

Relativistic contraction and related effects in noninertial frames

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Although there is no relative motion among different points on a rotating disk, each point belongs to a *different* noninertial frame. This fact, not recognized in previous approaches to the Ehrenfest paradox and related problems, is exploited to give a correct treatment of a rotating ring and a rotating disk. Tensile stresses are recovered, but, contrary to the prediction of the standard approach, it is found that an observer on the rim of the disk will see equal lengths of other differently moving objects as an inertial observer whose instantaneous position and velocity are equal to that of the observer on the rim. The rate of clocks at various positions, as seen by various observers, is also discussed. Some results are generalized for observers arbitrarily moving in flat or curved spacetime. The generally accepted formula for the space line element in a non-time-orthogonal frame is found inappropriate in some cases. The use of Fermi coordinates leads to the result that for any observer the velocity of light is isotropic and is equal to c , providing that it is measured by propagating a light beam in a small neighborhood of the observer.

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I. INTRODUCTION

If a body moves with a constant velocity, then, as is well known, the body is relativistically contracted in the direction of motion, whereas its length in the normal direction is unchanged. A naive generalization to a rotating disk leads to the conclusion that the circumference of the disk is contracted, whereas the radius of the disk is unchanged. This paradox is known as the Ehrenfest paradox. Obviously, the paradox is a consequence of the application of the constant-velocity result to a system with a nonconstant velocity.

The standard resolution (see Refs. [1,2] and references therein) of the Ehrenfest paradox is as follows: One introduces the coordinates of the rotating frame S'

$$\varphi' = \varphi - \omega t, \quad r' = r, \quad z' = z, \quad t' = t, \quad (1)$$

where φ , r , z , t are cylindrical coordinates of the inertial frame S and ω is the angular velocity. The metric in S' is given by

$$ds^2 = (c^2 - \omega^2 r'^2) dt'^2 - 2\omega r'^2 d\varphi' dt' - dr'^2 - r'^2 d\varphi'^2 - dz'^2. \quad (2)$$

It is generally accepted that the space line element should be calculated by the formula [3]

$$dl'^2 = \gamma'_{ij} dx'^i dx'^j, \quad i, j = 1, 2, 3, \quad (3)$$

where

$$\gamma'_{ij} = \frac{g'_{0i} g'_{0j}}{g'_{00}} - g'_{ij}. \quad (4)$$

This leads to the circumference of the disk

$$L' = \int_0^{2\pi} \frac{r' d\varphi'}{\sqrt{1 - \omega^2 r'^2/c^2}} = \frac{2\pi r'}{\sqrt{1 - \omega^2 r'^2/c^2}} \equiv \gamma(r') 2\pi r'. \quad (5)$$

The circumference of the same disk as seen from S is $L = 2\pi r = 2\pi r'$. If the disk is constrained to have the same radius r as the same disk when it does not rotate, then L is not changed by the rotation, but the proper circumference L' is larger than the proper circumference of the nonrotating disk. This implies that there are tensile stresses in the rotating disk.

However, there is something wrong with this standard resolution of the Ehrenfest paradox. Consider a slightly simpler situation: a rotating ring in a rigid nonrotating circular gutter with the radius $r = r'$. The statement that Eq. (5) represents the proper circumference implies that the *proper* frame of the rotating ring is given by Eq. (1). This means that an observer on the ring sees that the circumference is $L' = \gamma L$. The circumference of the gutter seen by him cannot be different from the circumference of the ring seen by him, so the observer on the ring sees that the circumference of the relatively moving gutter is *larger* than the proper circumference of the gutter, whereas we expect that he should see that it is smaller. This leads to another paradox. It cannot be resolved by saying that the observer on the ring accelerates, because one can consider a limit $r \rightarrow \infty$, $\omega \rightarrow 0$, $r\omega \equiv u = \text{constant}$, which implies that the acceleration $a = r\omega^2$ becomes zero, whereas the paradox remains.

Before explaining how we resolve this paradox, we give some general notes on the physical meaning of various coordinate frames in the theory of relativity. In practice, one usually uses the coordinates that simplify the technicalities of the physical problem considered. For example, when one describes physical effects in a rigid body, it may be convenient to use a comoving coordinate frame, i.e., a frame in which all particles of the rigid body have constant spatial coordinates. The coordinates of S' in Eq. (1) may be interpreted in this way. However, *the choice of the coordinate*

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frame is more than a matter of convenience. The main lesson we have learned from Lorentz coordinate frames is the fact that what an observer observes (time intervals, space intervals, components of a tensor, etc.) depends on how the observer moves. The main purpose of theoretical physics is to predict what will be *observed* under given circumstances. Therefore, unless stated otherwise, in this paper *by a coordinate frame we understand a coordinate frame that is inherent to an observer*, not to a set of physical particles. Our criticism of some earlier treatments originates from such an interpretation of coordinate frames. To avoid a possible misunderstanding, we note that coordinate frames do not necessarily need to be interpreted in this way, in which case our criticism does not apply.

We resolve the paradox by recognizing that, according to our interpretation, the frame defined by Eq. (1) is the proper frame *only* of the observer at $r=r'=0$. This observer has no velocity relative to S , so the corresponding coordinate transformation (1) does not depend on any velocity. As will become clear from the discussion of Sec. II, the frame defined by Eq. (1) is actually the Fermi frame of an observer who rotates, but has no velocity with respect to the frame S . Observers at different positions on the rotating disk have different velocities, so one has to use a different coordinate transformation for each of them. In other words, *although there is no relative motion among different points on a rotating disk, each point belongs to a different noninertial frame*. This is not strange to those who are familiar with the theory of Fermi coordinates [4,5], but it seems that many relativity theorists are not.

Note also that since we do not interpret the coordinates of S' in Eq. (1) as something inherent to the disk as a whole, r' can be arbitrarily large in Eq. (2), although there is a coordinate singularity at $r'=c/\omega$. It resembles the Schwarzschild singularity of a black hole, where the radial coordinate is not restricted to be larger than the Schwarzschild radius. However, to avoid a possible misunderstanding, note that the coordinate singularity in Eq. (2) does not correspond to an event horizon, because a rotating observer at $r'=0$ can receive information from $r' \geq c/\omega$.

There is also another paradox connected with the standard approach to rotating frames. Let us consider how the nonrotating gutter appears to a rotating observer in the center. His proper frame is given by Eq. (1). If Eq. (3) is the correct definition of the space line element, then he should see that the circumference of the gutter is larger than the proper circumference of the gutter by a factor $\gamma(r')$. However, $\omega r'/c$ can be arbitrarily large, so $\gamma(r')$ can be not only arbitrarily large, but also even imaginary. On the other hand, we know from everyday experience that the apparent velocity $\omega r'$ of stars, due to our rotation, can exceed the velocity of light, but we see neither a contraction, nor an elongation of the stars observed.

We resolve this paradox by examining the assumptions under which formula (3) is obtained. We find that this formula should be used with great care and show that it is not applicable in our case. The correct definition of the space line element depends on how it is measured, and we find that, in our case, γ'_{ij} should be replaced by $-g'_{ij}$ in Eq. (3).

It is fair to note that there are also some other “nonstandard” approaches to the Ehrenfest paradox (see Refs. [6,7], and references therein), but none of these approaches is similar to ours. In particular, the crucial fact that each point of the rotating ring belongs to a different frame has not been taken into account in any of these approaches. Formula (3) has already been criticized [7], but our criticism of Eq. (3) is quite different and more general.

The paper is organized as follows: In Sec. II we find the correct coordinate transformation that leads to the frame of an observer moving in flat spacetime. In Sec. III we explain why Eq. (3) is not always a correct definition of a space line element and show that in a frame that corresponds to an observer in flat spacetime it is more appropriate to calculate the space line element by $-g'_{ij}$. We also make some general remarks on the physical meaning of general coordinate transformations. In Sec. IV we study the relativistic contraction as seen by various observers and resolve the Ehrenfest paradox. In Sec. V we study the rate of clocks at various positions, as seen by various observers. In Sec. VI we discuss the velocity of light as seen by various observers. In Sec. VII we discuss our results, resolve some additional physical problems, and give some generalizations. Section VIII is devoted to concluding remarks, where the relevance of our results to general relativity is emphasized.

II. FRAME OF AN OBSERVER MOVING IN FLAT SPACETIME

The generalized Lorentz transformations for a local Fermi frame of an observer that has arbitrary time-dependent velocity and angular velocity in flat spacetime are found in Ref. [8]. We present the final results, using slightly different notation. Let S be an inertial frame and let S' be the frame of the observer whose velocity and angular velocity are $u^i(t')$ and $\omega^i(t')$, respectively, as seen by an observer in S . The coordinate transformation between these two frames is given by

$$x^i = -A_j^i(t')x'^j + \int_0^{t'} \gamma(t')u^i(t') dt' + \frac{1}{\mathbf{u}^2(t')} [\gamma(t') - 1] \times [u^k(t')A_{jk}(t')x'^j]u^i(t'), \quad (6)$$

$$t = \int_0^{t'} \gamma(t') dt' + \frac{1}{c^2} \gamma(t') [u^k(t')A_{jk}(t')x'^j], \quad (7)$$

where $\gamma(t') = 1/\sqrt{1 - \mathbf{u}^2(t')/c^2}$ and $A_{ji}(t') = -A_j^i(t')$ is the rotation matrix evaluated at $\mathbf{x}'=0$. The rotation matrix satisfies the differential equation

$$\frac{dA_{ij}}{dt} = -A_i^k \omega_{kj}, \quad (8)$$

where $\omega_{ik} = \varepsilon_{ikl}\omega^l$, $\varepsilon_{123} = 1$. The metric tensor in S' is

$$g'_{ij} = -\delta_{ij}, \quad g'_{0j} = -(\boldsymbol{\omega}' \times \mathbf{x}')_j, \quad (9)$$

$$g'_{00} = c^2 \left(1 + \frac{\mathbf{a}' \cdot \mathbf{x}'}{c^2} \right)^2 - (\boldsymbol{\omega}' \times \mathbf{x}')^2,$$

where

$$\boldsymbol{\omega}'^i = \gamma(\boldsymbol{\omega}^i - \Omega^i), \quad a'^i = \gamma^2 \left[a^i + \frac{1}{\mathbf{u}^2} (\gamma - 1) (\mathbf{u} \cdot \mathbf{a}) u^i \right], \quad (10)$$

Ω^i is the time-dependent Thomas precession frequency

$$\Omega_i = \frac{1}{2\mathbf{u}^2} (\gamma - 1) \varepsilon_{ikj} (u^k a^j - u^j a^k), \quad (11)$$

and $a^i = du^i/dt$ is the time-dependent acceleration. The transformations (6),(7) are chosen such that the space origins of S and S' coincide for $t=t'=0$. If \mathbf{u} is time independent and $\boldsymbol{\omega}=0$, then Eqs. (6),(7) reduce to the well-known ordinary Lorentz boosts. If $\mathbf{u}=\mathbf{0}$ and $\boldsymbol{\omega}$ is time independent, then Eqs. (6),(7) reduce to Eq. (1). It is important to emphasize that $\mathbf{u}(t')$ is the velocity of the *space origin* $\mathbf{x}'=\mathbf{0}$ of S' . If S' is a rotating frame, then other space points of S' have a different velocity. (Remember that rotation is *not* a motion along a circle, but rather a change of orientation of the axes with respect to an inertial frame.) Therefore, in general, S' is the proper frame *only* of the observer at $\mathbf{x}'=\mathbf{0}$. Note also that $g'_{\mu\nu} = \eta_{\mu\nu}$ only at $\mathbf{x}'=\mathbf{0}$, which is another confirmation that S' is the frame of the observer at $\mathbf{x}'=\mathbf{0}$ only. The metric (9) is also consistent with a more general theory of Fermi coordinates [4], which are coordinates of an observer arbitrarily moving in curved spacetime, and also have the property that $g_{\mu\nu} = \eta_{\mu\nu}$ at the space origin, i.e., at the position of the observer. Note also that if \mathbf{a}' and $\boldsymbol{\omega}'$ vanish, then Eq. (9) is a metric of an inertial frame and is equal to $\eta_{\mu\nu}$ everywhere, so, in this case, S' can be considered as a frame of an observer at *arbitrary* constant \mathbf{x}' .

It is interesting to note that the geometrical construction of Fermi coordinates is well established [4,5], but no analog of Eqs. (6),(7) is known for curved spacetime. The transformations (6),(7) are obtained by summation of infinitesimal Lorentz transformations (and rotations). It is not so easy to find an analog of Lorentz transformations in curved spacetime, because they correspond to the coordinate transformation between Fermi frames of two different free-falling observers. We can, however, write the transformations (6),(7) in a more elegant form, which could be illuminating for a generalization to curved spacetime. Let

$$x^\mu = f^\mu(t', \mathbf{x}'; \mathbf{u}) \quad (12)$$

denote the ordinary Lorentz transformations, i.e., the transformations between two inertial frames specified by the relative velocity \mathbf{u} , which can be considered as the relative velocity between two inertial (free-falling) observers at the instant when they have the same position. The differential of Eq. (12) is

$$dx^\mu = f^\mu_{, \nu}(t', \mathbf{x}'; \mathbf{u}) dx^\nu. \quad (13)$$

The transition to a noninertial frame introduces a time-dependent velocity: $\mathbf{u} \rightarrow \mathbf{u}(t')$. The transformations (6),(7) may be obtained by integrating Eq. (13) in the following way:

$$x^\mu = \int_0^{t'} f^\mu_{,0}(t', 0; \mathbf{u}(t')) dt' + \int_C f^\mu_{,i}(t', \mathbf{x}'; \mathbf{u}(t')) dx'^i, \quad (14)$$

where C is an arbitrary curve with constant t' , starting from 0 and ending at $-A_j^i(t')x'^j$. The subintegral function in the second term of Eq. (14) is a total derivative, so this term does not depend on the curve C and can be easily integrated. The time derivative in the first term is taken with $\mathbf{u}(t')$ kept fixed, so $f^\mu_{,0}$ in this term is not a total derivative.

Let us now apply the general formalism described in this section to a uniformly rotating ring. We assume that the ring is put in a rigid nonrotating circular gutter with the radius R , which provides that the radius of the rotating ring is the same as the radius of the same ring when it does not rotate, and is equal to R , as seen by an observer in S . This allows us not to worry about the complicated dynamical forces that tend to change the radius of the ring as seen by the observer in S , and pay all our attention to the kinematic effects resulting from the transformations (6),(7).

The ring can be considered as a series of independent short rods, uniformly distributed along the gutter. (By a short rod we understand a rod with a length much shorter than R .) We assume that the gutter is placed at the $z=0$ plane. We put the space origin of S at a fixed point on the gutter, such that the y axis is tangential to the gutter and the x axis is perpendicular to the gutter at $\mathbf{x}=\mathbf{0}$. (In the rest of this section, as well as in Secs. IV and V, $\mathbf{x} \equiv (x, y)$ and the z coordinate is suppressed.) We study a single short rod initially placed at $\mathbf{x}=\mathbf{0}$ and uniformly moving along the gutter in the counterclockwise direction. (This mimics a uniform motion of an electron in a synchrotron.) The gutter causes a torque that provides that the rod is always directed tangentially to the gutter. Therefore, $\boldsymbol{\omega} = u/R$, where $u = \sqrt{\mathbf{u}^2}$ is time independent. Now, $\gamma = 1/\sqrt{1 - \omega^2 R^2/c^2}$ is also time independent. Since a clock in S' is at $\mathbf{x}'=\mathbf{0}$, the clock rate between a clock in S and a clock in S' is given by $t = \gamma t'$, as seen by an observer in S . We assume that, initially, the axes x', y' are parallel to the axes x, y , respectively. Therefore the velocity

$$\mathbf{u}(t') = \omega R (-\sin \gamma \omega t', \cos \gamma \omega t') \quad (15)$$

is always in the y' direction and the solution of Eq. (8) is

$$A_{ij}(t') = \begin{pmatrix} \cos \gamma \omega t' & \sin \gamma \omega t' \\ -\sin \gamma \omega t' & \cos \gamma \omega t' \end{pmatrix}. \quad (16)$$

The transformations (6),(7) become

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \gamma \omega t' & -\gamma \sin \gamma \omega t' \\ \sin \gamma \omega t' & \gamma \cos \gamma \omega t' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + R \begin{pmatrix} \cos \gamma \omega t' - 1 \\ \sin \gamma \omega t' \end{pmatrix}, \quad (17)$$

$$t = \gamma t' + \frac{\gamma}{c^2} \omega R y'. \quad (18)$$

In particular, at $t' = 0$ these transformations become

$$x = x', \quad y = \gamma y', \quad t = \frac{\gamma u}{c^2} y', \quad (19)$$

which coincide with the ordinary Lorentz boost at $t' = 0$ for the velocity in the y direction.

III. GENERAL COORDINATE TRANSFORMATIONS AND THE SPACE LINE ELEMENT IN A NON-TIME-ORTHOGONAL FRAME

A non-time-orthogonal frame is a frame in which g'_{0i} is different from zero. It is generally accepted that the space line element in such a frame is given by Eq. (3). However, if we assume that this formula can be applied to calculate the space distance as seen by a local observer, then, as we have found in Sec. I, Eq. (3) leads to an imaginary length of a distant unaccelerated object as seen by a rotating observer. In order to resolve this puzzle, we examine the assumptions under which formula (3) is derived.

In Ref. [3], formula (3) is derived by assuming that the space distance between two points is measured by measuring the time $\Delta t'$ that light needs to travel from point A to point B and then back to point A . It is also assumed that the time is measured by a clock that does not change its position x'^i . The definition of the space distance $l' = c \Delta t' / 2$ leads to Eq. (3).

In order to perform the described measurement in a rotating frame, the clock must be positioned at point A . However, according to our interpretation of Eq. (1), this point can be far away from the center of the rotation, so the required velocity of point A can exceed c , as seen in S . Therefore, in general, such a measurement cannot be performed.

In practice, we measure space distances between distant objects in a completely different way, namely, by measuring the angles under which we see the objects. [We assume that we know the radial distance of these objects from us. The radial distance is not problematic in the theoretical sense, because $g'_{0r} = 0$ in Eq. (2).] Our rotation does not influence this angle. Therefore, the apparent velocity of distant objects can exceed the velocity of light owing to our rotation, but a pure rotation (without velocity) will not lead to relativistic contraction, nor to elongation. The effect is that, in a rotating frame, it is more appropriate to calculate the space line element as

$$dl'^2 = -g'_{ij} dx'^i dx'^j, \quad (20)$$

despite the fact that g'_{0i} is different from zero. This formula should be used to calculate the space distance between two arbitrary points which have the same t' coordinate, no matter how far these points are from the observer at $x'^i = 0$. Of course, if these points are end points of a body, then, in general, the distance calculated in this way will not be equal

to the proper length of the body, but merely to the length seen by the observer. Formula (20) is also correct for frames that are both accelerated and rotating, defined by Eqs. (6),(7).

To clarify the meaning of formula (3) completely, note that in Ref. [9] this formula is derived in a completely different way, without referring to any particular method of measurement. However, what is actually derived in Ref. [9] is the fact that the quantity (3) does not change under coordinate transformations of the form

$$t'' = f^0(t', x'^1, x'^2, x'^3), \quad x''^i = f^i(x'^1, x'^2, x'^3). \quad (21)$$

We refer to such transformations as *internal transformations*. Obviously, Eq. (1) is not an internal transformation. Regular internal transformations form a subgroup of the group of all regular coordinate transformations. Note that the invariant quantity $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ can always be written as

$$ds^2 = d\eta^2 - \gamma_{ij} dx^i dx^j, \quad (22)$$

where

$$d\eta^2 = \left[\frac{g_{0\mu} dx^\mu}{\sqrt{g_{00}}} \right]^2, \quad (23)$$

so $d\eta^2$ also does not change under internal transformations. The quantity $d\eta^2$ is nothing else but a time line element [3], defined by a measuring procedure similar to the measuring procedure used to define the space line element (3).

Let us illustrate the power of Eqs. (3), (21), and (23) on the example that has already been discussed at some length in Ref. [9]. The Galilei transformation $t'' = t$, $x'' = x - ut$ can also serve as a correct coordinate transformation needed to describe the relativistic effects related to a frame moving with a constant velocity u . The metric in these coordinates is given by

$$ds^2 = c^2(1 - u^2/c^2) dt''^2 - 2udx'' dt'' - dx''^2, \quad (24)$$

where it has been assumed that the metric of S is given by $ds^2 = c^2 dt^2 - dx^2$. From Eq. (3) and $dt = 0$ one can obtain the relativistic contraction $dl = dx = dl'' / \gamma$, where $\gamma = 1/\sqrt{1 - u^2/c^2}$. Similarly, from Eq. (23) and $dx'' = 0$ one can obtain $dt = \gamma d\eta''$. The frame S'' is physically equivalent to the frame S' which would be obtained from S by the ordinary Lorentz transformations, in the sense that S'' and S' are connected by an internal coordinate transformation

$$x' = \gamma x'', \quad t' = t'' / \gamma - \gamma u x'' / c^2. \quad (25)$$

Note that the non-time-orthogonal metric (24), unlike Eqs. (2) and (9), can be transformed to a time-orthogonal metric by an *internal* transformation. Note also that the metric (24), unlike Eqs. (2) and (9), is not a metric of a Fermi frame.

In Ref. [9], internal transformations are interpreted as transformations that correspond to a redefinition of the coordinates of the same *physical* observer. However, there is something unphysical about internal transformations; if t' is a measure of the physical time for the observer in S' , then t'' is not, because it corresponds to a “time” of the same ob-

server which depends on the space point x'^i . Therefore, we introduce a more restrictive class of coordinate transformations, which could be better suited to interpret them as transformations that correspond to a redefinition of the coordinates of the same physical observer:

$$t'' = f^0(t'), \quad x''^i = f^i(x'^1, x'^2, x'^3). \quad (26)$$

We refer to such transformations as *restricted internal transformations*. Regular restricted internal transformations form a subgroup of the group of all regular internal transformations. The quantities $g'_{00} dt'^2$ and Eq. (20) do not change under restricted internal transformations.

Now we have two definitions of the space line element, Eqs. (3) and (20), and related to this, two types of restricted coordinate transformations, internal and restricted internal. The space line element (3) reduces to Eq. (20) if $g_{0i} = 0$. However, as we have shown in this section, Eq. (20) is more appropriate in some cases, even if $g_{0i} \neq 0$. How can one know in general what is the suitable definition of the space line element?

We can immediately formulate one rule which is certainly suitable: If the metric of a frame can be transformed to a time-orthogonal frame by an internal transformation, then the space line element should be calculated by Eq. (3).

According to the results of this section, we can also formulate another rule: If the metric of a frame in flat spacetime can be obtained from $g_{\mu\nu} = \eta_{\mu\nu}$ by a transformation of the form of Eqs. (6),(7) followed by an arbitrary restricted internal transformation, then the space line element should be calculated by Eq. (20). Such coordinate transformations can be interpreted as the most general coordinate transformations in flat spacetime that correspond to a physical observer who has a positive mass.

We still do not have a general rule. However, one can be satisfied to have a rule for Fermi frames only, or for Fermi frames modified by an arbitrary restricted internal transformation, because only such frames have a direct physical interpretation. One can be tempted to guess that for all such frames the space line element should be calculated by Eq. (20), but such a conjecture requires further investigation.

For the sake of completeness, let us make a few remarks on general coordinate transformations in curved spacetime. The most general coordinate transformation that corresponds to a physical observer who has a positive mass is a transformation that leads to Fermi coordinates, followed by an arbitrary restricted internal transformation. Other coordinate frames may be useful for some physical calculations, for example, because it is easier to solve some covariant equations of motion in these coordinates. However, if one is interested in how the physical system appears to a physical observer, one must transform the results to the coordinates specific for this observer.

To summarize this section, we conclude that the correct definition of the space line element depends on how it is measured. Formula (3) is not incorrect, but its applicability is limited and it should be used with great care. In our case of accelerated, rotating frames, it is more appropriate to calculate the space line element with $-g'_{ij}$ instead of with γ'_{ij} .

IV. RELATIVISTIC CONTRACTION

In Sec. II we have found the coordinate transformation that describes the frame of a short rod uniformly moving along the circular gutter. Let us assume for a while that the length of the rod is infinitesimally small and that the rod is rigid (i.e., its proper length dL' is equal to the proper length of the same rod when it does not accelerate). Let us determine the relativistic contraction of the rod, as seen by an observer in S . The observer in S sees both ends of the rod at the same instant, so $dt = 0$. From symmetry it is obvious that the relativistic contraction cannot depend on t , so, in order to simplify the calculations, we evaluate this at $t = 0$. Since the rod is at $x' = y' = 0$, Eq. (18) implies that $t' = 0$. Taking the differential of Eqs. (17) and (18) with respect to space and time coordinates, and then putting $x' = y' = t' = dt = 0$, we find that the observer in S sees the length

$$dL = dy = \frac{dy'}{\gamma} = \frac{dL'}{\gamma}, \quad (27)$$

which is the expected relativistic contraction.

Let us now turn our attention to the concept of the proper length of a body. Traditionally, it is defined as a length of the body as seen from the proper frame of the body. However, as we have seen, in general, there is no such thing as a proper frame of the body as a whole. Such a thing exists only for a nonrotating, inertially moving body in flat spacetime. The concept of a proper length of a large body does not have any fundamental meaning, simply because a ‘‘large body’’ is not actually one object, but a set of many interacting particles. However, the proper length of an infinitesimally small part of a body is well defined. Therefore, we can define the proper length of a whole body as the sum of the proper lengths of its infinitesimal parts. Applying this to Eq. (27), we see that the relativistic contraction of a short (but not infinitesimal) rigid rod uniformly moving along the circular gutter is given by $L = L_0/\gamma$, as seen by the observer in S . Here L_0 is the proper length defined as above.

Now, as in Sec. II, assume that the rotating ring is a series of independent short rods, uniformly distributed along the gutter. Each rod is relativistically contracted, but the ring is not. This means that the distances between the neighboring ends of the neighboring rods are larger than those for a nonrotating ring, so the proper length of the ring is also larger than that of a nonrotating ring. This is concluded also in Ref. [2]. This situation mimics a more realistic ring made of elastic material, where atoms play the role of short rigid rods. Owing to the rotation the distances between neighboring atoms increase, so there are tensile stresses in the material. However, it is important to emphasize that the rotation is not essential for understanding of the origin of these tensile forces, because a similar effect also occurs in a linear relativistic motion [10].

The same relativistic contraction of short rods will be seen by a rotating observer in the center, because his frame is given by the Galilei transformation (1) and the lengths are calculated by g_{ij} , as explained in Sec. III.

Let us now study how the nonrotating gutter appears from the point of view of an observer on the rotating ring. Without losing on generality, we evaluate this at $t' = 0$. We calculate the length of an infinitesimal part of the gutter lying near the observer, so $x = y = 0$. Both ends are seen at the same instant, so $dt' = 0$. Taking the differential of Eq. (17) with respect to space coordinates, and then putting $t' = 0$, we find that the observer in S' sees the length

$$dL' = dy' = \frac{dy}{\gamma} = \frac{dL}{\gamma}, \quad (28)$$

which is the expected relativistic contraction.

It is important to emphasize that Eq. (28) is correct only in the infinitesimal form. The observer on the ring will not see other distant parts of the gutter contracted in the same way; for him, the gutter and the ring do not look azimuthally symmetric. In the following we study how other parts of the ring appear from the point of view of the observer on the ring. We introduce polar coordinates (r, φ) , defined by

$$y = r \sin \varphi, \quad R + x = r \cos \varphi, \quad (29)$$

which are new space coordinates for S , with the origin in the center of the circular gutter. The angle φ is a good label of the position of any part of the ring even in S' . (To visualize this, one can draw angular marks on the gutter. The number of marks separating two points on the gutter or on the ring is a measure of the ‘‘angular distance’’ in any frame.) Let S'' be the frame of another part of the ring. The position of that part of the ring is $x'' = y'' = 0$. The relative position of the space origin of S'' with respect to that of S' is given by the constant relative angle $\Delta\varphi_0$, as seen by an observer in S . In analogy with Eqs. (17),(18), we find that S'' is determined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\gamma\omega t'' + \Delta\varphi_0) & -\gamma \sin(\gamma\omega t'' + \Delta\varphi_0) \\ \sin(\gamma\omega t'' + \Delta\varphi_0) & \gamma \cos(\gamma\omega t'' + \Delta\varphi_0) \end{pmatrix} \begin{pmatrix} x'' \\ y'' \end{pmatrix} + R \begin{pmatrix} \cos(\gamma\omega t'' + \Delta\varphi_0) - 1 \\ \sin(\gamma\omega t'' + \Delta\varphi_0) \end{pmatrix}, \quad (30)$$

$$t = \gamma t'' + \frac{\gamma}{c^2} \omega R y''. \quad (31)$$

The observer in S' will see the other part of the ring at the relative ‘‘angular distance’’ $\Delta\varphi$, which, owing to the relativistic effects, differs from $\Delta\varphi_0$. Let the labels A, B denote the coordinates of the part of the ring that lie at S' and S'' , respectively. Since the rotation is uniform, the relative ‘‘angular distance’’

$$\Delta\varphi = \varphi_B(t_B'') - \varphi_A(t_A') = \Delta\varphi_0 + \gamma\omega t_B'' - \gamma\omega t_A', \quad (32)$$

cannot depend on t' , so without losing on generality, we evaluate this at $t' = 0$. Since the observer sees both parts of the ring at the same instant, we have $t_A' = t_B' = 0$. Since $x_B'' = y_B'' = 0$, from Eq. (30) we find

$$y_B = R \sin(\gamma\omega t_B'' + \Delta\varphi_0), \quad (33)$$

and from Eq. (31)

$$t_B = \gamma t_B''. \quad (34)$$

From $t_B' = 0$ and Eq. (19) it follows $t_B = \omega R y_B / c^2$, which, because of Eq. (34), can be written as $\gamma t_B'' = \omega R y_B / c^2$. This, together with Eq. (33), leads to the equation that determines t_B'' :

$$\gamma\omega t_B'' = \beta^2 \sin(\gamma\omega t_B'' + \Delta\varphi_0), \quad (35)$$

where $\beta^2 \equiv \omega^2 R^2 / c^2$. From $t_A' = 0$ and Eq. (32) we see that $\Delta\varphi = \gamma\omega t_B'' + \Delta\varphi_0$, so Eq. (35) can be written as

$$\Delta\varphi - \Delta\varphi_0 = \beta^2 \sin \Delta\varphi. \quad (36)$$

Equation (36) determines the relative ‘‘angular distance’’ $\Delta\varphi$ between two points on the ring as seen by the observer at one of the points, if the relative angle between these two points, as seen by the observer in S , is $\Delta\varphi_0$. In other words, Eq. (36) determines how the ring appears to the observer on the ring. For an inertial observer whose instantaneous position and velocity are equal to that of the observer on the ring, the same equation (36) is found in Ref. [1], where the solution is graphically depicted. This means, contrary to the conclusion of Ref. [1], that the inertial and the noninertial observers see the ring in the same way.

If the two points on the ring are very close to each other, then $\Delta\varphi_0$ and $\Delta\varphi$ are very small. By expanding Eq. (36) for small angles we find the approximative solution $\Delta\varphi = \gamma^2 \Delta\varphi_0$. The factor γ^2 is easy to understand; one factor of γ appears because the part of the gutter close to the observer on the ring looks shorter for that observer than it really is, and the other factor of γ appears because the part of the ring close to the observer on the ring is longer than that of the same ring when it does not rotate.

V. RATE OF CLOCKS

Assume that there are two clocks at different positions on the ring. Assume also that they show the same time, as seen by an observer in S . Then, as shown in Sec. II, both clocks show the time $t' = t/\gamma$, as seen from S .

These two clocks do not show the same time as seen by an observer on the ring. If the position of the observer coincides with the position of one of the clocks, then the time shift of the other clock is given by Eq. (35).

Let us calculate the time shift of the clock at the fixed position (x, y) , as seen by the observer in S' . From Eq. (17) we express y' as a function of x, y , and t' , and put this in Eq. (18). The result is

$$t = \gamma t' + \frac{\omega R}{c^2} [y \cos \gamma\omega t' - (x + R) \sin \gamma\omega t']. \quad (37)$$

For comparison, if Eqs. (17) and (18) are replaced by the ordinary Lorentz boosts for a constant velocity in the y direction, then Eq. (37) should be replaced by

$$t = \frac{t'}{\gamma} + \frac{u}{c^2}y. \quad (38)$$

To understand the physical meaning of Eq. (37), we explore some special cases. If $\gamma\omega t' = 2k\pi$, then $t = \gamma t' + \omega R y / c^2$. In this case, the rate of clocks $\Delta t / \Delta t' = \gamma$ is the same as that for the observer in S . This can also be understood as a time-averaged rate, because the oscillatory functions in Eq. (37) vanish when they are averaged over time. Therefore, the observer in S' agrees with the observer in S that the clock in S' is slower, but only in a time-averaged sense. At some instants the observer in S' sees that the clock in S is slower than his clock. For example, by putting $x = 0$ and expanding Eq. (37) for small t' , we recover formula (38), with $u = \omega R$. If the clock in S is in the center, which corresponds to $x = -R$, $y = 0$, then Eq. (37) gives $t = \gamma t'$, so in this case there is no oscillatory behavior.

VI. VELOCITY OF LIGHT

Let us also make some comments on the velocity of light. The Sagnac effect is usually interpreted as a dependence of the velocity of light on the direction of light propagation in a rotating frame (see, for example, Refs. [11,12] and references therein). However, such an interpretation is based on the interpretation of the frame S' defined by Eq. (1) as a proper frame of all observers on a rotating platform. Now we know that each observer belongs to a different local Fermi frame, and from Eq. (9) we see that in the *vicinity* of any observer the metric is equal to the Minkowski metric $\eta_{\mu\nu}$. This implies that for any local observer the velocity of light is isotropic and is equal to c , providing that it is measured by propagating a light beam in a *small* neighborhood of the observer, using Einstein synchronized clocks. This is also true for an observer in curved spacetime, because his proper frame is given by the appropriate Fermi coordinates, which also have a property that $g_{\mu\nu} = \eta_{\mu\nu}$ at the position of the observer. The phrases ‘‘local’’ and ‘‘small’’ denote spatial dimensions inside which the metric tensor does not change significantly.

Of course, the velocity of light does not have to be equal to c for an observer which is not at the same position as the light. However, this is not only a property of non-time-orthogonal frames. For example, if the acceleration of a uniformly accelerated observer and the propagation of light are both in the x' direction, then from Eq. (9) one can find that the accelerated observer sees the velocity of light as $|dx'/dt'| = c\sqrt{1 + a'x'/c^2}$, being equal to c only at $x' = 0$. A similar effect occurs for a radial motion of light in the vicinity of the Schwarzschild radius of a black hole, as seen by a static observer far away from the Schwarzschild radius.

Concerning the Sagnac effect, we do not claim that the standard prediction for the phase shift is incorrect. It can also be derived by performing calculations in the nonrotating frame S [11], and such a derivation, based on the well-understood Minkowski spacetime, is perfectly correct. We have nothing new to say about the phase shift, which appears when clockwise and counterclockwise propagated light

beams finally meet. However, as seen by an observer on the rim of a rotating disk, the velocity of the light beam will be a complicated function of time t' , or equivalently, of the position (x', y') of the beam. The trajectory of the light beam expressed in S coordinates takes a simple form

$$y = R \sin \omega_L t, \quad x = R(-1 + \cos \omega_L t), \quad (39)$$

where $\omega_L = \pm c/R$. The plus and minus signs refer to the counterclockwise and clockwise propagated beams, respectively. Using Eqs. (17), (18), and (39), one can eliminate x, y, t and express x', y' as functions of t' . The speed of light as seen by the observer in S' is

$$v'_L = \sqrt{\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2}. \quad (40)$$

Expanding Eqs. (17) and (39) for small t' and t , respectively, one can easily find $y' = \pm ct' + O(t'^2)$, $x' = O(t'^2)$, which means that the observer sees the velocity of light equal to c when the light is at the same position as the observer, just as expected.

VII. DISCUSSION

From the experience acquired by careful calculations in the preceding sections, we can generalize some of the results without much effort, using qualitative and intuitive arguments.

If an observation in S is performed at the instant t , then the solution of Eq. (8) can always be chosen such that at t the axes x'^i are parallel to the corresponding axes x^i . Therefore, for a small range of values of t' , the transformations (6),(7) can be approximated by the ordinary Lorentz boosts [see Eq. (19)]. From this fact we conclude that if a moving rigid body is short enough, then its relativistic contraction in the direction of the instantaneous velocity, as seen from S , is simply given by $L(t) = L'/\gamma(t)$, i.e., it depends only on the instantaneous velocity, not on its acceleration and rotation. (‘‘Short enough’’ means that $L' \ll c^2/a'_\parallel$, where a'_\parallel is the component of the proper acceleration parallel to the direction of the velocity [13].)

By a similar argument we may conclude that an arbitrarily accelerated and rotating observer sees equal lengths of other differently moving objects as an inertial observer whose instantaneous position and velocity are equal to that of the arbitrarily accelerated and rotating observer.

So far we have studied a rotating ring. A rotating disk is a more complicated object, with some additional dynamical effects related to elastic and inertial forces. However, a disk can be modeled as a series of concentric rings, each of them being constrained to have a fixed radius. In this case, the analysis of a rotating disk becomes essentially the same as that of a rotating ring.

Let us also give some additional arguments why our resolution of the Ehrenfest paradox is correct. Our method, based on coordinate transformations (6),(7), is really a generalization of the well-known derivation of the Lorentz contraction for constant velocities. In our approach the origin of the rela-

tivistic contraction lies in the non-Galilean transformation, not in the nontrivial metric, whereas in the standard approach the transformation is Galilean and the contraction is due to the nontrivial metric (2). Note finally that our approach allows a generalization to a more complicated motion, whereas the standard approach does not.

Finally, let us make some comments on the observability of the relativistic contraction. In principle, it could be observed by photographing a rod with a very short exposition, such that both ends are observed at the same instant. Since the velocity of the incoming information (velocity of light) is finite, both ends of the rod should be positioned at the same distance from the observer. Therefore, the ideal setup for such a measurement is a rod in a uniform circular motion and a camera in the center, providing that we can achieve a short enough exposition. It is assumed that in this experiment the only object that moves circularly is a rod (with two ends); there is neither a rotating disk, nor a rotating ring.

An indirect, but easier-to-perform experimental verification of the relativistic contraction could perhaps be obtained by measuring the velocity of a rotating ring in a rigid circular gutter, needed to achieve the break of the ring, and comparing it with the elongation needed to achieve the break of the ring caused by ordinary stretching.

Of course, in both types of experiments the problem is to achieve a relativistic velocity of macroscopic objects, so these can be considered merely as gedanken experiments.

VIII. CONCLUSION

In this paper a new resolution of the Ehrenfest paradox has been provided by taking into consideration the fact that although there is no relative motion among different points on a rotating disk, each point belongs to a different noninertial local Fermi frame. If a rotating ring (or a disk) is constrained to have a fixed radius from the point of view of an inertial observer, it has been found that there are tensile stresses in the disk, in agreement with the prediction of the standard approach. However, contrary to the prediction of the standard approach, it has been found that an observer on the rim of the disk will see equal lengths of other differently

moving objects as an inertial observer whose instantaneous position and velocity are equal to that of the observer on the rim, providing that the observations of different events are simultaneous. This also generalizes to observers arbitrarily moving in flat spacetime.

The paper deals mainly with flat spacetime, with particular attention paid to circular motion. However, it gives several results which are of very general relevance, not only for arbitrary motion in flat spacetime, but also for general relativity and curved spacetime.

First, it has been demonstrated that the generally accepted formula (3) is not always correct. The correct definition of the space line element depends on how it is measured, so Eq. (3) should be used with great care. In some cases, the ‘‘naive’’ formula (20) is more appropriate. One such case is a metric of a frame in flat spacetime that can be obtained from $g_{\mu\nu} = \eta_{\mu\nu}$ by a transformation of the form of Eqs. (6),(7), followed by an arbitrary restricted internal transformation. Further investigation is needed in order to generalize this result.

Second, the paper demonstrates the importance of the use of Fermi coordinates. One of the consequences of their use is the result that for any local observer the velocity of light is isotropic and is equal to c , providing that it is measured by propagating a light beam in a small neighborhood of the observer. This fact should be used for a correct treatment of the Sagnac effect if one wants to explore the general relativistic corrections. Fermi coordinates should also be used in order to understand the physical effects related to a rotating black hole, to give a correct treatment of the Hawking radiation, as well as for any other physical effect, whenever it is intended to describe how the world appears to a particular observer.

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