# Partial-wave dispersion relations: Exact left-hand *E*-plane discontinuity computed from the Born series

D. Bessis[1] and A. Temkin[2]
[1]*Center for Theoretical Studies of Physical Systems, Clark-Atlanta University, Atlanta, Georgia 30314*
[2]*Laboratory for Astronomy and Solar Physics, Goddard Space Flight Center, National Aeronautics and Space Administration, Greenbelt, Maryland 20771*

(Received 22 March 1999; published 9 February 2000)

We show that for a superposition of Yukawa potentials, the *exact* left-hand cut discontinuity in the complex-energy plane of the (*S*-wave) scattering amplitude is given, in an interval depending on *n*, by the discontinuity of the Born series stopped at order *n*. This establishes an *inverse* and unexpected correspondence of the Born series at positive high energies and negative low energies. With the discontinuity on the left-hand axis elucidated, we can construct a viable dispersion relation (DR) for the partial (*S*-) wave amplitude. The DR is numerically verified for the exponential potential at zero scattering energy. Generalization to higher partial waves, and extension of these ideas to field theory are discussed.

PACS number(s): 34.90.+q, 34.80.−i, 11.55.Fv

The main limitation in the application of dispersion relation (DR) to quantum physics comes from the absence of any reliable and systematic method for computing the *left-hand cut* discontinuity in the complex-energy plane. In a DR in the complex-energy plane, one has to deal with two different problems. The evaluation of the *right-hand cut discontinuity* (so-called *physical*) and the *left-hand cut discontinuity* (so-called *unphysical*). While the evaluation of the right-hand cut discontinuity involves only the knowledge of physical quantities such as the partial-wave cross sections that can be measured experimentally, this is not the case for the left-hand cut discontinuity that must be computed *theoretically*. Presently, no *reliable* and systematic method to solve this problem exists. We believe this paper provides a very elegant and powerful solution to it.

By way of introducing the problem of computing the left-hand cut discontinuity in energy dispersion relations, let us first discuss, since this is our primary interest, the importance of dispersion relations in electron-atom scattering. A correct dispersion relation has not been derived, even after about 40 years of trying [1]. The major difficulty has been the understanding of the analytic structure of the scattering amplitude caused by the exchange (i.e., the identity of the incident and orbital electrons). Blum and Burke [2] attempted to resolve the problem whose importance is evident from the quotation, ''The future of dispersion relations in atomic physics depends critically on a better understanding of the nature of the singularities of the left-hand cut'' (in the complex energy plane).

In this paper we initiate an alternative attack on this problem. The key idea in the solution lies in the derivation of dispersion relations for *partial waves*. The contribution to the left-hand cut due to the exchange will be dealt with in another paper [3] (see also the discussion in the conclusion of the extension of our results to the full three-body problem and to field theory). In nonrelativistic quantum mechanics, the Born series is an approximation good at high energies. In our case, we show that it provides *exact* results in the unphysical *low*-energy region.

We now provide a complete proof for the basic case of the two-body nonrelativistic problem with a potential that is a *superposition* of Yukawa potentials

$$V(r)=\int_{\mu}^{+\infty} C(\alpha)e^{-\alpha r}d\alpha. \tag{1}$$

In Refs. [4], [5], it has been shown for a linear superposition of Yukawa potentials that the solution of the *S*-wave Schrödinger equation (SE), can be written as usual as

$$\psi_k(r)=S(k)f(-k,r)e^{ikr}-f(k,r)e^{-ikr}, \tag{2}$$

where $E=k^2$, and

$$f(\pm k,r)\rightarrow 1 \quad \text{for} \quad r\rightarrow +\infty, \tag{3}$$

while the $S$ matrix, $S(k)$, is given in terms of the Jost functions, $f(\pm k,r)$, by

$$S(k)=\frac{f(k,0)}{f(-k,0)}. \tag{4}$$

The Jost function is the solution of the modified SE

$$\left[\frac{d^2}{dr^2}-2ik\frac{d}{dr}-\lambda V(r)\right]f(k,r)=0, \tag{5}$$

where $\lambda$ is the coupling constant. The solution of Eq. (5) can be analyzed analytically in terms of Laplace transforms [4]. Writing

$$f(k,r)=1+\int_{\mu}^{+\infty}\rho_k(\alpha)e^{-\alpha r}d\alpha \tag{6}$$

one can show that $\rho_k(\alpha)$ can be calculated in segments [5], the general equation for which is

$$\alpha(\alpha+2ik)\rho_k(\alpha)=\lambda C(\alpha)+\int_{\mu^-}^{\alpha-\mu}\lambda C(\alpha-\beta)\rho_k(\beta)d\beta. \tag{7}$$

1050-2947/2000/61(3)/032702(3)/$15.00

Equation (7) (where we define $\mu^-$ to be $\lim_{\epsilon\to 0^+}\mu-\epsilon$) determines $\rho_k(\alpha)$ for any *finite* value of $\alpha$ in a *finite* number of steps. In particular, $\rho_k(\alpha)=0$ for $\alpha<\mu$ and $\alpha(\alpha+2ik)\rho_k(\alpha)=\lambda C(\alpha)$ for $\mu^-\leqslant\alpha<2\mu$, etc. When $k$ is outside the cut $i(\mu/2)\to i\infty$, $\rho_k(\alpha)$ is well defined for all values of $\alpha$. Note the fundamental fact that $\rho_k(\alpha)$ on the interval $\mu^-\leqslant\alpha\leqslant(n+1)\mu^-$, $n=1,2,\ldots$ is a *polynomial of degree exactly n in the coupling constant* $\lambda$ (with no constant term).

Furthermore, the Jost function $f(k,0)$ has a cut along the imaginary positive axis running from $i(\mu/2)\to i\infty$ and *no other singularities*. It tends to 1 at infinity, sufficiently rapidly that one can write a simple dispersion relation for it

$$f(k,0)=1-i\int_{\mu/2}^{+\infty}\frac{\omega(\chi)}{k-i\chi}d\chi, \tag{8}$$

where

$$\omega(\chi)=\frac{-1}{2i\pi}\mathcal{D}f(k,0)|_{k=i\chi},\quad \chi\geqslant\frac{\mu}{2}. \tag{9}$$

($\mathcal{D}$ represents the discontinuity of a function cut along the positive imaginary axis; it is defined as being the difference between the right and the left values on the cut. For a function having its cut along the real axis, it will be the difference between the upper and lower values on the cut.)

From Eq. (4), the $S$ matrix can be written in the form

$$S(k)=\frac{1-i\int_{\mu/2}^{+\infty}\frac{\omega(\chi)}{k-i\chi}d\chi}{1+i\int_{\mu/2}^{+\infty}\frac{\omega(\chi)}{k+i\chi}d\chi}. \tag{10}$$

Let $2\pi iD(\chi)$ be the discontinuity of $S(k)$ across the upper $k$ cut which corresponds to the left-hand cut in the $E=k^2$ plane. It has been shown [5] that

$$-2\chi D(\chi)=\lambda C(2\chi)+\int_{\mu^-}^{2\chi-\mu^-}\lambda C(2\chi-\beta)\rho_{i\chi}(\beta)d\beta. \tag{11}$$

Equation (11) shows clearly, just as Eq. (7) does for $\rho_k(\alpha)$, that the discontinuity of the left-hand cut, in the interval $[-(n+1)^2(\mu^2/4)\leqslant E\leqslant-(n)^2(\mu^2/4)]$ is *exactly a polynomial of degree n in* $\lambda$. It then follows that the exact left-hand cut discontinuity of the $S$ matrix on the interval $[-(n+1)^2(\mu^2/4)\leqslant E\leqslant-(n)^2(\mu^2/4)]$ has to be exactly equal to that of the Born series stopped at order $n$. To demonstrate this statement, let us first assume that the Born series converges ($|\lambda|<\lambda_c$). We compute the left-hand cut discontinuity, $\Delta_{\text{left}}(E)$, in the interval $[-(n+1)^2(\mu^2/4)\leqslant E\leqslant-(n)^2(\mu^2/4)]$, in two ways. First, using the convergent Born series

$$\Delta_{\text{left}}(E)=\sum_{p=1}^{\infty}\lambda^p\Delta^{(B)}_{\text{left},p}(E) \tag{12}$$

and second, using the Martin algorithm defined by Eq. (11)

$$\Delta_{\text{left}}(E)=\sum_{p=1}^{p=n}\lambda^p\Delta^{(M)}_{\text{left},p}(E). \tag{13}$$

Since Eq. (12) has to be identical to Eq. (13) for every $|\lambda|<\lambda_c$, it follows that the coefficients of each $\lambda^p$ must be identical. This implies that

$$\Delta^{(B)}_{\text{left},p}(E)=\Delta^{(M)}_{\text{left},p}(E)\quad 1\leqslant p\leqslant n \tag{14}$$

all the $\Delta^{(B)}_{\text{left},p}(E)$ are identically zero for $p>n$ when $E$ belongs to the interval $[-(n+1)^2(\mu^2/4)\leqslant E\leqslant-(n)^2(\mu^2/4)]$.

Even if the Born series has a zero radius of convergence (field theory), it is still an asymptotic series and its left-hand cut discontinuity in the energy can be computed as an asymptotic series. Now, we know from the previous argument that this asymptotic series terminates at the $n$th contribution on the interval $[-(n+1)^2(\mu^2/4)\leqslant E\leqslant-(n)^2(\mu^2/4)]$ because it has to be a polynomial of exact degree $n$. Therefore, all the higher order contributions vanish identically while the lower ones must give the exact result. We illustrate this result for the first Born $S$ wave. We have

$$S^{(B)}(k)=1-2ikF_1^{(B)}(k), \tag{15}$$

where

$$F_1^{(B)}(k)=\frac{1}{k^2}\int_0^{+\infty}\sin^2(kr)V(r)dr. \tag{16}$$

Substituting for $V(r)$ from Eq. (1) and carrying out the integration over $r$ gives

$$F_1^{(B)}(k)=\frac{\lambda}{2}\int_{\mu^-}^{+\infty}\frac{C(\alpha)}{\left(\frac{\alpha}{2}\right)^2+k^2}\frac{d\alpha}{\alpha}$$
$$=-\frac{\lambda}{2}\int_{\mu^-}^{+\infty}\left[\frac{1}{\frac{\alpha}{2}+ik}+\frac{1}{\frac{\alpha}{2}-ik}\right]\frac{C(\alpha)}{\alpha^2}d\alpha. \tag{17}$$

Only the first term of in the second expression of Eq. (17) contributes to the left-hand cut discontinuity in the $E$ complex plane. From it we derive that the discontinuity (up to a factor of $2i\pi$) is

$$D^{(B)}(\chi)=-\lambda\frac{C(2\chi)}{2\chi}. \tag{18}$$

Also, directly from Eq. (11), in the first interval $[\mu/2<\chi<\mu]$, the exact discontinuity (up to a factor of $2i\pi$) is exactly what is given by Eq. (18).

Our results have two major implications: (1) they show that it is possible (and how) to calculate results on the unphysical cut from finite order perturbation theory (i.e., the Born series) and (2) they are both theoretical and practical in nature. Because the left-hand cut discontinuity in the $E$ plane can now be evaluated from the Born series, while the right-hand cut discontinuity can be evaluated from physical measurements, self-consistent evaluations of the amplitude can be achieved using *dispersion relations* for the partial wave amplitudes. To illustrate this point, let us consider the exponential potential $V(r) = -\exp(-r)$. For this potential the $S$ wave SE is analytically solvable [6]. In particular, the left-hand cut reduces to an infinite series of poles, which is reflected in the sum term of the dispersion relation for the scattering length $A$, (i.e., the scattering amplitude at $E=0$) given below

$$-A = \sum_{n=0}^{+\infty} \mathrm{Res}\left[\frac{f(E_n)}{E_n}\right] + \frac{1}{\pi}\int_0^{+\infty} \frac{\mathrm{Im}\, f(E')}{E'}\, dE'. \quad (19)$$

The terms of the residue sum can be evaluated explicitly (in this particular case either from the analytic solution or, as we have shown, from the Born series)

$$\mathrm{Res}\left[\frac{f(E_n)}{E_n}\right] = \frac{2}{(n+1)[(n+1)!]^2}. \quad (20)$$

They are seen to vanish even faster than $[(n+1)!]^{-2}$. Although this convergence will diminish somewhat with increasing $E$, Eq. (20) is fairly representative of the convergence of the Born series *on the left-hand cut* for exponentially damped potentials. A simple heuristic argument shows that one can expect a decrease of the left-hand discontinuity on the interval $[-(n+1)^2(\mu^2/4) \leqslant E \leqslant -(n)^2(\mu^2/4)]$, to be of the order of $[n^2(\mu^2/4)]^{-n}$, corresponding to an enlargement of the domain of analyticity towards the left of $n^2(\mu^2/4)$.

Numerical values for the DR Eq. (19) are given in Table I. The positive energy term was computed from the formula $\mathrm{Im}\, f(E) = (\sqrt{E}/4\pi)\sigma_s(E)$, where $\sigma_s(E)$ is the $S$-wave cross section at energy $E$. The entries in Table I are exact up to the significant figures given. Note the convergence in $N$ of the residue sum (coming from the Born series!) which is the counterpart of the left-hand cut discontinuity in this case.

We conclude by pointing out some possible generalizations. From Ref. [7], it is clear that our result can be extended readily to any physical partial waves as well as to complex angular momentum partial waves and coupled channels. It has been shown [8] that for any field theory (e.g., a single positive mass $\mu$) the analytic continuation of the (on-shell or off-shell) $t$-channel scattering function has discontinuities (i.e., absorptive parts) in the crossed $s$ and $u$ channels whose values in any given energy strip $(n\mu)^2 < s$ (respectively, $u$) $< (n+1)^2\mu^2$ (at $t<0$) are completely determined in terms of the first $n$ iterates of the underlying $t$-channel Bethe-Salpeter type kernel of the theory. The symbols $s$, $t$, and $u$ have their usual meaning, $s = 4\mu^2 + 4k^2$, $t = -2k^2(1-\cos\theta)$, $u = -2k^2(1+\cos\theta)$, where $E = k^2$ is the kinetic energy and $\theta$ is the scattering angle.

TABLE I. $S$-wave dispersion relation for the exponential potential $V(r) = -\exp(r)$ at $E=0$, Eq. (19).

| $N$ | Exact value ($-A$) | Residue sum | Integral | Total |
|---|---|---|---|---|
| 0 | | 2 | 3.7376476295 | 5.737647 |
| 1 | | 2.25 | | 5.987647 |
| 2 | | 2.25815 | | 5.89500 |
| 3 | | 2.26937 | | 6.007018 |
| ∞ | 6.00706264 | 2.26941500 | | 6.00706264 |

Finally, the possibility of extending the same type of results to the $N$-body collision theory is also indicated by some field theoretical results for the 6-point function and 3 to 3 collision amplitudes, where similar structures of subchannel or crossed channel discontinuities are exhibited [9]. By using the close structural similarity between the Lippman-Schwinger and the (exact) Bethe-Salpeter type integral equations in off-shell momentum space, one can make use of the common properties to Yukawa-type potential theory and local quantum field theory [10], to make a direct extension of those results in the nonrelativistic case.

We wish to thank Dr. Richard J. Drachman for valuable discussions and for using MATHEMATICA to evaluate the integral in Table I. One of us (A.T.) would like to thank Professor Alfred Msezane for the kind hospitality extended to him at the Center for Theoretical Studies of Physical Systems, Clark Atlanta University. Both of us thank him for a careful reading of the manuscript.

[1] E. Gerjuoy and N. A. Krall, Phys. Rev. **119**, 705 (1960).
[2] K. Blum and P. G. Burke, Phys. Rev. A **16**, 163 (1977).
[3] A. Temkin and R. J. Drachman (unpublished).
[4] A. Martin, Nuovo Cimento **14**, 403 (1959).
[5] A. Martin, Nuovo Cimento **19**, 1257 (1961).
[6] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions*, 3rd ed. (Oxford University, Oxford, 1965), p. 143.
[7] A. Martin, Nuovo Cimento **21**, 157 (1961).
[8] J. Bros, in *Analytic Methods in Mathematical Physics*, edited by R. P. Gilbert and R. P. Newton (Gordon & Breach, New York, 1970), p. 85.
[9] J. Bros and M. Lassalle, Ann. Inst. Henri Poincare, Sect. A **27**, 279 (1977); (unpublished); (unpublished).
[10] J. Bros (private communication).