\nu_0 + y)\cos(2\pi y t)$.

Taking the Fourier transform ($x \leftrightarrow k$) of the time-dependent Schrödinger equation

$$i\hbar\,\partial_t \Psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + V_\psi, \tag{8}$$

one obtains

$$i\hbar\,\partial_t C(k,t) = \frac{\hbar^2 k^2}{2m} C(k,t) + W(k,t) \otimes C(k,t), \tag{9}$$

where $C(k,t)$ and $W(k,t)$ are the Fourier transforms of ψ and V, and $\otimes$ is the convolution product. Because of the special form of $V(x,t)$ [Eq. (7a)] this convolution product is easily calculated,

$$W \otimes C = \int d\nu\, H(\nu)\{\exp[2\pi i \nu(t - t_0)] C(k + \kappa, t) + \exp[-2\pi i \nu(t - t_0)] C(k - \kappa, t)\}, \tag{10}$$

where $\kappa = 2\pi/\lambda$.

Let us set

$$C(k,t) = \Gamma(k,t) \exp\left(-i\frac{\hbar k^2}{2m} t\right) \tag{11}$$

[so that for a free wave $\Gamma(k,t)$ remains identical to $C(k,0)$]. For atoms at thermal velocity the de Broglie wavelength is extremely short (1 Å or less) and even for very slow atoms it

is smaller than 1 μm. On the other hand, λ is a parameter of macroscopic size. Therefore in most cases κ is extremely small compared to any k value in the momentum spectrum, which suggests the use of an expansion of $C(k\pm\kappa,t)$ in powers of κ. However because the relevant variable is $[i(\hbar k/m)\kappa t]$, the infinite sum must be evaluated. A simplification arises because the k derivatives of $C(k,t)$ are largely dominated by those of the exponential factor [Eq. (11)]. Indeed one has

$$\partial_k C=\left(\partial_k\Gamma+i\frac{\hbar k}{m}t\Gamma\right)\exp\left(-i\frac{\hbar k^2}{2m}t\right).$$

An order of magnitude of the first term is $|\Gamma|/\Delta k$, where Δk is the width of $C(k,0)$, whereas that of the second term is $|\Gamma|v_g t$; therefore, the second term dominates as soon as the wave packet has moved over a distance that is large compared to its initial spatial width (cf. the Rayleigh zone in light optics).

Under such conditions, neglecting the derivatives $\partial_k^n\Gamma$, one finally gets

$$i\hbar\partial_t\Gamma(k,t)=\left\{\int dv\,H(v)\cos\left[2\pi\left(v-\frac{\hbar k}{\lambda m}\right)t-2\pi v t_0\right]\right\}\times\Gamma(k,t), \tag{12}$$

which readily gives

$$\Gamma(k,t)=\Gamma(k,0)\exp[i\delta\varphi(k,t)]=C(k,0)\exp[i\delta\varphi(k,t)], \tag{13a}$$

where

$$\delta\varphi(k,t)=-\frac{1}{\hbar}\int_0^t dt'\int d\nu\,H(\nu)\cos[2\pi(\nu^* t'-\nu t_0)]$$

$$=-\frac{1}{\hbar}\int d\nu\,H(\nu)t\left\{\cos(2\pi\nu t_0)\,\mathrm{sinc}(2\pi\nu^* t)+\sin(2\pi\nu v t_0)\frac{1-\cos(2\pi\nu^* t)}{(2\pi\nu^* t)}\right\} \tag{13b}$$

and $\nu^*=\nu-\hbar k/\lambda m$.

In the special case of a monochromatic potential $V(x,t)=V_0\cos[2\pi(\nu_0 t-x/\lambda)]$ synchronized with the matter wave ($t_0=0$), Eq. (13b) gives

$$\delta\varphi(k,t)=-\frac{1}{\hbar}V_0 t\,\mathrm{sinc}\left[2\pi\left(\nu_0-\frac{\hbar k}{\lambda m}\right)t\right]. \tag{14}$$

Insofar as the sinc function takes a negligible value for an argument large compared to π, this expression clearly shows that at sufficiently large times the phase shift induced by the perturbation only concerns the ''resonant'' wave number such as the velocity $\hbar k/m=\lambda\nu_0$ is equal to the velocity of the comoving potential. In spite of a great similarity with the result already obtained in a much simpler way [Eq. (6)], the expression (14) is more general since it is valid for any value of k within the momentum spectrum of $\psi(x,t)$ whereas Eq. (6) only involves the central group velocity of the wave packet $v_g(k_0)=\hbar k_0/m$.

Let us now return to the case of a continuous frequency spectrum (Eqs. (13a) and (13b)) [15], which implies that the potential is pulsed in time [cf. Eq. (7b)]. For sufficiently large values of t the sinc in Eq. (13b) becomes a very narrow function of ν of width $\delta\nu(t)=1/(2t)$, centered at the value $\hbar k/m$, of which the integral on ν over the real axis is $1/(2t)$. At the same time the integral of the odd term in $(1-\cos x)/x$ vanishes. Consequently $\delta\varphi(k,t)$ has a limit at infinite t, which is

$$\delta\varphi^{\infty}(k)=-\frac{1}{2\hbar}H\left(\frac{\hbar k}{\lambda m}\right)\cos\left(2\pi\frac{\hbar k}{\lambda m}t_0\right). \tag{15}$$

Therefore, with an appropriate choice of the spectrum $H(\nu)$ and t_0, any dependence of the phase shift on k is in principle realizable. The criteria to get this limiting value is that t is large compared to t_0 and large enough to make $\delta\nu(t)$ much smaller than the width $\Delta\nu$ of the spectrum $H(\nu)$; in other words t must be large compared to the duration of the envelope $h(t)$. Notice that if t_0 is very large then Eq. (15) is no longer valid and, from Eqs. (13a) and (13b), $\delta\varphi^{\infty}$ tends to zero: the perturbation has missed the particle. More generally a total control of the phase shift induced on the matter-wave packet by the pulsed external potential implies the control of t_0, i.e., a synchronization, e.g., by means of a controlled emission of the wave packets (pulsed source). To get some idea about the accuracy needed in the definition of t_0 (around $t_0=0$), one has to examine the stationarity of the cosine term in Eq. (15), which implies $(\hbar k/m)|t_0|\ll\lambda$, for any k value in the wave-packet spectrum (in particular $v_g t_0\ll\lambda$). From this view point the present control of the phase by an external field is different from that based upon a nonlinear process that implies an internal and automatic synchronization driven by the wave packet itself.

IV. BALANCING THE VACUUM DISPERSION

We shall consider now a remarkable application of this genericity property in atom optics: balancing the vacuum dispersion of matter-wave packets. The natural spreading of a wave packet $\psi(x,t)$ *in vacuo* comes from the fact that the kinetic energy $E=\hbar\omega$ is quadratic in the momentum $\hbar k$. As a consequence, the k spectrum $c(k)$ is identical to $C(k,0)$, the Fourier transform of $\psi(x,0)$, not that of $\psi(x,t)$, which is rather $C(k,t)$ [see Eq. (11)] [16]. If one compensates for the (constant) concavity of $\omega(k)$ by adding an appropriate phase shift, then one is able to balance the vacuum dispersion, at least at specific times (or positions of the wave-packet center). This can be achieved by using a frequency spectrum of a parabolic form (see Fig. 1), such as

$$H(\nu)=\begin{cases}-\dfrac{A}{2}(\nu-\nu_0)^2 & \text{for } |\nu-\nu_0|\leqslant\dfrac{\Delta\nu}{2},\\ 0 & \text{elsewhere,}\end{cases} \tag{16}$$

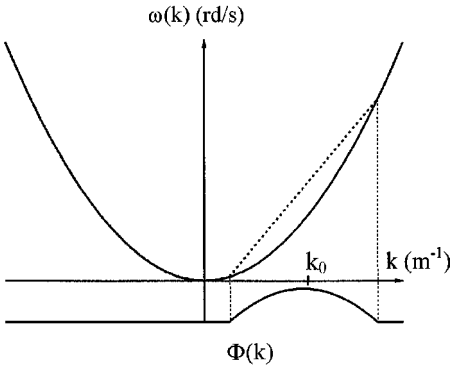

FIG. 1. Upper solid line: dispersion curve of matter waves. Lower solid line: extra phase needed to balance the vacuum dispersion. Dotted line: total phase accumulated at time t_A (see text) showing curvature's compensation.

where $\Delta\nu$ is such that $m\Delta\nu/\hbar$ is large compared to the width of the k spectrum of $\psi(x,t)$; A is a positive constant and $\nu_0=\hbar k_0/(\lambda m)$. Assuming that $t_0=0$, which is a perfect synchronization of the potential and the wave packet, one obtains

$$\delta\varphi^{\infty}=\frac{\hbar A}{4\lambda^2 m^2}(k-k_0)^2. \tag{17}$$

It is then readily verified that at time $t_A=A/2\lambda^2 m$, and provided that this time is sufficiently large compared to the pulse duration, one has

$$C(k,t_A)=C(k,0)\exp[i\tfrac{1}{2}(k_0-k)X(t_A)], \tag{18}$$

where $X(t_A)=(\hbar k_0/m)t_A$ is the position of the center of the wave packet at time t_A. As a consequence

$$\psi(x,t_A)=\exp\left[-i\frac{\hbar k_0^2}{2m}t_A\right]\Psi(x-X(t_A),0). \tag{19}$$

Therefore, apart from a phase factor, the wave function at time t_A is identical to the initial one translated by $X(t_A)$. This result holds for any initial shape, which means that any wave packet can be shifted as a whole by an arbitrary amount. Afterwards the (free-) wave packet recovers its natural evolution, i.e., it spreads again. Here the need of synchronization appears from the fact that when $t_0\neq 0,\delta\varphi^{\infty}$ is multiplied by $\cos[2\pi(\hbar k/\lambda m)t_0]$ and has no longer a parabolic dependence on k.

Obviously the process can be repeated. Having this in view let us first consider a potential consisting of a train of N pulses of period T:

$$V(x,t)=\sum_{n=0}^{N-1}\int d\nu\, H(\nu)\cos\left[2\pi\left(\nu(t-nT)-\frac{x}{\lambda}\right)\right]. \tag{20}$$

The expression obtained for the phase shift is similar to that given by Eq. (13b), $\cos(2\pi\nu t_0)$ being replaced by $\Sigma_n\cos(2\pi\nu nT)$, and a similar sum for the sine term. Let us assume that the potential is *synchronized* with the motion of the center of the wave packet, that is $T=\lambda m/\hbar k_0$ (see Fig. 2). Then at a time t such that $t-NT$ is large compared to the duration of h, the accumulated phase shift is [cf. Eq. (15)]

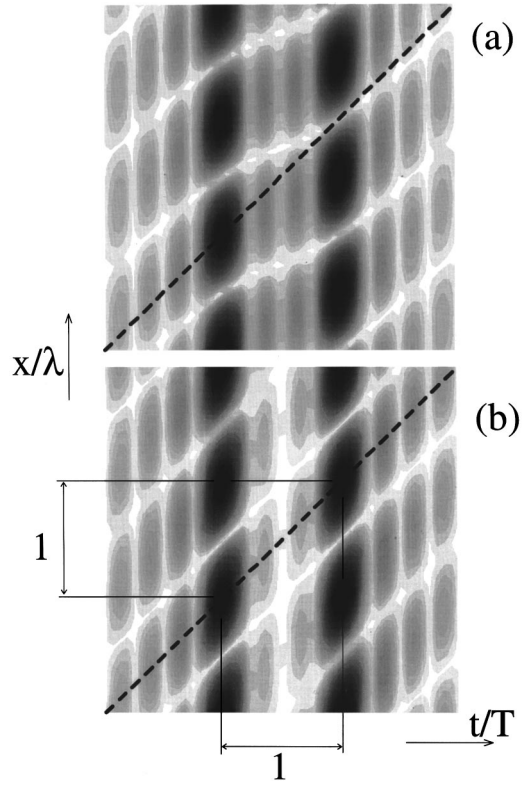

FIG. 2. Behavior in a reduced-coordinate diagram ($x/\lambda, t/T$) of a comoving potential of spatial period λ created by a pair of pulses separated by a time interval T. The broken line is a resonant atomic trajectory synchronized with the first pulse and (a) not synchronized with the second pulse (b) synchronized with the second pulse.

$$\delta\varphi_N^{\infty}(k)=-\frac{1}{2\hbar}H\left(\frac{\hbar k}{\lambda m}\right)\sum_n\cos\left(2\pi n\frac{k}{k_0}\right). \tag{21}$$

The sum is similar to the diffraction amplitude of a N-slit grating. In so far as $N\Delta k/k_0\ll 1$, it is close to N. For larger-N values and/or a wider spectral range the synchronization becomes less and less efficient on the borders of the k spectrum $C(k,NT)$. An exceptional situation is encountered when $T=t_A$, since in that case one obtains, up to a phase factor and a translation by $NX(t_A)$ along $\hat{x}$, a periodical recovering of $\psi(x,0)$. It might be expected that this phenomenon gives rise to an observable effect in an atom interferometer, e.g., in a device similar to that used in Refs. [5] and [7], which could provide a new experimental approach to problems already discussed at length [17,18] but rarely investigated experimentally [19]. While physically different, this periodic revival of the wave packet reminds us of solitons in optical fibers: in this latter case, due to the Kerr effect, a light pulse generates a local index perturbation that moves together with it and, for very special shapes, compensates for the dispersion. Here the perturbation is imposed from outside, with the

advantage that the dispersion is balanced whatever the shape of the pulse and the disadvantage that a synchronization is necessary.

V. CONCLUSION

We have shown the common encounter of the comoving configuration in which a wave is submitted to an external perturbation that moves in the same direction, with about the same velocity (the relevant velocity depending on the nature of the wave under consideration). We then demonstrated the genericity property of such a configuration: the resonant character of the interaction between wave perturbation enables a detailed control of the phase since the phase shift accumulated within the medium essentially reproduces the frequency spectrum of the perturbation. It is then easily adjustable to any purpose dealing not solely with matter waves, as it has been shown, but also with light waves. In light optics comoving configurations correspond to traveling-wave electro-optic modulators.

In order to use the general scheme presented before [see Eq. (2)], one has to make u dependent on the frequency ν_{RF} of an exciting radio-frequency field: $u=u(\nu_{\mathrm{RF}})$. For example, in the case of an electro-optic modulator this simply means that the crystal is inserted in a dispersive waveguide (e.g., operating in the cutoff region). The phase velocity of each monochromatic component of light in the crystal depends on its wavelength λ through the refractive index n: $v=c/n(\lambda)$. Therefore the resonance condition $u=v$ imposes a relation between v_{RF} and the resonant wavelength [20]

$$\lambda_{\mathrm{res}}=n^{-1}\left(\frac{c}{u(\nu_{\mathrm{RF}})}\right), \tag{22}$$

where n^{-1} means the inverse function of $n(\lambda)$. Such an apparatus provides a tunable dispersive optical medium. As mentioned before, such a device could find applications, for instance, in ultrafast pulse reshaping as it enables us to individually rephase each spectral component, providing an optimized compression of the pulse. It is interesting to note that such a ''programmable'' dispersive filter has been independently proposed in Ref. [21] in a slightly different context. It uses an acousto-optic modulator driven by a properly designed excitation spectrum. In this case, however, the perturbing acoustic wave is not comoving with the light wave since their velocities are orders of magnitude apart.

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