Quantum-state estimation by self-learning measurements

Dietmar G. Fischer, Stefan H. Kienle, and Matthias Freyberger
Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany
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We describe strategies for quantum-state estimation based on self-learning algorithms. In contrast to the optimal estimation procedures proposed earlier, our schemes rely on measurements performed separately on each quantum system in a finite ensemble. We numerically simulate our strategies in the case of finite ensembles of qubits and compare the resulting average fidelities to the fidelity of optimal quantum-state estimation.

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I. INTRODUCTION

Quantum mechanics is a theory based on ensembles of objects. Usually these ensembles are assumed to be infinite. In this case the corresponding quantum state, containing the complete information about the ensemble, can be determined exactly, provided that we perform suitable measurements [1]. In practice, however, we never encounter infinite ensembles which would require infinite resources of energy and time. Therefore, the question arises how much information can be extracted form a finite ensemble of quantum objects. Are we able to identify the underlying quantum state [2]? In general this will be impossible, but we can estimate [3,4] the state guided by an appropriate strategy. Of course, the fidelity of the estimation strongly depends on this strategy. It was shown [5–9] that optimal estimation procedures must view the finite ensemble of quantum objects as a single composite system. It is then possible to design a universal algorithm, which leads to a finite positive operator valued measurement (POVM) for the combined system [7,8]. The result of such a generalized measurement gives the best possible estimation of the quantum state of the identically prepared quantum objects.

As a consequence, separate measurements on each quantum object will not result in an optimal estimation. However, separate measurements are in practice relatively easy to realize. Hence we can ask for the difference between optimal state estimation and estimation strategies based on measurements that are performed on each quantum object separately. This is the scope of the discussion in the present paper. In particular, we shall analyze strategies which rest upon self-learning algorithms. That is, we optimally design the nth measurement by using the results of the $(n-1)$th, $(n-2)$th, etc., measurement. We shall exemplify these algorithms for finite ensembles of N spin 1/2 systems (qubits).

We emphasize that such a state estimation is not of academic interest only. Let us imagine a situation in which a quantum algorithm has been implemented on a quantum computer. One would now start test runs. A test could be the output of a certain qubit which then has to be measured. Since we cannot start the quantum computer infinitely often, we will always be confronted with a finite ensemble of test qubits. Moreover, we will only get these qubits sequentially [10] and hence we have to measure them one after the other.

The paper is organized as follows. In Sec. II we introduce the quantum system we are dealing with and the corresponding notation. Section III contains the description of the self-learning algorithm that is used in connection with the different optimization strategies presented in Sec. IV. The results of our state estimation schemes are shown and discussed in Sec. V followed by some concluding remarks in Sec. VI.

II. QUANTUM SYSTEM AND MEASUREMENT OPERATOR

In this paper we will concentrate on the state estimation for two-level systems in order to keep the formalism and the presented ideas as simple as possible. We want to stress, however, that the discussed schemes can also be applied to arbitrary d-level quantum systems. The problem is the following. Suppose we are given N identical two-level systems prepared in the pure state $|\psi\rangle$. The task is to estimate $|\psi\rangle$ by using experimentally realizable measurements on the two-level systems.

Let us first define the notation and the key elements of our analysis. Any pure quantum state of a two-level system (qubit) can be written down in the Bloch-sphere representation

$$|\theta,\phi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle, \tag{1}$$

where $|0\rangle$ and $|1\rangle$ are the two basis states. The parameters $\theta\in[0,\pi]$, $\phi\in[0,2\pi)$ uniquely determine a state vector and hence we can associate $|\theta,\phi\rangle$ to the point (θ,ϕ) on the Bloch sphere.

By using these states $|\theta,\phi\rangle$ we can find a parametrization of general density matrices

$$\hat{\rho} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\, w(\theta,\phi)\,|\theta,\phi\rangle\langle\theta,\phi| \tag{2}$$

with the normalized probability distribution $w(\theta,\phi)$, i.e.,

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \;\; w(\theta,\phi) = 1. \tag{3}$$

To guarantee the experimental realizability of our proposed measurement strategy we restrict ourselves to a simple class of measurement operators. This class consists of von Neumann measurements, e.g., polarization or spin measurements

along a certain axis (θ_m, ϕ_m). The tunable parameters θ_m and ϕ_m define the direction of the projection on the Bloch sphere surface. The corresponding projection operator reads

$$\hat{\Pi}(\theta_m, \phi_m) = |\theta_m, \phi_m\rangle\langle\theta_m, \phi_m| \tag{4}$$

with

$$|\theta_m, \phi_m\rangle = \cos\frac{\theta_m}{2}|0\rangle + \sin\frac{\theta_m}{2}e^{i\phi_m}|1\rangle. \tag{5}$$

We therefore find two measurement results. Either the system described by $|\theta, \phi\rangle$, Eq. (1), is polarized in the direction (θ_m, ϕ_m) or in the opposite direction given by $(\pi - \theta_m, \pi + \phi_m)$. We encode the first result by the number 1 and the second by the number 0.

The two possible outcomes of the measurement occur with the probabilities

$$\begin{aligned} P_1(\theta, \phi|\theta_m, \phi_m) &= |\langle\theta_m, \phi_m|\theta, \phi\rangle|^2 \\ &= \cos^2\frac{\theta - \theta_m}{2}\cos^2\frac{\phi - \phi_m}{2} \\ &\quad + \cos^2\frac{\theta + \theta_m}{2}\sin^2\frac{\phi - \phi_m}{2} \end{aligned} \tag{6a}$$

and

$$\begin{aligned} P_0(\theta, \phi|\theta_m, \phi_m) &= 1 - P_1(\theta, \phi|\theta_m, \phi_m) \\ &= \sin^2\frac{\theta - \theta_m}{2}\cos^2\frac{\phi - \phi_m}{2} \\ &\quad + \sin^2\frac{\theta + \theta_m}{2}\sin^2\frac{\phi - \phi_m}{2} \end{aligned} \tag{6b}$$

and clearly depend on the chosen direction and on the measured quantum state.

III. SELF-LEARNING ALGORITHMS

We now propose a self-learning measurement strategy [11] to optimize the estimation of a quantum state

$$|\psi\rangle = \cos\frac{\Theta}{2}|0\rangle + \sin\frac{\Theta}{2}e^{i\Phi}|1\rangle \tag{7}$$

from a finite ensemble of N identically prepared quantum systems. Note that the Bloch angles (Θ, Φ) are the same for all N systems. Contrary to the optimal measurement schemes presented in Refs. [6–9], in which simultaneous measurements on all N quantum systems are used, we do restrict ourselves to simple projection measurements on single quantum systems because of their experimental realizability. We will show that it is possible to come close to the theoretical optimal estimation limit by using a measurement strategy based on the following self-learning algorithm, cf. Fig. 1, which consists of five steps.

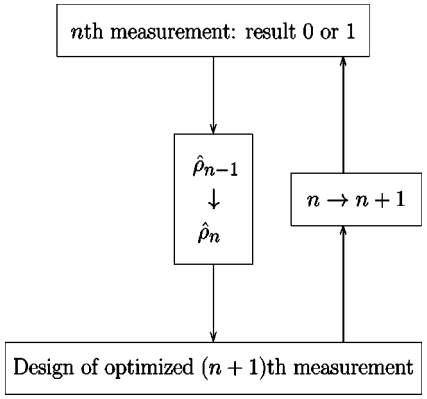

FIG. 1. Schematic picture showing the sequence of steps in the self-learning algorithm. The algorithm starts with performing the nth experiment. The measurement result is then used to update the estimated density operator $\hat{\rho}_{n-1}$. With the help of the updated density operator $\hat{\rho}_n$ the measuring operator for the $(n+1)$th measurement is selected and the algorithm restarts.

(1) We take the first, $n=1$, of the N quantum systems and perform a measurement with randomly chosen direction (θ_m, ϕ_m).

(2) Each measurement yields one of the two possible outcomes that we denote by $i=0$ and $i=1$. By using this information we modify the distribution $w_{n-1}(\theta, \phi)$ of the estimated density operator

$$\hat{\rho}_{n-1} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi w_{n-1}(\theta, \phi)|\theta, \phi\rangle\langle\theta, \phi| \tag{8}$$

according to Bayes' rule [12]

$$w_n(\theta, \phi) = Z^{-1}P_i(\theta, \phi|\theta_m, \phi_m)w_{n-1}(\theta, \phi) \tag{9}$$

with normalization constant

$$Z = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi P_i(\theta, \phi|\theta_m, \phi_m)w_{n-1}(\theta, \phi) \tag{10}$$

and with the probabilities P_i, Eqs. (6a) and (6b). Hence we get the updated distribution w_n and the corresponding density operator $\hat{\rho}_n$, Eq. (2). Before we have acquired any information about the system, we assume our knowledge to be randomly distributed over the Bloch sphere; that is, we start from the initial distribution

$$w_0(\theta, \phi) = \frac{1}{4\pi}. \tag{11}$$

(3) This updated probability distribution $w_n(\theta, \phi)$ describes the present knowledge about the quantum state. With its help we determine the $(n+1)$th measuring operator, i.e., the parameters θ_m and ϕ_m by which it is quantified. The new measuring operator should be designed in such a way that it allows us to gain the maximum amount of information about the unknown quantum state $|\psi\rangle$. For this purpose we have to apply a criterion by which we quantify the notion of maxi-

mum information gain. The different criterions that we use in this context will be described in detail in the next section. This step reflects the self-learning aspect of our algorithm, because the choice of a measuring operator is based on $\hat{\rho}_n$ and thereby on the history of all previous measurement outcomes.

(4) Once having found the next measuring operator $\hat{\Pi}(\theta_m,\phi_m)$ we now take one of the remaining quantum systems, and measure it. If we still have some more quantum systems left, we continue with step 2.

(5) After we have used up all N qubits we analyze the final probability distribution $w_N(\theta,\phi)$. As we know that our unknown qubit was initially prepared in a pure state $|\psi\rangle = |\Theta,\Phi\rangle$ we now select the estimated state $|\psi_e\rangle = |\theta_e,\phi_e\rangle$ from $w_N(\theta,\phi)$.

As the measure of our state estimation quality, expressed by an estimated pure state $|\psi_e\rangle$, we will use the fidelity

$$F = |\langle\psi_e|\psi\rangle|^2 = |\langle\theta_e,\phi_e|\Theta,\Phi\rangle|^2, \tag{12}$$

which is just the absolute value squared of the scalar product between the initial and estimated state.

IV. OPTIMIZATION STRATEGIES

In this section we will describe the optimization strategies that we have used to find the sequence of projection operators $\hat{\Pi}(\theta_m,\phi_m) = |\theta_m,\phi_m\rangle\langle\theta_m,\phi_m|$ and the corresponding estimated state $|\psi_e\rangle$ finally estimated after N measurements.

A. Random selection

The easiest way to select a new measuring operator $\hat{\Pi}(\theta_m,\phi_m)$ is to choose the parameters θ_m and ϕ_m randomly on the Bloch sphere, independent of any knowledge about the state. That is, each infinitesimal surface element $\sin\theta_m d\theta_m d\phi_m$ occurs with the same probability $1/4\pi$. Thus this strategy is not self-learning because $\hat{\Pi}(\theta_m,\phi_m)$ does not depend on previous measurement outcomes. Nevertheless, the estimated density operator $\hat{\rho}_n$ can still be updated after each measurement as described in step 2 of our algorithm.

The random choice implements a measurement protocol lacking any constructive strategy. Thus the results of this method will serve as a reference to which we can compare the outcomes of the two optimization strategies described below.

Eventually, our knowledge about the initial quantum state $|\psi\rangle$ is stored in the final probability distribution w_N. Since we assume that the initial state was pure, our estimated state should be pure as well. For this reason we select the most probable pure state from w_N, which means nothing else than choosing the estimated state $|\psi_e\rangle = |\theta_e,\phi_e\rangle$ such that

$$w_N(\theta_e,\phi_e) = \max_{\{\theta,\phi\}} w_N(\theta,\phi). \tag{13}$$

B. Maximization of average information gain

If we look at the measurement procedure from an information theoretic point of view then our aim will be to maximize the average classical information that we can get in the measurement step from $n-1$ to n. This average information is defined in general as

$$S = -\sum_i p_i \ln p_i, \tag{14}$$

with p_i being the probability of the measurement outcome i. As we do not know the initial state, we can only calculate an estimated average information gain based on our already estimated density operator $\hat{\rho}_{n-1}$. In our case we have only two possible outcomes $i=0,1$ and the estimated probabilities p_i are given by

$$\begin{aligned} p_1(\theta_m,\phi_m) &= \langle\theta_m,\phi_m|\hat{\rho}_{n-1}|\theta_m,\phi_m\rangle \\ &= \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi w_{n-1}(\theta,\phi) \\ &\quad \times |\langle\theta_m,\phi_m|\theta,\phi\rangle|^2 \\ p_0(\theta_m,\phi_m) &= 1 - p_1(\theta_m,\phi_m). \end{aligned} \tag{15}$$

Using these expressions we get the estimated average information gain

$$\begin{aligned} S(\theta_m,\phi_m) = &- p_1(\theta_m,\phi_m)\ln p_1(\theta_m,\phi_m) \\ &- p_0(\theta_m,\phi_m)\ln p_0(\theta_m,\phi_m) \end{aligned} \tag{16}$$

that depends on the parameters θ_m, ϕ_m of the nth measurement. Hence we can maximize $S(\theta_m,\phi_m)$ by adjusting these parameters and select the measuring operator $\hat{\Pi}(\theta_m,\phi_m)$ that yields the maximum average information gain.

The estimated state $|\psi_e\rangle = |\theta_e,\phi_e\rangle$ is finally again determined as in the previous measurement method, namely, by choosing the most probable pure state from w_N.

C. Fidelity maximization

The criterion for the quality of our state estimation is the fidelity, Eq. (12). It is the target function that we want to maximize in the end. Therefore, we now consider a strategy that tries to maximize a mean fidelity after each measurement step.

Let us again assume that we have already performed $n-1$ measurements and possess an estimated density operator $\hat{\rho}_{n-1}$. With this knowledge we can now determine the estimated density operator

$$\hat{\rho}_n^{(i)} = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi w_n^{(i)}(\theta,\phi)|\theta,\phi\rangle\langle\theta,\phi| \tag{17}$$

after the nth measurement, assuming that the outcome of the measurement was $i=0$ or $i=1$. The probability distribution $w_n^{(i)}(\theta,\phi)$ can be expressed in terms of the already known $w_{n-1}(\theta,\phi)$ by using Bayes' rule

$$w_n^{(i)}(\theta,\phi) = Z^{-1} P_i(\theta,\phi|\theta_m,\phi_m) w_{n-1}(\theta,\phi) \tag{18}$$

with Z given by Eq. (10). Note that $w_n^{(i)}$ also explicitly depends on the measuring parameters (θ_m, ϕ_m). The expected mean fidelity $\overline{F}_n$ after the next measurement is then given by

$$\overline{F}_n = \sum_{i=0}^{1} p_i(\theta_m, \phi_m) \langle \theta_n^{(i)}, \phi_n^{(i)} | \hat{\rho}_n^{(i)} | \theta_n^{(i)}, \phi_n^{(i)} \rangle, \qquad (19)$$

where $p_i(\theta_m, \phi_m)$, given by Eq. (15), are the estimated probabilities to find result i, if the measuring operator $\hat{\Pi}(\theta_m, \phi_m)$ was used. Additionally, $|\theta_n^{(i)}, \phi_n^{(i)}\rangle$ is the pure state that would be estimated after the nth measurement, if the result i was found. As we are not restricted to a special estimated pure state by any criterion, the parameters $(\theta_n^{(i)}, \phi_n^{(i)})$ can be varied independently of each other and of the parameters (θ_m, ϕ_m).

Therefore, the expected mean fidelity

$$\overline{F}_n = \overline{F}_n(\theta_m, \phi_m, \theta_n^{(0)}, \phi_n^{(0)}, \theta_n^{(1)}, \phi_n^{(1)}) \qquad (20)$$

is a function of six parameters that can be chosen independently of each other.

We now want to select the next measuring operator $\hat{\Pi}(\theta_m, \phi_m)$ in such a way that $\overline{F}_n$ will be maximized. This leads to the optimal expected fidelity

$$\overline{F}_n^{opt}(\theta_m^{opt}, \phi_m^{opt}, \theta_n^{(0),opt}, \phi_n^{(0),opt}, \theta_n^{(1),opt}, \phi_n^{(1),opt}) = \max \overline{F}_n(\theta_m, \phi_m, \theta_n^{(0)}, \phi_n^{(0)}, \theta_n^{(1)}, \phi_n^{(1)}). \qquad (21)$$

Having found this $\overline{F}_n^{opt}$ we then perform the nth measurement using the operator $\hat{\Pi}(\theta_m^{opt}, \phi_m^{opt})$. The actual values of the states $|\theta_n^{(i),opt}, \phi_n^{(i),opt}\rangle$ are not yet needed as long as we still have some more quantum systems left. Only after the Nth measurement the actually estimated state will be chosen according to the parameters $(\theta_N^{(i),opt}, \phi_N^{(i),opt})$, if the Nth outcome had been i, i.e., $|\psi_e\rangle = |\theta_N^{(0),opt}, \phi_N^{(0),opt}\rangle$ for $i=0$ and $|\psi_e\rangle = |\theta_N^{(1),opt}, \phi_N^{(1),opt}\rangle$ for $i=1$. Hence in contrast to the two other methods described above the final selection of $|\psi_e\rangle$ is implemented in the fidelity maximization procedure itself.

V. NUMERICAL SIMULATIONS AND RESULTS

In this section we will describe the numerical simulations for the fidelity calculation and our results. Our aim is to calculate the average fidelities for the self-learning state estimation schemes described above. These average fidelities will depend on the number N of identically prepared quantum systems that we have at our disposal. Thus one state estimation experiment consists of a sequence of N measurements performed on N identical systems in state $|\psi\rangle$ and a subsequent estimation of a pure state $|\psi_e\rangle = |\theta_e, \phi_e\rangle$. The fidelity F, Eq. (12), of the state estimation is then calculated by comparing $|\psi_e\rangle$ to the state $|\psi\rangle$, Eq. (7).

However, we have to perform such a single run of the (numerical) experiment over and over again for different states $|\psi\rangle$ in order to get the average fidelity

$$\langle F \rangle = \langle |\langle \psi_e | \psi \rangle|^2 \rangle_\psi. \qquad (22)$$

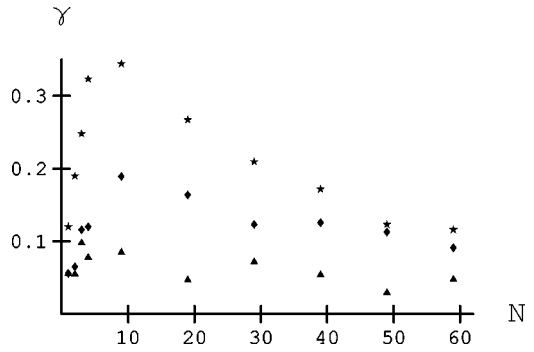

FIG. 2. Plot of the ratio $\gamma \equiv (f - f_{opt})/f_{opt}$ versus number N of quantum systems. The three curves show the results of numerical simulations for three optimization strategies. The biggest errors are found for the random selection strategy (stars). By optimizing the average information gain (rhombs) we already get smaller errors. The best results are found for the fidelity maximization strategy (triangles).

The initial states $|\psi\rangle$, Eq. (7), are chosen randomly weighted by an isotropic probability distribution $1/(4\pi)$ over the surface of the Bloch sphere. Such an average fidelity really quantifies the performance of an estimation strategy since it is not biased by any specific choice of initial states.

The average fidelity for each N was obtained by averaging over 10^4 experiments, i.e., 10^4 initial states equally distributed over the Bloch sphere. For the sake of a clear graphical presentation not the average fidelities themselves but the average errors

$$f = 1 - \langle F \rangle \qquad (23)$$

are calculated for different N [13]. In Fig. 2 the average errors f are compared to the average error

$$f_{opt} = 1 - \langle F_{opt} \rangle = 1 - \frac{N+1}{N+2} = \frac{1}{N+2} \qquad (24)$$

of the optimal measurement scheme [6–8] by plotting

$$\gamma := \frac{f - f_{opt}}{f_{opt}} = \frac{f}{f_{opt}} - 1 \qquad (25)$$

versus N. This quantity is the relative difference between the average error f of our state estimation scheme and the optimal one. The points for the information gain and fidelity optimization strategies are all positive but lie below the points for random selection. This means that we can indeed improve the quality of state estimation by applying these optimization strategies. Note that for random selection γ takes values up to 0.35. This means that f is up to 35% bigger than f_{opt} whereas this worst case value drops to 20% for the information gain and 10% for the fidelity optimization strategies. However, we clearly cannot reach the optimal limit, Eq. (24), for simultaneous measurements. Moreover, the fidelity optimization turns out to be better than the information gain optimization. This is a consequence of the fact

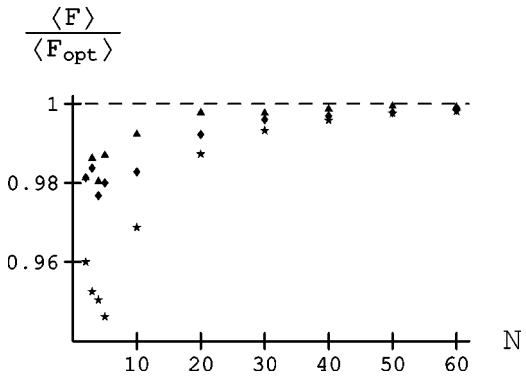

FIG. 3. Relative values of the fidelity $\langle F\rangle$ of separate measurement strategies divided by the fidelity $\langle F_{opt}\rangle$ of optimal measurements. Again stars denote the results of the random selection strategy, while rhombs and triangles symbolize information gain and fidelity optimization, respectively.

that the fidelity itself is the target function of our state estimation scheme. Therefore, maximizing this function has to lead to the maximum fidelity in the end. For this reason the fidelity maximization strategy sets a lower error bound for all schemes based on single measurements and optimizations with respect only to the next, namely, the nth measurement. In principle this bound could be further improved, if optimizations with respect to the $(n+1)$th, $(n+2)$th, ... measurement are taken into consideration. We will, however, not deal with these kinds of optimization strategies, because they only complicate the optimization expressions without yielding much insight or promising great improvements.

To explore the quantitative relation between the average fidelity $\langle F_{opt}\rangle$ of the optimal measurement and the fidelities of our self-learning algorithms we plotted the relative values $\langle F\rangle/\langle F_{opt}\rangle$ in Fig. 3. They show that it is possible to reach fidelities of more than $0.98\langle F_{opt}\rangle$ by applying our self-learning algorithms for all N. Moreover the fidelities for $N>10$ are even bigger than $0.99\langle F_{opt}\rangle$. This is a remarkable result and shows that one can avoid using the very complicated measurement prescript of the optimal measurement schemes. Instead one can use simple polarization measurements in connection with a clever choice of their direction to get a comparable estimation quality. It also illustrates the above argument that the consideration of optimizations with respect to more than just the next measurement can only lead to minor improvements of the average fidelities.

We also see that we get the largest deviations from $\langle F_{opt}\rangle$ for small $N<10$. Here the collective measurements of the optimal schemes offer the biggest advantage compared to our single ones. However, in this range of N we can also get the biggest improvement of the state estimation by the self-learning optimization strategies compared to the random selection method.

VI. CONCLUSION

We have presented a state estimation method based on a self-learning algorithm and simple single-quantum-system measurements and demonstrated it for pure qubits. The algorithm is used to update the knowledge about the true quantum state after each measurement and to choose the best measuring operator for the next measurement. With this scheme we are able to get average fidelities that are very close to the optimal upper limit. Moreover, we do not need to perform a complicated collective measurement that would be required to reach this upper limit, but can restrict ourselves to simple separate measurements. An additional advantage of our scheme is that there is no need to have all N quantum systems at our disposal at the same time. In contrast to the optimal measurement schemes it can also be used if the N quantum systems can only be prepared one after the other. These features ensure the applicability of our scheme to experiments and practical state estimation problems in quantum information theory.

ACKNOWLEDGMENTS

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