# Multipath interference in a multistate Landau-Zener-type model

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(Received 1 June 1999; published 10 February 2000)

The bow-tie model is a particular generalization of the famous two-state Landau-Zener model to an arbitrary number of states. Recently, solution of the bow-tie model was found by the contour integral method [V. N. Ostrovsky and H. Nakamura, J. Phys. A **30**, 6939 (1997)]. However, its physical interpretation remained unclear, because the model implies simultaneous strong interaction of all the states apparently irreducible to any simpler pattern. We introduce here a generalized bow-tie model that contains an additional parameter $\varepsilon$ and an additional state. It includes the conventional bow-tie model as a particular limiting case when $\varepsilon \to 0$. In the generalized model all the state-to-state transitions are reduced to the sequence of pairwise transitions and the two-state Landau-Zener model is applied to each of them. Such a reduction is well justified at least in the opposite limit of large parameter $\varepsilon$. Several paths connect initial and final states; the contributions of different paths are summed up coherently to obtain the overall transition probability. The special $ET$ (energy and time) symmetry intrinsic for the generalized bow-tie model results in a particular property of interference phases: only purely constructive or purely destructive interference is operative. The complete set of transition probabilities is obtained in a closed form. Importantly, the results do not depend on the parameter $\varepsilon$. In the limit of conventional bow-tie model all previously derived results are reproduced. This amounts to rationalization of the bow-tie model by its interpretation in terms of multipath successive two-state transitions.

PACS number(s): 34.10.+x, 32.80.Bx, 32.60.+i

## I. INTRODUCTION

Many quantum problems of practical significance are reducible to the nonstationary Schrödinger equation considered in the finite basis of $N$ states. Among them, the case when the Hamiltonian matrix $H$ depends linearly on time $t$,

$$H(t) = Bt + A, \tag{1.1}$$

is distinguished by its basic simplicity combined with capability to model general situation. In Eq. (1.1) $A$ and $B$ are time-independent Hermitian $N \times N$ matrices that generally do not commute. The expression (1.1) can be considered as a linear approximation to the Hamiltonian matrix with general time dependence. We choose the basis of states in which the matrix $B$ is diagonal:

$$B_{jk} = \beta_j \delta_{jk}. \tag{1.2}$$

The diagonal elements of the Hamiltonian matrix

$$H_{jj} = \beta_j t + \varepsilon_j, \quad \varepsilon_j \equiv A_{jj} \tag{1.3}$$

are named *diabatic potential curves*. They are slanted straight lines with the slopes $\beta_j$. The nondiagonal elements

$$H_{jk} = A_{jk} \equiv V_{jk} \quad (j \neq k) \tag{1.4}$$

describe the *coupling* between the diabatic basis states. These coupling matrix elements are time independent. The number of states $N$ depends on the approximation used to describe physical system that usually is infinite dimensional. The Hamiltonian $H(t)$ (1.1) is a straightforward generalization of the famous two-state ($N=2$) Landau-Zener model [1]. Actually almost simultaneously with Landau and Zener a similar two-state model was suggested and solved also by Mayorana [2]. Later the model was rediscovered by Wannier [3] and Horwitz [4] and directly tested experimentally by Chigir [5].

In many applications one is interested in a description of the time propagation in a general quantum problem when its Hamiltonian $H(t)$ varies slowly with time [6], i.e., when it depends on $\tau = vt$ where $\tau$ is reduced time variable and $v$ is small velocitylike parameter. Description of this *adiabatic* case essentially uses the *adiabatic potential curves* $\mathcal{E}_n(t)$ and the adiabatic states $\chi_n(t)$ that are defined, respectively, as the time-dependent eigenvalues and eigenstates of the instantaneous Hamiltonian $H(t)$: $H(t)\chi_n(t) = \mathcal{E}_n(t)\chi_n(t)$. The additional condition $\langle \chi_n | \dot{\chi}_n \rangle = 0$ introduced originally by Born and Fock [7] fixes the time-dependent phases of diabatic states; in case of real $H(t)$ it implies that the eigenfunctions $\chi_n(t)$ are essentially real (within a time-independent phase factor). For the Hamiltonian (1.1) at large $|t|$ the adiabatic potential curves asymptotically approach the diabatic curves.

The crossings of the diabatic potential curves correspond to the *avoided crossings*, or *pseudocrossings* of the adiabatic potential curves, with the splittings increasing as $V_{jk}$ grow. Born and Fock [7] assumed *continuous* time-dependence of the Hamiltonian $H(t)$ and showed that as $v \to 0$, the probability of transition between the adiabatic states tends to zero, being proportional to $v$ in nondegenerate case or to some power of $v$ in degenerate case. Landau [8] noticed that if the Hamiltonian $H(t)$ is an *analytical* function of time, that holds in the most interesting cases, then the transition probability is exponentially small. This is exemplified by the two-state Landau-Zener model, as discussed in Sec. IV A.

The transitions between the adiabatic states are known to be localized in the vicinities of the avoided crossings. This circumstance allows one to develop an approximate method to construct solution of nonstationary Schrödinger equation applicable when only pairwise crossings occur between the

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diabatic potential curves. The two-state Landau-Zener model is employed in the vicinity of each avoided crossing. In between the pseudocrossings adjacent on the time scale, the purely adiabatic time propagation is operative. This means that the phase $\int^t \mathcal{E}_n(t')dt'$ is gained in the $n$th adiabatic state. This scheme implies a simple factorized form of the time-propagation operator being extensively used in the atomic collision theory [9].

The factorization scheme could be conveniently cast as a *multipath interference* problem. The time evolution is associated with propagation along some path in the network formed by potential curves. Each path presents some sequence of pieces of potential curves; the leaps from one curve to another occur at the crossing points. The factorization approach allows one to ascribe some amplitude to propagation along each path. Generally, several paths connect initial and final states; only the paths with propagation forward in time should be taken into account. The contributions of different paths are summed coherently, i.e., with account for the phases gained along each path. This implies interference effects.

The multipath propagation scheme certainly is valid when different pseudocrossing regions do not overlap that occurs in the limit of small couplings $V_{jk}$. Fortunately, it seems that the factorization approach often works well in a much broader range. An important insight is provided by the exactly solvable generalized Landau-Zener models.

Discussing the possibility to find an analytical solution to the multistate generalized Landau-Zener model, when $N>2$, one naturally appeals to the fact that commonly the problems with linear time dependence are efficiently solved by using the Laplace transformation. The latter plainly reduces the problem to single first-order differential equation in the case when only one coefficient $\beta_j$ is nonzero. This gives the Demkov-Osherov [10,11] model (see also bibliography in Ref. [10]): one slanted diabatic potential curve crosses parallel set of horizontal curves (of course, a trivial phase transformation makes the horizontal curves also slanted but with equal slopes). The remarkable property of the Demkov-Osherov model is that for it the factorization scheme provides exact transition probabilities for arbitrarily large couplings, i.e., when the strong-coupling regions for various pseudocrossings overlap substantially. Note, however, that in this model one important physical effect is not operative, namely there is no multipath transitions and interference.

A less obvious case when the Laplace transformation also reduces problem to the first-order differential equation corresponds to the so-called *bow-tie model* (see Sec. II for details). Originally it was solved by Carroll and Hioe [12] for the particular case $N=3$, but the method employed could not be extended to larger $N$. Harmin [13] discussed the model in connection with transitions within a manifold of Rydberg states. Brundobler and Elser [14] presented a solution in form of contour integral for an arbitrary number of states $N$ but did not find the transition probabilities. The latter goal was achieved recently by Ostrovsky and Nakamura [15]. However, the appealing physical interpretation was not given. Resolution of the latter problem appeared to be hopeless, since the bow-tie model implies *simultaneous strong coupling of all N states* that apparently could not be disentangled and reduced to some simpler situation such as a sequence of two-state transitions. Therefore the factorization scheme was not applicable.

In the present paper we show that the bow-tie model could be considered as a particular limiting case of the generalized bow-tie model (Sec. III). The latter allows an appealing reduction to the sequence of pairwise transitions between the states. We succeeded (Sec. IV) in finding the closed-form expressions for all transition probabilities without actually solving the nonstationary Schrödinger equation but by using the factorization scheme discussed above. The arguments presented in Sec. VI strongly suggest that this generally approximate scheme provides exact results in the particular case of the generalized bow-tie model as well as in some other situations (Sec. VI B). This conclusion is fairly supported by the fact that we are able to reproduce all the results obtained earlier for the bow-tie model [15]. For this we plainly consider an appropriate limit (Sec. V) of the probabilities derived for the generalized bow-tie model. This development amounts to physical rationalization of the bow-tie model via the factorization scheme, i.e., by interpretation of the model solution as a sequence of pairwise transitions intervened by plain ''phase gain'' time-propagation regions without transitions.

As mentioned above, the important feature of the exactly solvable Demkov-Osherov model is that only one (or none) path connects any initial and final state. This means that no interference is operative in this model. It was tempting to suggest that the absence of interference is a necessary property of any exactly solvable model. The present study shows that this guess is incorrect by providing a counterexample: the exactly solvable generalized bow-tie model includes multipath interference (Sec. IV C), albeit in a quite peculiar situation. Namely, the special $ET$ symmetry intrinsic for the model leads to a particular property of interference phases: only purely constructive or purely destructive interference is operative. The full account for the multipath interference in the exactly solvable model seems to be an achievement of general interest in the present study.

Before concluding the introductory discussion we mention another subclass of Landau-Zener-type models where some important advancements were made. In these models a set of parallel diabatic potential curves is crossed by another set of parallel curves. In the simplest case both sets contain only two curves ($N=4$) [16,17]. The larger sets were also considered with particular attention paid to the crossing of two infinite bands of equidistant potential curves [20–22]. Two essentially different approaches were employed: direct construction of the time evolution operator in the spirit of factorization scheme [18,19] and *ab initio* solution of the non-stationary quantum problem [20–22]. Interestingly, the similar mathematical models are considered in apparently very different theory of transitions in quantum networks [23]. Among the vast realm of nonlinear nonstationary models we mention the recent solution of the time-dependent quadratic problem with the inclusion of phase interference effects by Teranishi and Nakamura [24].

## II. BOW-TIE MODEL

In the bow-tie model the diabatic potential curves are (i) linear in time and (ii) all of them cross at some moment of time, that could be chosen as zero. The energy at which the multiple crossing occurs is naturally chosen as the origin of the energy scale that corresponds to $A_{jj}\equiv 0$ in Eq. (1.1). Hence in the diabatic basis the diagonal matrix elements of the Hamiltonian $H$ are

$$H_{jj}=\beta_j t. \tag{2.1}$$

The nondiagonal matrix elements are time independent, most of them being zero. The nonzero matrix elements couple one particular state, labeled as $j=0$, to all other states:

$$H_{j0}=H_{0j}=V_j. \tag{2.2}$$

By a trivial phase transformation one always can turn horizontal the diabatic potential curve with $j=0$, i.e., make $\beta_0=0$. The convenient way to label all other states is to ascribe positive (negative) indices $j$ to the states with $\beta_j>0$ ($\beta_j<0$) in such a way that larger $|j|$ correspond to larger $|\beta_j|$, i.e.,

$$\ldots<\beta_{-3}<\beta_{-2}<\beta_{-1}<\beta_0=0<\beta_1<\beta_2<\beta_3<\ldots. \tag{2.3}$$

The diabatic potential curves are shown in Fig. 1(a).

The general solution $|\Psi(t)\rangle$ of the nonstationary Schrödinger equation is expanded over the basis of diabatic states $|j\rangle$:

$$|\Psi(t)\rangle=\sum_j c_j(t)|j\rangle, \tag{2.4}$$

where the expansion coefficients $c_j(t)$ obey the following system of equations:

$$i\frac{dc_0}{dt}=\sum_{n\neq 0} V_n c_n, \tag{2.5a}$$

$$i\frac{dc_j}{dt}=\beta_j t c_j+V_j c_0 \quad (j\neq 0) \tag{2.5b}$$

(unfortunately the latter equation was cited previously [15] with a misprint). Solution of these equations carried out by the contour integral method [15] produced probabilities of transitions between all combinations of initial and final states.

## III. GENERALIZED BOW-TIE MODEL

The diabatic potential curves of the bow-tie model Fig. 1(a), as well as their adiabatic counterparts (considered in Ref. [15], see also Appendix A) are symmetrical with respect to the simultaneous $ET$ reflection: $\{E\Rightarrow -E,\ t\Rightarrow -t\}$. Here we suggest a generalization of the bow-tie model in which this key property is retained. Namely, we replace the single state $|0\rangle$ with two states $|0^+\rangle$ and $|0^-\rangle$. Just as the original $|0\rangle$ state, these two states correspond to horizontal potential curves that, however, are shifted respectively by $\pm\frac{1}{2}\varepsilon$:

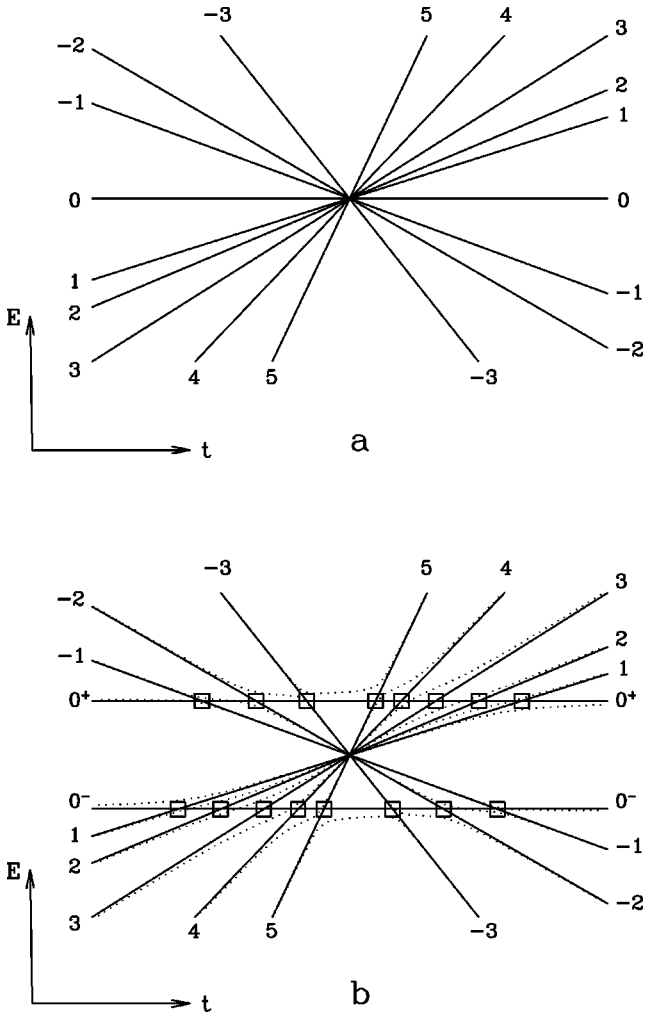


FIG. 1. Diabatic potential curves for the bow-tie model (a) and the generalized bow-tie model (b). In both cases the potential curves are linear. The slanted potential curves are labeled by integer quantum numbers $j$ with $j>0$ for the curves with a positive slope $\beta_j$ and $j<0$ for the curves with a negative slope $\beta_j$, larger values of $|j|$ corresponding to larger values of $|\beta_j|$. All these curves cross simultaneously at the point $t=0$, $E=0$. In the bow-tie model all the diabatic states are coupled only with the single state that is labeled as 0 and corresponds to horizontal potential curve. In the generalized bow-tie model the horizontal potential curve is ''doubled,'' i.e., replaced by two curves $0^-$ and $0^+$ that correspond, respectively, to the energies $-\frac{1}{2}\varepsilon$ and $\frac{1}{2}\varepsilon$. The coupling with the slanted potential curves is ''shared'' between $0^-$ and $0^+$ states (see text for detail). The pairwise crossings between horizontal and slanted diabatic potential curves are marked by blocks. Note that due to the well-known optical illusion a human eye perceives horizontal lines $0^+$ and $0^-$ as slightly curved. The adiabatic potential curves are shown by dots.

$$H_{0^+0^+}=\frac{1}{2}\varepsilon, \quad H_{0^-0^-}=-\frac{1}{2}\varepsilon. \tag{3.1}$$

It is assumed that these states are not coupled to each other:

$$H_{0^+0^-}=H_{0^-0^+}=0. \tag{3.2}$$

The slanted diabatic potential curves are not changed as compared with the bow-tie model. The related states $|j\rangle$ are coupled to the horizontal states $|0^+\rangle$ and $|0^-\rangle$ by identical couplings parametrized as

$$H_{0^+j}=H_{j0^+}=H_{0^-j}=H_{j0^-}=\frac{1}{\sqrt{2}}V_j. \tag{3.3}$$

The diabatic potential curves for the generalized bow-tie model are shown in Fig. 1(b). Equations (2.5) are now replaced by their generalizations:

$$i\frac{dc_{0^+}}{dt}=\frac{1}{\sqrt{2}}\varepsilon c_{0^+}+\sum_{n\neq 0^\pm}\frac{1}{\sqrt{2}}V_n c_n, \tag{3.4a}$$

$$i\frac{dc_{0^-}}{dt}=-\frac{1}{\sqrt{2}}\varepsilon c_{0^-}+\sum_{n\neq 0^\pm}\frac{1}{\sqrt{2}}V_n c_n, \tag{3.4b}$$

$$i\frac{dc_j}{dt}=\beta_j t c_j+\frac{1}{\sqrt{2}}V_j c_{0^+}+\frac{1}{\sqrt{2}}V_j c_{0^-}\quad (j\neq 0). \tag{3.4c}$$

In the limit $\varepsilon\to 0$ the horizontal states $|0^+\rangle$ and $|0^-\rangle$ become degenerate, and it is natural to introduce the linear combinations

$$|0\rangle=\frac{1}{\sqrt{2}}[|0^+\rangle+|0^-\rangle], \tag{3.5a}$$

$$|d\rangle=\frac{1}{\sqrt{2}}[|0^+\rangle-|0^-\rangle]. \tag{3.5b}$$

One can straightforwardly see that for $\varepsilon=0$ the state $|d\rangle$ is fully *decoupled* from all the other states,

$$H_{jd}=H_{dj}=H_{0d}=H_{0d}=0, \tag{3.6}$$

whereas for the state $|0\rangle$ all the Hamiltonian matrix elements $H_{0j}$ are the same as in the conventional bow-tie model [see Eq. (2.2)] [note that the factor $1/\sqrt{2}$ in the parametrization (3.3) is chosen so as to reproduce this limiting case with the notations of Sec. II retained]. This means that in the limit $\varepsilon\to 0$ the generalized bow-tie model is essentially reduced to the original bow-tie model.

Concerning the adiabatic potential curves in the generalized bow-tie model we note, first of all, that they could be found analytically in case of symmetric four-state model ($N=4$) when two slanted diabatic potential curves correspond to equal couplings ($V_1=V_{-1}$) and slopes of opposite sign ($\beta_{-1}=-\beta_1$):

$$\mathcal{E}(t)=\pm\frac{1}{\sqrt{2}}\left[\beta_1^2t^2+\frac{1}{4}\varepsilon^2+2V_1^2 \pm\sqrt{\left(\beta_1^2t^2-\frac{1}{4}\varepsilon^2+2V_1^2\right)^2+2\varepsilon^2V_1^2}\right]^{1/2}. \tag{3.7}$$

All four curves correspond to different branches of unique analytical function $\mathcal{E}(t)$ (3.7) defined on the four-sheet Riemann surface of complex-valued $t$ variable. The choice of the sheet (i.e., of the particular potential curve) corresponds to the choice of $\pm$ signs in the expression (3.7). It is worthwhile to note that two adiabatic potential curves *cross exactly* at the central point $C$: $t=0$, $E=0$ just as the diabatic curves; two other curves correspond to $0^+$ and $0^-$ diabatic curves repelled by the indirect coupling via the slanted curves: $\mathcal{E}(0)=\pm\sqrt{\varepsilon^2/4+2V_1^2}$. It is easy to see how this property is generalized in case of arbitrary number of states $N$. Here for $t=0$ two adiabatic potential curves have energies

$$\mathcal{E}(0)=\pm\sqrt{\frac{1}{4}\varepsilon^2+2\sum_n V_n^2}. \tag{3.8}$$

All other $(N-2)$ adiabatic potential curves *cross exactly* at the point $t=0$, $E=0$.

Generally, as discussed in the Introduction, the crossings of diabatic potential curves correspond to the pseudocrossings of adiabatic curves. In the generalized bow-tie model this refers to the crossings between the slanted diabatic curves and either $0^+$ or $0^-$ curve [these crossings are marked by blocks in Fig. 1(b)]. If the couplings $V_j$ are small, all the pseudocrossings are well separated. Otherwise the adjacent pseudocrossing regions might strongly overlap. The pattern of adiabatic potential curves in the generalized bow-tie model is illustrated by Fig. 1(b). The plot was constructed by actual diagonalization of the Hamiltonian matrix $H(t)$ for some choice of couplings $V_j$ that are not listed here. We only mention that both weak- and strong-coupling cases are represented in Fig. 1(c). For instance, the coupling $V_{-1}$ is small and the related pseudocrossings [within $(-1,0^+)$ and $(-1,0^-$ pairs)] are well isolated. The coupling $V_3$ is large and the related broad pseudocrossing exhibits strong overlap with its neighbors.

In the present study we do not pursue solution of the dynamic Eq. (3.4). Instead we reduce the multistate problem to the sequence of pairwise transitions and find transition probabilities under this presumption. Such a reduction is possible only within the *generalized* bow-tie model where the *interacting* potential curves cross only in pairs (as seen from the model statement, the interacting pairs include any slanted curve and one of two horizontal curves). On the contrary, in the conventional version of this model only single multiple crossing is present [see Fig. 1(a)] that apparently does not allow disentangling to some simpler pattern. Of course, in the generalized model, in addition to pairwise crossings between diabatic slanted and horizontal curves, there exists also multiple crossing of slanted curves at the central point $C$ [see Fig. 1(b)]. However, this crossing does not induce any transitions since the slanted curves *do not interact with each other*. This is clear even from the simplest example (3.7): all four branch points of the function $\mathcal{E}(t)$ are associated with the crossings between diabatic slanted and horizontal curves and there is no branch point stemming from the central point $C$.

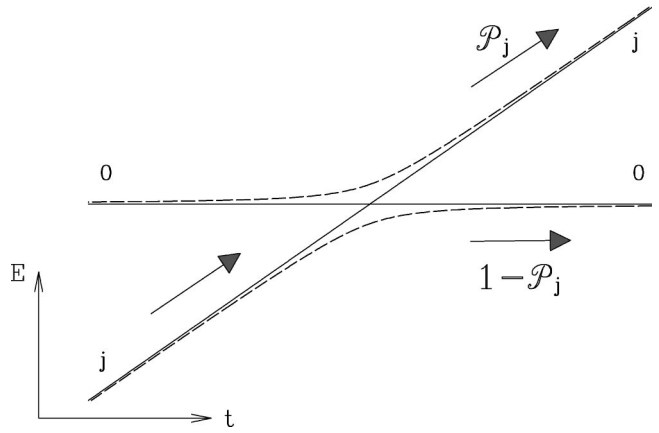


FIG. 2. Diabatic (solid) and adiabatic (dashed) potential curves in the conventional two-state Landau-Zener model. The slanted diabatic curve $j$ crosses the horizontal curve 0. The diabatic passage has probability $\mathcal{P}_j$ and the adiabatic passage has probability $1-\mathcal{P}_j$ (see text).

## IV. TRANSITION PROBABILITIES IN GENERALIZED BOW-TIE MODEL

### A. Transitions at the crossing of two diabatic curves

According to the standard time-dependent two-state Landau-Zener model, the probability of *diabatic passage* (i.e., the nonadiabatic transition probability) is

$$\mathcal{P}_j = \exp\left(-\frac{2\pi V_j^2}{|\beta_j|}\right), \tag{4.1}$$

assuming that one diabatic potential curve is horizontal, the other one has the slope $\beta_j$, and the coupling between the diabatic states is $V_j$. The probability of *adiabatic* passage (i.e., staying in the same adiabatic state) is $\mathcal{Q}_j=(1-\mathcal{P}_j)$ (see Fig. 2). The diabatic passage amounts to transition between adiabatic states; its probability is exponentially small, with the argument of exponent in Eq. (4.1) often referred to as the Massey parameter.

As discussed above (Sec. III), in the generalized bow-tie model the coupling $V_j$ is ''shared'' between two horizontal states $0^+$ and $0^-$. Namely, each of these two states is coupled to the slanted state $j$ by the coupling $(1/\sqrt{2})V_j$. Hence, if all other states except $0^+$ and $j$ would be absent, the probabilities of transition for diabatic passage ($0^+\Rightarrow 0^+$ or $j\Rightarrow j$) and adiabatic passage ($0^+\Rightarrow j$ or $j\Rightarrow 0^+$) were, respectively, $p_j$ and $(1-p_j)$ with

$$p_j = \exp\left(-\frac{\pi V_j^2}{|\beta_j|}\right). \tag{4.2}$$

The same probabilities describe transitions within the isolated pair of states $0^-$ and $j$. It is important to emphasize that the definitions of $\mathcal{P}_j$ and $p_j$ differ by a factor 2 in the exponential. This circumstance leads to the remarkable relation

$$\mathcal{P}_j = p_j^2. \tag{4.3}$$

Our task is to describe transitions within the generalized bow-tie model in terms of independent pairwise transitions in the spirit of factorization scheme discussed in the Introduction. One has to consider some path starting from the initially populated potential curve and terminating at the curve that corresponds to the final state of the specific transition considered. The path goes along some sequence of intermediate diabatic potential curves: the jump from one curve to another occurs at the point of crossing of these two curves. Hence the path represents a broken, piecewise-straight line. The propagation can proceed only in the direction of increasing time, but not backwards. Each passage of individual crossing is characterized by the probability $p_j$ of *diabatic* passage and the complementary probability $(1-p_j)$ of sudden turn (that means *adiabatic* passage). For brevity we denote the crossings of slanted potential curve $j$ with the horizontal potential curves $0^+$ and $0^-$, respectively, as $J^+$ and $J^-$ crossings. In the case when several paths are possible for given initial and final states the related amplitudes are to be summed up coherently.

### B. Single-path transitions

We start with the simplest case of transition when only one path connects particular initial and final states, hence a coherent summation does not emerge. There are three essentially distinct situations when such a single-path regime is operative. For definiteness below we consider the initial slanted state $j$ with a positive slope, i.e., the state $j>0$, or initial $0^-$ horizontal state. All the results are straightforwardly extended to the initial $j<0$ or $0^+$ states (see Sec. IV D).

#### *1. $j\to 0^-$ transitions*

As seen from Fig. 3(a), the only path possible for the $j\to 0^-$ transition implies adiabatic passage of the $J^-$ crossing followed by diabatic propagation along $0^-$ horizontal potential curve to $t\to\infty$. The latter means diabatic passage of all the crossings $N^-$ with $n>j>0$ and with $n<0$. The transition probability is

$$P_{j\to 0^-} = (1-p_j)\left(\prod_{n>j} p_n\right)\left(\prod_{n<0} p_n\right). \tag{4.4}$$

Note that hereafter in all expressions that include the product sign $\Pi$ the running index is denoted as $n(n\neq 0^{\pm})$.

#### *2. $0^-\to 0^-$ transitions*

The only way to achieve the $0^-\to 0^-$ transition is to follow along $0^-$ horizontal potential curve passing diabatically all the $N^-$ crossings with slanted potential curves. The transition probability for this simplest case reads

$$P_{0^-\to 0^-} = \prod_n p_n \equiv \left(\prod_{n>0} p_n\right)\left(\prod_{n<0} p_n\right). \tag{4.5}$$

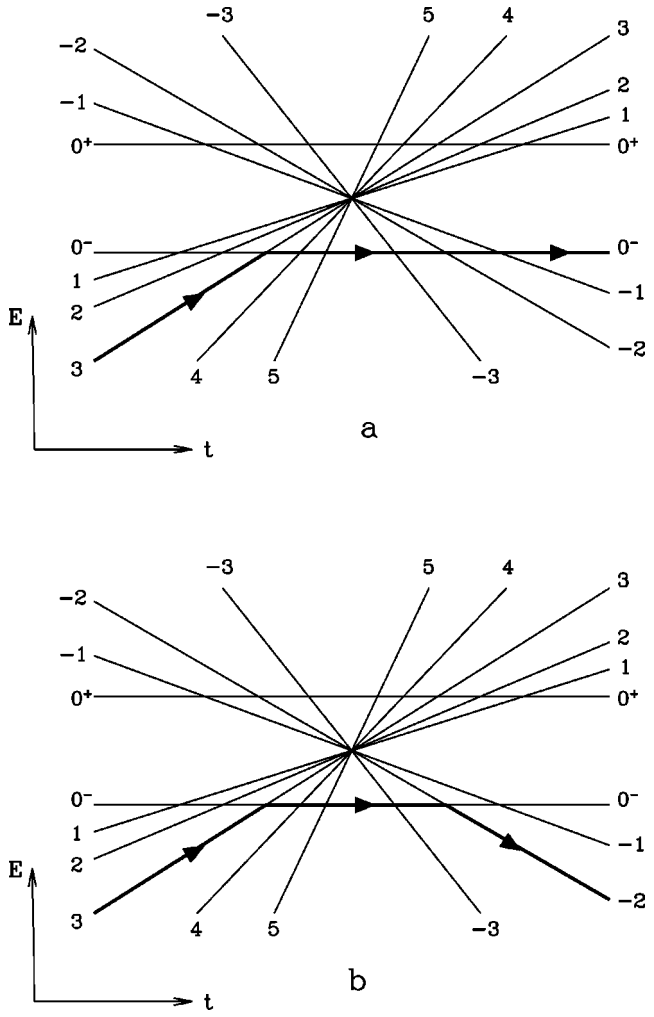


FIG. 3. Single-path transitions in the generalized bow-tie model. The transition paths are shown by thick broken line with arrows: (a) $j\to 0^-$ transitions (the case $j=3$ is shown); (b) $j\to k$ transitions with $j>0$, $k<0$ (the case $j=3$, $k=-2$ is shown).

#### *3. j→k transitions (j>0, k<0)*

The path shown in Fig. 3(b) implies adiabatic transition of the $J^-$ crossing, development along $0^-$ potential curve with diabatic passage of all the crossings $N^-$ with $n>j>0$ and with $n<k<0$ and finally adiabatic passage of the $K^-$ crossing. The overall transition probability is

$$P_{j\to k}=(1-p_j)(1-p_k)\left(\prod_{n>j} p_n\right)\left(\prod_{n<k} p_n\right). \qquad (4.6)$$

### C. Multipath transitions and interference

If several paths connect initial and final potential curves, then at first the transition amplitude is to be evaluated for each individual path. The amplitudes characterizing various paths are to be summed up coherently. The latter step requires an information about the phases of different contributions. The phases are generally comprised of ''geometrical'' and ''dynamical'' parts. The geometrical phases arise from the standard integral $\int E(t)dt$ over the path under consideration.

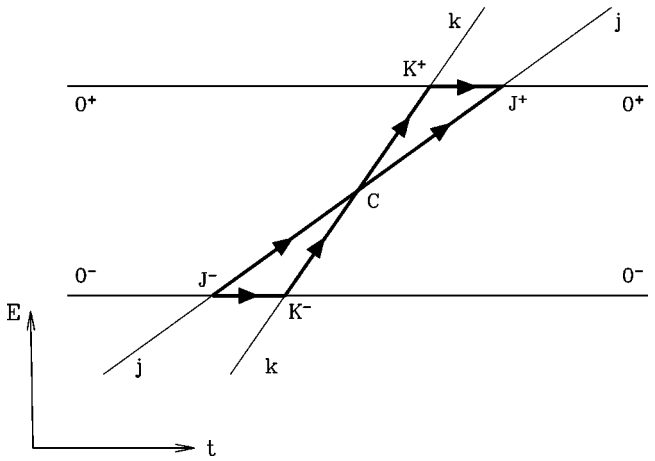


FIG. 4. Diabatic geometrical phases in the generalized bow-tie model. The paths $J^-\Rightarrow(C)\Rightarrow J^+$ and $J^-\Rightarrow K^-\Rightarrow(C)\Rightarrow K^+\Rightarrow J^+$ correspond to the same geometrical phase. This is seen from the fact that the phase difference is proportional to the sum of areas of two congruent triangles $J^-K^-C$ and $K^+J^+C$. The opposite signs are to be ascribed to these areas since the first path lies higher at $t<0$ whereas the second path lies higher in the $t>0$ domain.

The *difference* of diabatic geometrical phases between two paths is given by the integral $\oint E(t)dt$ over the loop formed by one path passed in the direction of increasing time and another path passed in the opposite direction. Generally this phase difference has a simple *geometrical* meaning that justifies our terminology: it equals the *area of the loop* in the $(E,T)$ plane. In the case of the generalized bow-tie model *this phase difference is always zero*, since the part of the loop that lies at $E>0$ exactly compensates the part lying at $E<0$ (see Fig. 4). It should be stressed that this property of the phases follows directly from the special $ET$ symmetry of the generalized bow-tie model, namely its symmetry under the reflection $\{E\Rightarrow -E,\ t\Rightarrow -t\}$. From this we conclude that the difference of adiabatic geometrical phases also is zero in this model.

The *dynamic* phase is the phase increment due to the strong interaction in the domain of intensive nonadiabatic transitions. Conventionally this phase is derived [9] for description in the basis of *adiabatic* states when the related potential curves $\mathcal{E}(t)$ are used for evaluation of phase integrals. These results are not applicable to our scheme based on diabatic representation. We use the following simple rules to incorporate the dynamic phase: for the diabatic passage of the crossing the phase is zero; each adiabatic passage induces increment of phase by $\frac{1}{2}\pi$. In order to support this choice we consider crossing within an isolated pair of states, the horizontal one and the slanted one with the slope $\beta_j$. Within the first-order nonstationary perturbation theory in the basis of diabatic states the amplitude of adiabatic passage is evaluated as

$$\mathcal{A}_j=i\int_{-\infty}^{\infty} V_j \exp\left(i\frac{1}{2}\beta_j t^2\right)dt=i\sqrt{\frac{2\pi}{|\beta_j|}}V_j \exp\left(i\frac{\pi}{4}\mathrm{sgn}(\beta_j)\right) \qquad (4.7)$$

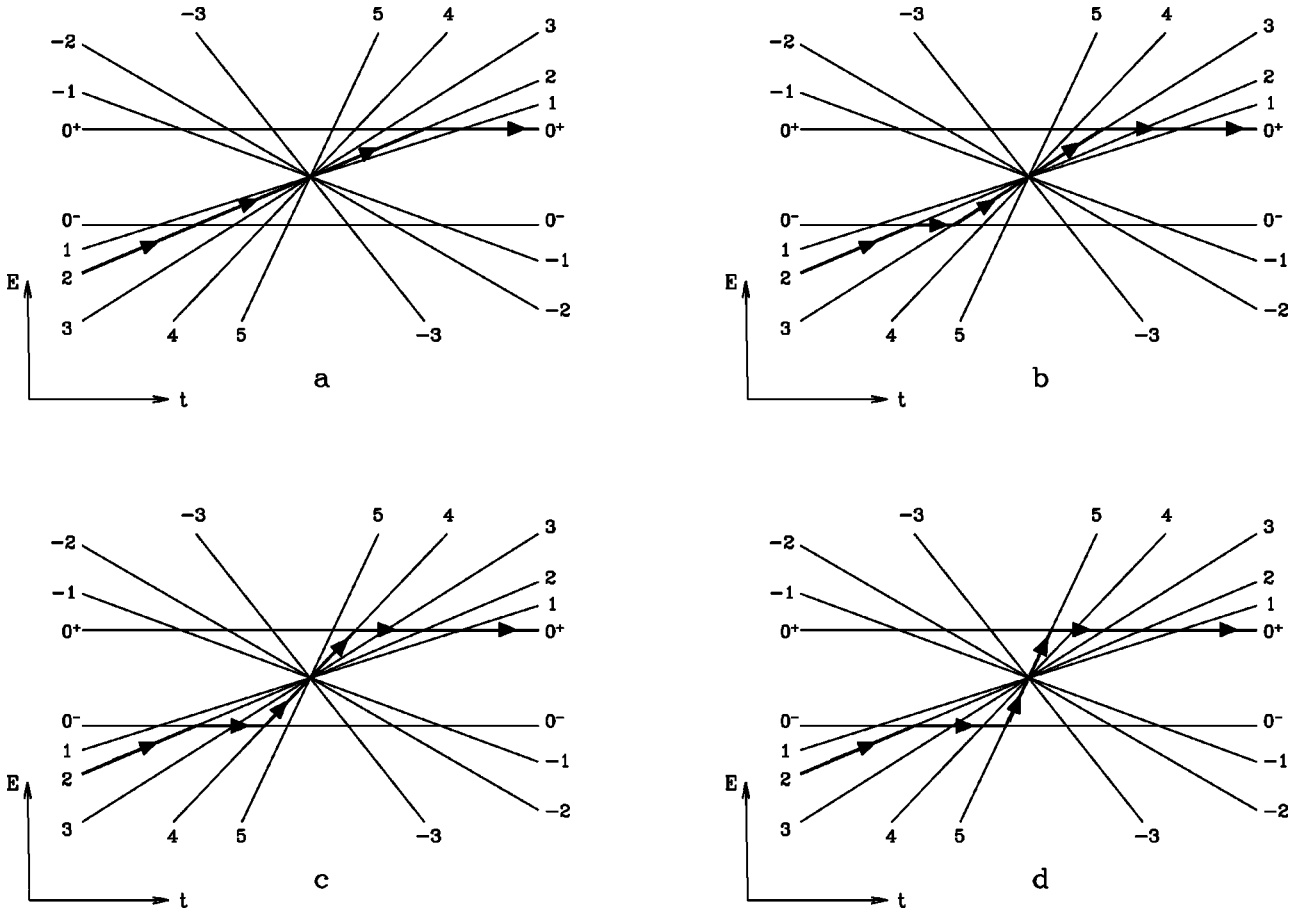


FIG. 5. Multipath transition $j\to 0^+$ in the generalized bow-tie model (the case $j=2$ is shown). Each plot (a), (b), (c), (d) pictures an individual transition path (thick broken line with arrows) corresponding to intermediate states respectively $m=2,3,4,5$.

(it is related to the probability of adiabatic passage discussed in Sec. IV A as $\mathcal{Q}_j=1-\mathcal{P}_j\approx|\mathcal{A}_j|^2$). One can easily check that in all application below the phase in Eq. (4.7) is equivalent to the phase $\frac{1}{2}\pi$. Indeed, when one evaluates the phase difference between the interfering paths, the factor $\exp[i(\pi/4)\mathrm{sgn}(\beta_j)]$ in the right-hand side of Eq. (4.7) does not play role (the turns clockwise and counterclockwise compensate each other). Therefore only the factor $i$ in Eq. (4.7) is effectively operative. Since the number of turns is even, the interference phase could accept only values $\pi$ or $2\pi$ depending on the specific path as detailed below.

#### *1. $j\to 0^+$ transitions*

Various paths for the $j\to 0^+$ transition are shown in Fig. 5. The path shown in Fig. 5(a) presents a particular, most simple case. It implies diabatic passage of the $J^-$ crossing followed by adiabatic passage of the $J^+$ crossing and subsequent propagation along the $0^+$ potential curve with diabatic passage of the crossings $N^+$ with $j>n>0$. We denote the related probability as $P^{(j)}_{j\to 0^+}$. It is straightforwardly evaluated as

$$P^{(j)}_{j\to 0^+}=p_j(1-p_j)\left(\prod_{j>n>0}p_n\right)\equiv(1-p_j)\left(\prod_{j\geqslant n>0}p_n\right). \tag{4.8}$$

The other paths proceed via the intermediate slanted potential curve with the positive slope $\beta_m$ larger than $\beta_j$ [see Figs. 5(b)–5(d)]. We label this intermediate curve as $m(m>j)$. The same index $m$ is employed also to label the path for the $j\to 0^+$ transition. The propagation along the $m$th path implies adiabatic passage of the crossing $J^-$ followed by diabatic passage of the crossings $N^-$ with $n$ lying in the interval $m>n>j$, then adiabatic passage of the crossing $M^-$, adiabatic passage of the crossing $M^+$ and finally diabatic passage of the crossings $N^+$ with $m>n>0$. The related transition probability reads

$$P^{(m)}_{j\to 0^+}=(1-p_j)\left(\prod_{m>n>j}p_n\right)(1-p_m)(1-p_m)\left(\prod_{m>n>0}p_n\right). \tag{4.9a}$$

It is conveniently rewritten as

$$P^{(m)}_{j\to 0^+}=(1-p_j)\left(\prod_{j\geqslant n>0}p_n\right)\left(\prod_{m>n>j}p_n\right)^2(1-p_m)^2. \tag{4.9b}$$

The amplitude of transition along the $m$th path has modulus $\sqrt{P^{(m)}_{j\to 0^+}}$. The phase of this amplitude is denoted as $\varphi_m$.

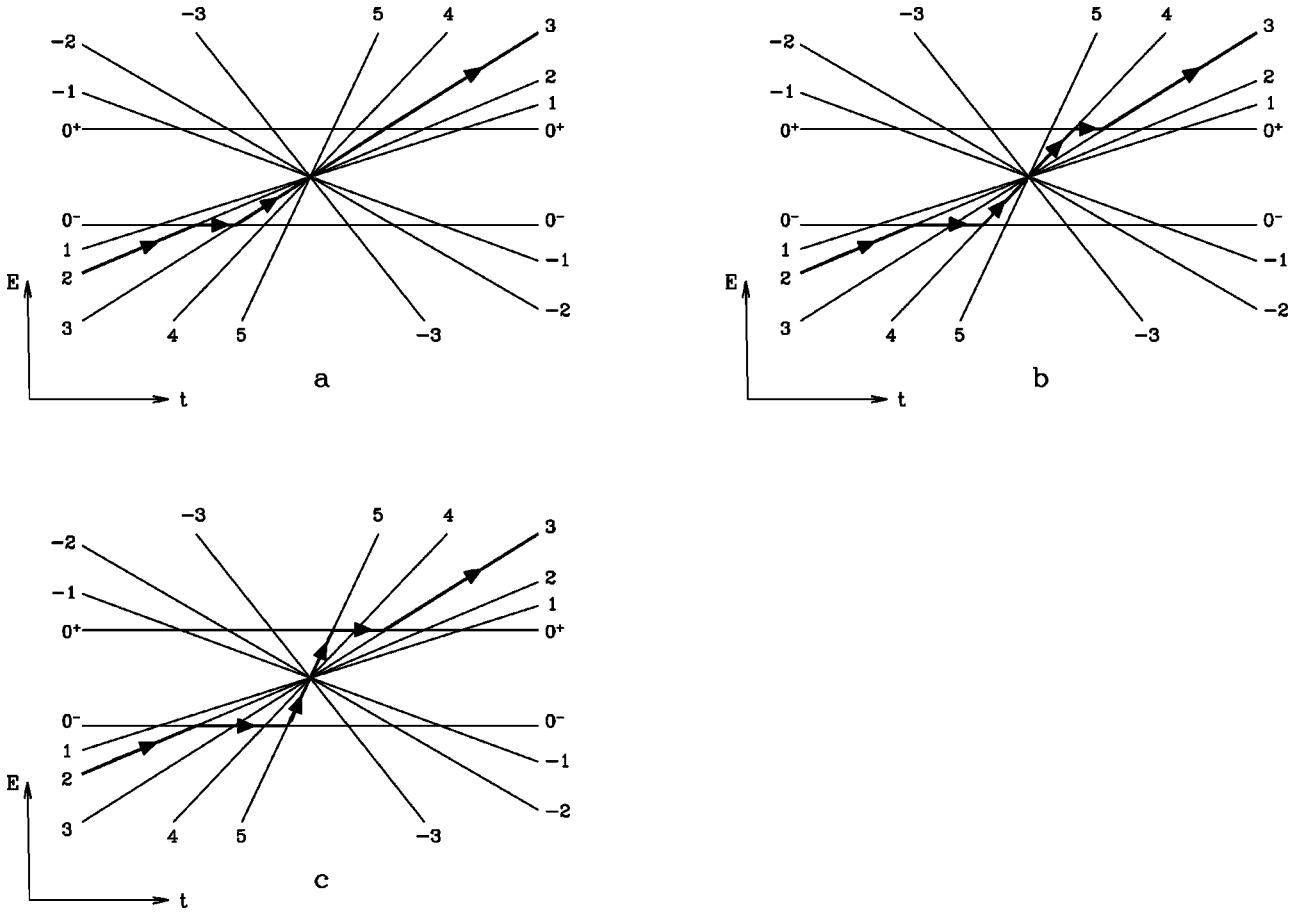


FIG. 6. Same as in Fig. 3, but for multipath transition $j\to k(j>0,\ k>0)$ (the particular case $j=2,\ k=3$ is shown). Plots (a), (b), (c) correspond to intermediate states, respectively, $m=3,4,5$.

The coherent summation of contributions from different paths to find the probability of transition $i\to f$ generally implies that

$$P_{i\to f}=\left|\sum_m \sqrt{P_{i\to f}^{(m)}}\exp(i\varphi_m)\right|^2. \tag{4.10}$$

In the present case $(i=j, f=0^+)$ we extend the summation in Eq. (4.10) over $m\geqslant j$ in order to embrace contribution from the path $m=j$ [Eq. (4.8)]. The latter path includes only one adiabatic passage, whereas all other paths with $m>j$ imply three adiabatic passages. Consequently, according to the rules formulated above, all the phases $\varphi_m$ with $m>j$ coincide $(\varphi_m=\varphi)$, whereas the phase $\varphi_j$ differs by $\pi$: $\varphi_j=\varphi-\pi$. Using these phases and the representations (4.8) and (4.9b) we obtain from Eq. (4.10)

$$P_{j\to 0^+}=(1-p_j)\left(\prod_{j\geqslant n>0} p_n\right)\times\left|1-\sum_{m>j}\left(\prod_{m>n>j} p_n\right)(1-p_m)\right|^2. \tag{4.11}$$

The summation over $m$ is carried out employing the useful formula (cf. Ref. [15])

$$\sum_{m>j}\left(\prod_{m>n>j} p_n\right)(1-p_m)=1-\prod_{n>j} p_n. \tag{4.12}$$

Finally we obtain

$$P_{j\to 0^+}=(1-p_j)\left(\prod_{j\geqslant n>0} p_n\right)\left(\prod_{n>j} p_n\right)^2, \tag{4.13a}$$

or, in equivalent form,

$$P_{j\to 0^+}=(1-p_j)\left(\prod_{n>j} p_n\right)\left(\prod_{n>0} p_n\right). \tag{4.13b}$$

#### *2. $j\to k$ transitions $(j>0,\ k>0)$*

As in the previous case, the $j\to k$ transitions proceed via various intermediate slanted potential curves $m$. Figure 6 corresponds to the case when $k>j$; the feasible intermediate states have $m\geqslant k$.

At first we consider separately the intermediate state $m=k$ [Fig. 6(a)]. The related path implies adiabatic passage of the $J^-$ crossing followed by diabatic passage of the crossings $N^-$ with $j<n<k$. After that the crossing $K^-$ is passed adiabatically and the crossing $K^+$ diabatically. This sequence leads to the following expression for the probability of transition along the path $m=k$:

$$P_{j\to k}^{(k)}=(1-p_j)\left(\prod_{k>n>j}p_n\right)(1-p_k)p_k$$

$$=(1-p_j)(1-p_k)\left(\prod_{k\geqslant n>j}p_n\right). \tag{4.14}$$

The case $m>k$ corresponds to more complicated path [Figs. 6(b) and 6(c)]. It passes the $J^-$ crossing adiabatically, then follows the $0^-$ potential curve passing diabatically the crossings $N^-$ with $j<n<m$. After that the crossings $M^-$ and $M^+$ are passed adiabatically. Subsequently the crossings $N^+$ with $m>n>k$ are passed diabatically. The propagation is completed by adiabatic passage of the crossing $K^+$. The probability of transition along this path is

$$P_{j\to k}^{(m)}=(1-p_j)\left(\prod_{m>n>j}p_n\right)(1-p_m)(1-p_m)$$

$$\times\left(\prod_{m>n>k}p_n\right)(1-p_k). \tag{4.15a}$$

The following equivalent representation is convenient for further calculations:

$$P_{j\to k}^{(m)}=(1-p_j)(1-p_k)\left(\prod_{k\geqslant n>j}p_n\right)(1-p_m)^2\left(\prod_{m>n>k}p_n\right)^2. \tag{4.15b}$$

As in the previous case, we carry out coherent summation (4.10) ($i=j,f=k$) over $m\geqslant k$, taking into account that the phases $\varphi_m=\varphi$ coincide for $m>k$, whereas for $m=k$ one has $\varphi_k=\varphi-\pi$ (respectively, two or four adiabatic passages along the path). Using formulas (4.14) and (4.15b) we obtain

$$P_{j\to k}=(1-p_j)(1-p_k)\left(\prod_{k\geqslant n>j}p_n\right)$$

$$\times\left[1-\sum_{m>k}(1-p_m)\left(\prod_{m>n>k}p_n\right)\right]^2. \tag{4.16}$$

The summation formula (4.12) allows us to cast expression (4.16) as

$$P_{j\to k}=(1-p_j)(1-p_k)\left(\prod_{k\geqslant n>j}p_n\right)\left(\prod_{n>k}p_n\right)^2, \tag{4.17a}$$

or, equivalently, as

$$P_{j\to k}=(1-p_j)(1-p_k)\left(\prod_{n>j}p_n\right)\left(\prod_{n>k}p_n\right), \tag{4.17b}$$

that coincides with Eq. (4.6). This is a remarkable and quite unexpected result: the analytical expressions for the transition probabilities in case of the single-path transition $j\to k$ ($j>0$, $k<0$) proves to be the same as these for the multipath transition $j\to k$ ($j,k>0$).

The symmetry of the expression (4.17b) with respect to permutation of $j$ and $k$ labels indicates that it applies not only for $k>j$ (as it was presumed in the present derivation), but also for $k<j$. This conclusion could be checked by a similar direct calculation that we omit here for brevity. However the transitions $j\to j$ are to be considered separately.

#### *3. $j\to j$ transitions*

The calculations for the $j\to j$ transitions differ from the previous case only in some details that are emphasized below. For the intermediate state $m=j$ (i.e., for direct propagation along the initial potential curve $j$) only two crossings are met, $J^-$ and $J^+$, both of them being passed diabatically. The probability is

$$P_{j\to j}^{(j)}=p_j^2. \tag{4.18}$$

For the path with intermediate state $m(m>j)$ the sequence is as follows: the $J^-$ crossing is passed adiabatically, the crossings $N^-$ with $m>n>j$ are passed diabatically, the crossings $M^-$ and $M^+$ are passed adiabatically, the crossings $N^+$ with $m>n>j$ are passed diabatically, and finally the crossing $J^+$ is passed adiabatically. The overall transition probability is

$$P_{j\to j}^{(m)}=(1-p_j)\left(\prod_{m>n>j}p_n\right)(1-p_m)(1-p_m)$$

$$\times\left(\prod_{m>n>j}p_n\right)(1-p_j)$$

$$=(1-p_j)^2(1-p_m)^2\left(\prod_{m>n>j}p_n\right)^2. \tag{4.19}$$

The coherent summation (4.10) ($i=f=j$) over all feasible paths ($m\geqslant j$) gives

$$P_{j\to j}=\left[p_j+\sum_{m\geqslant j}(1-p_j)(1-p_m)\left(\prod_{m>n>j}p_n\right)\right]^2$$

$$=\left\{p_j+(1-p_j)\left[1-\left(\prod_{n>j}p_n\right)\right]\right\}^2$$

$$=\left[1+\left(\prod_{n>j}p_n\right)p_j-\left(\prod_{n>j}p_n\right)\right]^2, \tag{4.20}$$

where we used the same summation formula (4.12) as before. Note that all the phases $\varphi_m$ effectively coincide in the present case, since zero or four adiabatic passages are met along the path $m=j$ or $m>j$, respectively.

#### *4. $0^-\to 0^+$ transitions*

Here all the slanted potential curves with $m>0$ play the role of intermediate states. For the $m$th path the crossings $N^-$ with $m<n$ are passed diabatically, the crossings $M^-$ and $M^+$ are passed adiabatically and then the crossings $N^+$ with $m<n$ are passed diabatically. The probability of transition via such a path is

$$P_{0^-\to 0^+}^{(m)}=\left(\prod_{m<n}p_n\right)(1-p_m)(1-p_m)\left(\prod_{m<n}p_n\right). \tag{4.21}$$

All the phases $\varphi_m$ coincide. The coherent summation is carried out as above:

$$P_{0^-\to 0^+}=\left[\sum_{m>0}(1-p_m)\left(\prod_{m<n}p_n\right)\right]^2=\left[1-\left(\prod_{n>0}p_n\right)\right]^2. \tag{4.22}$$

### D. Time reversal and detailed balance

The time-dependent Schrödinger equation in the generalized bow-tie model is invariant under the transformation $\{t\Rightarrow -t;\varepsilon\Rightarrow-\varepsilon\}$. From this we obtain the following ''detailed balance'' relations:

$$P_{j\to k}=P_{k\to j}\quad (j,k\neq 0^{\pm}), \tag{4.23a}$$

$$P_{0^{\pm}\to j}=P_{j\to 0^{\mp}}\quad (j\neq 0^{\pm}), \tag{4.23b}$$

$$P_{0^+\to 0^+}=P_{0^-\to 0^-}. \tag{4.23c}$$

Our derivation above was carried out for initial states with positive slope, or for initial $0^-$ state, and for all final states. The detailed balane relations (4.23) extend these results to the initial states with negative slope and to the initial $0^+$ state. Thus the complete set of transition probabilities is obtained. Note that the probability $P_{0^+\to 0^-}$ is unrelated to the probability $P_{0^-\to 0^+}$ [Eq. (4.22)]

$$P_{0^+\to 0^-}=\left[1-\left(\prod_{n<0}p_n\right)\right]^2. \tag{4.24}$$

The fact that $P_{0^-\to 0^+}$ and $P_{0^+\to 0^-}$ are different is obvious, since the first probability is governed by the intermediate states with positive slope, whereas in the second case the negative slope intermediate states are operative.

### E. Probability conservation

The following probability conservation conditions are satisfied:

$$P_{j\to 0^-}+P_{j\to 0^+}+\sum_{k\neq 0^{\pm}}P_{j\to k}=1, \tag{4.25a}$$

$$P_{0^-\to 0^-}+P_{0^-\to 0^+}+\sum_{k\neq 0^{\pm}}P_{0^-\to k}=1. \tag{4.25b}$$

They are easily checked using the summation formulas similar to Eq. (4.12). The complete formulation of *S*-matrix unitarity (i.e., orthogonality of its columns and rows) also could be checked but it needs identification of phases of transition amplitudes. We postpone analysis of this issue to the future mathematical paper [25].

## V. BOW-TIE MODEL PROBABILITIES AS A LIMITING CASE OF GENERALIZED BOW-TIE MODEL

The set of transition probabilities in the generalized bow-tie model (4.4), (4.5), (4.6), (4.13b), (4.17b), (4.20), (4.22) gives the major result of the present study. For any $j,k\neq 0^{\pm}$ our results for the $j\to k$ transitions probabilities as given by formulas (4.6), (4.17b), (4.20) coincide with those obtained earlier [15] for the conventional bow-tie model. Further, the following relation is easily checked by using the formulas (4.4), (4.13b):

$$P_{j\to 0^+}+P_{j\to 0^-}=\left(\prod_{n>j}p_n\right)(1-p_j)\left(\prod_{n>0}p_n+\prod_{n<0}p_n\right)\equiv P_{j\to 0}, \tag{5.1}$$

where the right-hand side is the transition probability in the bow-tie model [15]. The formula (5.1) reflects the fact that 0 state of the bow-tie model absorbs transitions $j\to 0^+$ and $j\to 0^-$ in the limit $\varepsilon\to 0$ of the generalized bow-tie model.

At last, for the transitions within the $\{0^+0^-\}$ manifold we obtain by using formulas (4.5), (4.22),

$$P_{0^+\to 0^+}+P_{0^-\to 0^-}+P_{0^+\to 0^-}+P_{0^-\to 0^+}=1+P_{0\to 0}, \tag{5.2}$$

$$P_{0\to 0}=\left[1-\prod_{n>0}p_n-\prod_{n<0}p_n\right]^2, \tag{5.3}$$

where $P_{0\to 0}$ is the probability of the $0\to 0$ transition found earlier [15] within the bow-tie model. The relation (5.2) reflects the fact that in the limit of the bow-tie model the sum in the left-hand side of Eq. (5.2) reduces to the sum of probabilities of transitions $0\to 0$ and $d\to d$, where for the fully decoupled state $d$ [Eq. (3.5b)] one has identically $P_{d\to d}\equiv 1$. The sum

$$\frac{1}{2}[P_{0^+\to 0^+}+P_{0^-\to 0^-}+P_{0^+\to 0^-}+P_{0^-\to 0^+}] \tag{5.4}$$

can be interpreted as a probability to remain within the $\{0^+,0^-\}$ manifold averaged over initial states. According to Eqs. (5.2), (5.3) it cannot be less than $\frac{1}{2}$. The latter value is achieved when $P_{0\to 0}=0$, i.e.,

$$\prod_{n>0}p_n+\prod_{n<0}p_n=1. \tag{5.5}$$

It is worthwhile to mention that when one considers continuous transition from adiabatic to diabatic regime at the individual pseudocrossings, i.e., augments the probabilities $p_j$ from 0 to 1 (that is equivalent to decreasing the couplings $V_j$ from $\infty$ to 0), one inevitably passes the borderline where the condition (5.5) is satisfied. Note also that among the entire set of transition probabilities in the bow-tie or generalized bow-tie model only one, namely the probability $P_{0\to 0}$ (5.3) in the bow-tie model, can turn zero for some choice of parameters. Zero of transition probability is to be attributed to the interference effects, albeit their manifestation in the generalized bow-tie model as a minimum in the averaged probability (5.4) does not look very lucid.

Thus the relation between the transition probabilities in the generalized bow-tie model and the original bow-tie model is quite simple and straightforward. The generalized bow-tie model could be also related to the Demkov-Osherov

model [10] in some limiting situation. Namely, one has to augment $\varepsilon$ and simultaneously vary the slopes of the slanted diabatic potential curves. If all the slopes tend to two limiting values (one for positive slopes and another for negative ones), then we obtain patterns characteristic of the Demkov-Osherov model: crossing of the diabatic potential curve ($0^+$ or $0^-$) by the set of (almost) parallel linear potential curves. Such a pattern is repeated four times (crossings of $0^+$ and $0^-$ curves by the slanted curves with positive and negative slopes).

## VI. ON JUSTIFICATION OF LANDAU-ZENER-TYPE MODEL REDUCTION TO SEQUENCE OF PAIRWISE TRANSITIONS

### A. Generalized bow-tie model

A remarkable property of the probabilities derived above for the bow-tie model is that they are independent of the parameter $\varepsilon$. It leads to far-reaching consequences. Indeed, for the fixed couplings $V_j$ one can always choose $\varepsilon$ large enough to ensure that the crossings of diabatic potential curves (or pseudocrossings of adiabatic potential curves) are well separated in $E$ and $t$ scales to justify reduction of the problem to the sequence of independent pairwise transitions. Since the expressions for the transition probabilities thus derived prove to be $\varepsilon$ independent, one can argue that the formulas are applicable also for small values of $\varepsilon$ and in the limit $\varepsilon \to 0$.

This heuristic mode of reasoning looks quite convincing being complementary to rigorous mathematical treatment, as discussed more in Sec. VII. Here we mention some subtlety in application of this kind of arguments. To see it consider the *shifted bow-tie model* that differs from the conventional bow-tie model of Sec. II in one respect: the horizontal potential curve has some nonzero energy $\epsilon$, while the multiple crossing of slanted potential curves lies, as before, at $E=0$. One can consider the simplest three-state case when only two slanted potential curves are present with the slopes of opposite sign. At first it seems that only a single path connects each initial and final state in this model, since the slanted curves are not coupled directly, as presumed in the bow-tie model. If this would be true, then the interference effects were absent and the transition probabilities were independent of $\epsilon$ just as in the case of the generalized bow-tie model. However, in fact the slanted curves are coupled, albeit *indirectly* (via intermediate horizontal curve), and the Landau-Zener-type transitions are actually operative in the vicinity of their crossing. This is seen, for instance, from the fact that the *adiabatic* potential curves exhibit *pseudocrossing* in the region where two slanted *diabatic* curves cross. Therefore, in fact, the shifted bow-tie model allows multipath interference. Consequently, the transition probabilities in this model depend on the value of $\epsilon$. The latter fact was tested in the numerical example by Brundobler and Elser [14].

### B. Probability of survival on the steepest diabatic potential curve

The similar reasoning could be applied to evaluation of some particular transition probabilities in the general Landau-Zener-type model discussed in Sec. I. Namely, we evaluate *the probabilities of survival* on the diabatic potential curves possessing the largest or the smallest value of the slope $\beta_j$. Consider the general Landau-Zener-type model of Sec. I defined by the set of $3N$ parameters $\{\varepsilon_j, \beta_j, V_j\}$ and let $\beta_M$ be the largest slope available. By increasing the value of the related parameter $\varepsilon_M$ one can always achieve situation where the crossings of the $M$th potential curve with all the other curves are well separated. In such a case reduction of the time propagation to the sequence of pairwise transitions is well justified.

The other point is that only a single propagation path is available for the $M \to M$ transition. Indeed, at the moment of time when the initially populated curve with the largest slope crosses any other curve the latter remains still unpopulated just because propagation in time proceeds only in forward direction. Therefore at each pseudocrossing between $M$th and $j$th diabatic curves only loss of the $M$th curve population occurs with the probability of diabatic transition described by a standard Landau-Zener expression:

$$p_{Mj} = \exp\left( -\frac{2\pi V_{Mj}^2}{|\beta_M - \beta_j|} \right). \tag{6.1}$$

It is to be emphasized again that the population of $M$th state does not receive any influx from the $j$th state because the latter is not populated as discussed above. As a result the probability of survival on the $M$th curve as $t$ varies from $-\infty$ to $\infty$, i.e., the probability of the single-path $M \to M$ transition is evaluated as a product of diabatic-propagation factors (6.1) for all $j$:

$$P_{M\to M} = \exp\left( -\sum_n \frac{2\pi V_{Mn}^2}{|\beta_M - \beta_n|} \right). \tag{6.2}$$

Since this result does not depend on $\varepsilon_M$ it could be extended to arbitrary values of this parameter. Previously the formula (6.2) for the states with extremal (maximal or minimal) slopes was tentatively suggested by Harmin [13] and Brundobler and Elser [14] and tested numerically for various choices of parameters in $H(t)$ including the situations with strong overlap of adjacent avoided crossing regions. Validity of this formula in the bow-tie model was demonstrated in Ref. [15].

Note that independence of the transition probabilities from the value of the parameter $\varepsilon$ reflects mathematically the fact that the probabilities do not depend on the geometrical phase difference. This situation emerges not only in case of single-path transitions as in Sec. VI B, but also for multipath transitions in the generalized bow-tie model, where the geometrical phase is always zero due to particular $ET$-symmetry property of the model.

## VII. CONCLUSION

In the present paper, among a variety of feasible multistate Landau-Zener-type problems we seek those which allow an exact solution with appealing physical interpretation. We show that the exactly solvable bow-tie model can be

embedded into the generalized bow-tie model as some particular limit of the latter. The dynamics of the generalized bow-tie model is interpreted as a sequence of two-state transitions each of which is described by the conventional Landau-Zener model. In the limit of the bow-tie model the multicrossing pattern is contracted into a point. It seems that the only way to provide an appealing interpretation of the transition probability in the bow-tie model is to consider it as a limiting case of the generalized bow-tie model as done above.

The rules used in evaluation of the transition amplitudes could be summarized as follows.

(i) Only forward propagation in time is to be considered.

(ii) Only the phase factors are gained in the course of time propagation between crossings of adiabatic potential curves adjacent on the time axis. The phases could be evaluated ''geometrically'' as areas in the $(E,t)$ plane (Sec. IV C). They are zero in the generalized bow-tie model due to its special $ET$ symmetry.

(iii) The crossing of two diabatic potential curves induces rearrangement within the related two-dimensional subspace of Hilbert space. It is described by the transformation matrix

$$\begin{pmatrix} \sqrt{p_j} & i\sqrt{1-p_j} \\ i\sqrt{1-p_j} & \sqrt{p_j} \end{pmatrix}, \tag{7.1}$$

where $p_j$ is the Landau-Zener probability of diabatic passage of the related crossing [in Fig. 1(b) these transformations are marked symbolically by small blocks around each crossing]. The expression (7.1) shows that the dynamic phase $\frac{1}{2}\pi$ is gained in the transition from one diabatic state to another.

According to these rules the matrix of transition amplitudes is constructed as a product of $N\times N$ unitary matrices of simple structure, namely either diagonal phase matrices of the item 2 above, or quasidiagonal matrices of item 3, that differ from the unit matrix only by one diagonal block of the form (7.1). The structure of these factor matrices is so simple that it is convenient to reformulate the problem in terms of propagation paths. The path is defined as a piecewise straight line that can change its slope only at the points where the diabatic potential curve intersect. The rules summarized above are straightforwardly reformulated as the rules that ascribe the transition amplitude to each path connecting given initial and final states. The contributions of all paths are to be added coherently.

Analysis of the generalized bow-tie model includes some features that hopefully might be of broad interest. Generally one has to account for the interference between various paths that connect initial and final states. It is universally known that in such a situation the interference phases play crucial role in the result. Commonly it is assumed that the phases vary smoothly when the system parameters are changed. This leads to the characteristic interference oscillations in the transition probabilities. The generalized bow-tie model presents an instructive exception from this common situation. Due to the particular $ET$ symmetry the phases cannot be altered continuously. Only two discrete values of the interference phase are realized, one possibility corresponds to purely constructive interference, another one gives purely destructive interference. One of these two possibilities takes place for each path independent of other parameters of the problem, such as couplings. Note that the calculated transition probabilities never turn zero due to interference effects. This feature also stems from the fact that the interference phase cannot be varied continuously thus preventing the probability oscillations characteristic of the conventional multipath interference problems.

Our scheme to describe the multipath interference follows simple rules outlined above. Since we do not resort to actual solution of the nonstationary Schrödinger equation one can question their validity. The latter is supported by two arguments. First, our scheme reproduces (Sec. V) all the results obtained by the contour integral method [15] in the limit of the conventional bow-tie model. This method provides a mathematically rigorous and complete solution of the nonstationary Schrödinger equation. Second, the arguments presented in Sec. VI, to our mind, strongly support our results for the generalized bow-tie model. In a sense we give here solution of the problem essentially without actual calculations, in contrast to the paper by Brundobler and Elser [14] who presented calculations (wave function in terms of contour integral), but not the solution (that implies derivation of transition probabilities) for the conventional bow-tie model. To make the treatment complete we plan to publish a full solution of the generalized bow-tie model [i.e., of Eqs. (3.4)] by the contour integral method [25]. The general scheme of calculations is similar to that employed in Ref. [15], but some important distinctions appear in realization. All the transition probabilities for the generalized bow-tie model are reproduced by this rather elaborate mathematical development that lies outside the scope of the present paper devoted to physically appealing aspects.

The important theoretical problem, still unresolved, is to find all cases within the generalized Landau-Zener model (Sec. I) which are exactly solvable. Probably this is the same as to find the cases when the matrix of transition amplitudes is composable from elementary matrices.

An attractive feature of the bow-tie and generalized bow-tie models is flexibility in the sense that the number of adjustable parameters is large (the slopes $\beta_j$, the couplings $V_j$, and additionally the parameter $\varepsilon$ in the generalized model). In this respect the model is similar to the Demkov-Osherov [10] model, whereas the exactly solvable multistate $SU(2)$ model considered by Hioe [26] effectively contains only one parameter. The physical realization of the bow-tie and generalized bow-tie models remains an important and not fully resolved question. The set of potential curves that are linear in time and cross at the same moment could be obtained for Rydberg atom placed into uniform electric and/or magnetic field with linear time dependence. The high-$l$ members of the Rydberg manifold are non-core-penetrating and therefore exhibit the hydrogenic degeneracy in the absence of external fields. When the fields are switched on, the energy splitting is linear in the field strengths within the range of applicability of the first-order perturbation theory. An additional state that interacts with all other states can be some state of different nature that occasionally is degenerate with the zero-field Rydberg manifold and interacts with it. Alternatively,

an extra state could correspond to the core-penetrating state, for instance, one with $l=0$. The experimentator can apply initially a strong external field, populate one of the levels under consideration by using the well developed laser technique, then change the external field strength linearly in time so as to obtain again the strong field but with opposite direction. After that repopulation of different levels could be investigated. The applicability of bow-tie model to this sort of problem needs an additional analysis.


## ACKNOWLEDGMENTS

The authors are thankful to Dr. P. N. Price for providing part of his unpublished thesis.


## APPENDIX: SLOPES OF ADIABATIC POTENTIAL CURVES IN THE BOW-TIE MODEL AT $t=0$

The equation defining the adiabatic potential curves $\mathcal{E}(t)$ for the bow-tie model was presented in Ref. [15] as

$$\left(\prod_j (\beta_j t-\mathcal{E})\right)\left(\mathcal{E}+\sum_j \frac{V_j^2}{\beta_j t-\mathcal{E}}\right)=0. \tag{A1}$$

In this appendix we derive an interesting property of the slopes of adiabatic potential curves at $t=0$. We seek for the adiabatic potential curves in the linear form $\mathcal{E}(t)=\gamma t$, where various solutions for $\gamma$ give the set of slopes under consideration. The equation for $\gamma$ follows from Eq. (A1) considered in the limit $t\to 0$:

$$F(\gamma)=0, \quad F(\gamma)=\sum_j \frac{V_j^2}{\beta_j-\gamma}. \tag{A2}$$

Obviously, $F(\gamma)$ behaves similarly to the tangent function increasing monotonically from $-\infty$ to $+\infty$ in each interval $\beta_{n-1}<\gamma<\beta_n$. In such an interval inevitably the function $F(\gamma)$ crosses the abscissa axis at some point $\gamma_n$ that gives one of the slopes of interest. Thus we conclude that the slopes of adiabatic potential curves at zero time lie in between the adjacent slopes at infinite time (the latter ones coincide with $\beta_j$). From Eq. (A1) it is readily seen that the adiabatic curves never cross the diabatic ones, except at the origin $t=0$, $E=0$. They stay in the sector between the straight lines $E=\beta_{n-1}t$ and $E=\gamma_n t$.

Equation (A2) has $(N-2)$ solutions for the slopes $\gamma_n$. As discussed in Ref. [15], there are also two adiabatic potential curves that take nonzero values at $t=0$:

$$\mathcal{E}(0)=\pm\sqrt{\sum_j V_j^2}. \tag{A3}$$

The related slopes are easily calculated using Eq. (A1). They coincide for both curves:

$$\gamma=\frac{1}{2[\mathcal{E}(0)]^2}\sum_j \beta_j V_j^2. \tag{A4}$$

Note that $\gamma$ turns zero in the symmetric case: $\beta_{-j}=-\beta_j, V_{-j}=V_j$.


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