Coulomb corrections to the e^+e^- pair production in ultrarelativistic heavy-ion collisions

R. N. Lee and A. I. Milstein
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia
(Received 21 September 1999; published 10 February 2000)

We manifest the origin of the wrong conclusion made by several groups of authors on the absence of Coulomb corrections to the cross section of the e^+e^- pair production in ultrarelativistic heavy-ion collisions. The source of the mistake is connected with an incorrect passage to the limit in the expression for the cross section. When this error is eliminated, the Coulomb corrections do not vanish, and agree with the results obtained within the Weizsäcker-Williams approximation.

PACS number(s): 12.20.Ds, 95.30.Cq

RHIC and LHC projects initiated a set of recent publications on the e^+e^- pair production in ultrarelativistic heavy-ion collisions. Using slightly different approaches, the authors of Refs. [1–3] calculated the cross section of the process exactly in the parameters $\alpha Z_{A,B}$ ($Z_{A,B}$ being the charge numbers of the nuclei A and B; α is the fine-structure constant). In these papers the nuclei were treated as sources of the external field, and the amplitude was calculated at a fixed impact parameter of the nuclei. After that the cross section was obtained by integration over the impact parameter. As a result, the conclusion was made that the exact cross section coincides with that calculated in the lowest-order perturbation theory with respect to $\alpha Z_{A,B}$ (Born cross section). On the other hand, in the Weizsäcker-Williams approximation with respect to one of the nuclei, the cross section of the process is proportional to the well-known cross section of the e^+e^- pair production by a photon in a Coulomb field [4] and, therefore, contains the Coulomb corrections. This obvious circumstance was observed in Ref. [5], where the Coulomb corrections in the process under discussion were calculated. Though the existence of the Coulomb corrections is not in doubt, the source of the disagreement between the results was not revealed so far. This question is important from the theoretical point of view, since the approach developed in [1–3] is used now in QCD. In the present paper we present the solution of this puzzle.

Let the ultrarelativistic nuclei A and B move in the positive and negative directions of the z axis, respectively. Then the expression for the cross section of the e^+e^- pair production, obtained in Refs. [1–3], reads

$$d\sigma=\frac{m^2d^3pd^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}\int\frac{d^2k}{(2\pi)^2}|F_B(\mathbf{k})|^2\times|F_A(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k})|^2|\mathcal{M}(\mathbf{k})|^2, \tag{1}$$

$$\mathcal{M}(\mathbf{k})=\bar{u}(p)\frac{\boldsymbol{\alpha}(\mathbf{k}-\mathbf{p}_\perp)+\gamma_0 m}{-p_+q_--(\mathbf{k}-\mathbf{p}_\perp)^2-m^2+i\epsilon}\gamma_-u(-q)+\bar{u}(p)\frac{-\boldsymbol{\alpha}(\mathbf{k}-\mathbf{q}_\perp)+\gamma_0 m}{-p_-q_+-(\mathbf{k}-\mathbf{q}_\perp)^2-m^2+i\epsilon}\gamma_+u(-q).$$

Here $\mathbf{p}$ and ε_p ($\mathbf{q}$ and ε_q) are the momentum and energy of the electron (positron), $u(p)$ and $u(-q)$ are positive- and negative-energy Dirac spinors, $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$, $\gamma_\pm=\gamma^0\pm\gamma^z$, γ^μ are the Dirac matrices, $p_\pm=\varepsilon_p\pm p^z$, $q_\pm=\varepsilon_q\pm q^z$, m is the electron mass, $\mathbf{k}$ is a two-dimensional vector lying in the xy plane, and the function $F(\boldsymbol{\Delta})$ is proportional to the electron eikonal scattering amplitude in the potential $V(\mathbf{r})$ of the corresponding nucleus:

$$F(\boldsymbol{\Delta})=\int d^2\rho\,\exp[-i\boldsymbol{\rho}\boldsymbol{\Delta}]\{\exp[-i\chi(\boldsymbol{\rho})]-1\}, \tag{2}$$

$$\chi(\boldsymbol{\rho})=\int_{-\infty}^{\infty}dzV(z,\boldsymbol{\rho}).$$

For the potential $V(\mathbf{r})=V_c(r)=-Z\alpha/r$, the integral in $\chi(\boldsymbol{\rho})$ becomes divergent and requires a regularization. This regularization can be made by using the potential $V(\mathbf{r})=-Z\alpha\,\exp(-r/a)/r$. Performing the integration in Eq. 2, and taking the limit $a\to\infty$ at fixed $\boldsymbol{\Delta}\neq0$, one obtains (up to the constant phase depending on a)

$$F(\boldsymbol{\Delta})=\mathcal{F}(\boldsymbol{\Delta})\equiv i\pi Z\alpha\frac{\Gamma(1-iZ\alpha)}{\Gamma(1+iZ\alpha)}\left(\frac{4}{\Delta^2}\right)^{1-iZ\alpha}. \tag{3}$$

Actually, to obtain this result one can use any regularization of the phase $\chi(\boldsymbol{\rho})$ for which $\chi(\boldsymbol{\rho})\to0$ at $\rho\to\infty$. Since $|\mathcal{F}(\boldsymbol{\Delta})|^2=(4\pi Z\alpha/\Delta^2)^2\propto Z^2$, then the substitution of Eq. (3) into Eq. (1) would lead to the wrong conclusion [1–3] that the exact cross section coincides with the Born result. Let us show that, in order to obtain the Coulomb corrections in Eq. (1), it is necessary first to take the integral over $\mathbf{k}$ using the functions $F(\boldsymbol{\Delta})$ with the regularized phase and then remove the regularization.

Consider the integral

$$G=\int\frac{d^2k}{(2\pi)^2}k^2[|F(\mathbf{k})|^2-|F^0(\mathbf{k})|^2], \tag{4}$$

where $F^0(\boldsymbol{\Delta})=-i\int d\boldsymbol{\rho}\,\exp(-i\boldsymbol{\Delta}\boldsymbol{\rho})\chi(\boldsymbol{\rho})$ is the first term of the expansion of $F(\boldsymbol{\Delta})$ with respect to the potential. For $F=\mathcal{F}$ and, correspondingly, $F^0=\mathcal{F}^0\equiv4i\pi Z\alpha/\Delta^2$, the integrand in Eq. (4) vanishes. Let us show that the integral G is

not equal to zero for the regularized F and is independent of the regularization method, if $V(\mathbf{r})\to -Z\alpha/r$ at $r\to 0$ [when $\chi(\boldsymbol{\rho})\to 2Z\alpha\ \ln(\rho)+\text{const}$ at $\boldsymbol{\rho}\to 0$]. For the sake of simplicity, we present the proof of this statement for a spherically symmetric potential $V(r)$. Taking the integral in Eq. (2) over the angle of $\boldsymbol{\rho}$, and integrating by parts over ρ, we obtain the following expression for F:

$$F(\mathbf{k})=\frac{2\pi i}{k}\int_0^\infty d\rho\,\rho J_1(k\rho)\chi'(\rho)\exp[-i\chi(\rho)], \qquad (5)$$

where $J_1(x)$ is the Bessel function. The function $F^0(\mathbf{k})$ can be obtained from Eq. (5) by omitting the exponent in the integrand. Substituting Eq. (5) into Eq. (4), and integrating over the angle of $\mathbf{k}$, we find

$$G=2\pi\int_0^\infty dq\,q\int_0^\infty\int_0^\infty d\rho_1\,d\rho_2\,\rho_1\rho_2 J_1(k\rho_1)J_1(k\rho_2)$$
$$\times\chi'(\rho_1)\chi'(\rho_2)\{\exp[-i\chi(\rho_1)+i\chi(\rho_2)]-1\}. \qquad (6)$$

If one naively changes the order of integration over k and $\rho_{1,2}$, and takes the integral over k, using the relation

$$\int_0^\infty dk\,kJ_1(k\rho_1)J_1(k\rho_2)=\frac{1}{\sqrt{\rho_1\rho_2}}\delta(\rho_1-\rho_2),$$

then, after the integration over ρ_1, the result will be zero. To demonstrate that the change of the integration order in Eq. (6) is invalid, we restrict the upper limit of the integral over k by Q. After that one can change the order of integration in triple integral in Eq. 6. Integrating over k, we obtain

$$G=2\pi\int_0^\infty\int_0^\infty d\rho_1\,d\rho_2\frac{Q\rho_1\rho_2}{\rho_1^2-\rho_2^2}[\rho_2 J_0(Q\rho_2)J_1(Q\rho_1)$$
$$-\rho_1 J_0(Q\rho_1)J_1(Q\rho_2)]\chi'(\rho_1)\chi'(\rho_2)$$
$$\times\{\exp[-i\chi(\rho_1)+i\chi(\rho_2)]-1\}. \qquad (7)$$

Substituting $\rho_{1,2}\to\rho_{1,2}/Q$, and taking the limit $Q\to\infty$ with the use of the asymptotics of χ, we find

$$G=8\pi(Z\alpha)^2\int_0^\infty\int_0^\infty\frac{d\rho_1\,d\rho_2}{\rho_1^2-\rho_2^2}\left\{\left(\frac{\rho_2}{\rho_1}\right)^{2iZ\alpha}-1\right\}$$
$$\times[\rho_2 J_0(\rho_2)J_1(\rho_1)-\rho_1 J_0(\rho_1)J_1(\rho_2)]. \qquad (8)$$

Making the change of variables $\rho_{1,2}=r\ \exp(\pm t/4)$, and integrating over r, we obtain the nonzero result for the quantity G:

$$G=8\pi(Z\alpha)^2\int_0^\infty dt\frac{\cos(Z\alpha t)-1}{\exp(t)-1}$$
$$=-8\pi(Z\alpha)^2[\mathrm{Re}\psi(1+iZ\alpha)+C]$$
$$=-8\pi(Z\alpha)^2 f(Z\alpha), \qquad (9)$$

where C is the Euler constant, and $\psi(x)=d\ \ln\Gamma(x)/dx$. Thus, we come to a remarkable conclusion: although the main contribution to the integral in Eq. (4) comes from the region of small k, where $|F(\mathbf{k})|$ differs from $|\mathcal{F}(\mathbf{k})|=4\pi Z\alpha/k^2$ and depends on the regularization parameters (the radius of screening), nevertheless the integral G itself is a universal function of $Z\alpha$. Note that the integral in Eq. (4) appears in the theory of multiple scattering (see Ref. [6], where the approximate formula for this integral was obtained).

Now it is clear how to derive the Coulomb corrections starting from expression (1). Let us calculate the Coulomb corrections related to the nucleus B (the contribution of the higher-order perturbation theory with respect to the parameter $Z_B\alpha$). For this purpose, in Eq. (1) one should replace the functions $|F_B|^2$ and $|F_A|^2$ with $|F_B|^2-|F_B^0|^2$ and $|\mathcal{F}_A^0|^2$, respectively, keeping the regularization in the functions F_B and F_B^0. The main contribution to the integral is given by the region of small $\mathbf{k}$. Therefore, we can neglect $\mathbf{k}$ in the argument of $\mathcal{F}_A^0$, and expand the matrix element $\mathcal{M}$ at small $\mathbf{k}$:

$$\mathcal{M}(\mathbf{k})\approx\mathbf{k}\mathbf{L},\quad \mathbf{L}=\bar{u}(p)\left\{\frac{\boldsymbol{\alpha}(\gamma_-/p_+-\gamma_+/q_+)}{(p_-+q_-)}\right.$$
$$\left.+\frac{2\gamma_-(\mathbf{p}_\perp/p_+-\mathbf{q}_\perp/q_+)}{(p_-+q_-)^2}\right\}u(-q). \qquad (10)$$

Using Eqs. (9) and (10), and performing the summation over electron and positron polarizations, we obtain the following expression for the Coulomb corrections related to the nucleus B:

$$d\sigma_B^c=\frac{2G_B d^3p\,d^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}\frac{|\mathcal{F}_A^0(\mathbf{p}_\perp+\mathbf{q}_\perp)|^2}{[p_+q_+(p_-+q_-)]^2}$$
$$\times\left\{p_+q_+(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\frac{2(\mathbf{p}_\perp q_+q_-+\mathbf{q}_\perp p_+p_-)^2}{(p_-+q_-)^2}\right\}. \qquad (11)$$

Here G_B denotes the function G in Eq. (9) at $Z=Z_B$. The Coulomb corrections related to the nucleus A can be obtained from Eq. (11) by the substitution $Z_A\leftrightarrow Z_B$ and the replacement of indices $-\leftrightarrow+$.

It is necessary to note the following circumstance. Actually, in the expansion over $Z_A\alpha$ and $Z_B\alpha$ of the differential cross section $d\sigma/d\mathbf{p}d\mathbf{q}$ in Eq. (1), only the lowest (Born) term is correct. As for the higher-order terms in Eq. (1) (Coulomb corrections), they give the correct result only after the integration over the directions of the positron (electron) momentum. This is due to the fact that the asymptotic form of the wave functions in Refs. [1–3] corresponds to the problem of scattering, but not to the problem of pair production. If one calculates the cross section integrated over the direction of $\mathbf{q}$, then, due to the completeness relation, it is possible to replace the set of functions containing in asymptotics the converging spherical wave with the set of functions containing the diverging spherical wave. Thus Eq. (11) should be integrated over the angles of $\mathbf{q}$ or $\mathbf{p}$. The same trick was

made at the recalculation of the bremsstrahlung cross section integrated over the photon momentum from the cross section of pair photoproduction integrated over the positron momentum [7]. This explains why the Coulomb corrections (11) are given by the region of small $\mathbf{k}$, while at the calculation of the Coulomb correction using the wave functions with the correct asymptotic behavior the main contribution would come from the region $k\sim m$. The same situation occurs at the calculation of bremsstrahlung and pair photoproduction cross sections, where the Coulomb corrections come from different regions of momentum transfers.

Let us calculate within the logarithmic accuracy the Coulomb corrections to the cross section $d\sigma/d\varepsilon_p d\varepsilon_q$ at $\varepsilon_{p,q} \gg m$. At the integration over the transverse momenta the main contribution comes from the region $\Delta=|\mathbf{p}_\perp+\mathbf{q}_\perp| \ll p_\perp, q_\perp \sim m$. The integral over Δ requires regularization at $\Delta\to 0$. It is obvious that the lower limit of integration over Δ coincides with that in the Weizsäcker-Williams method. In the rest frame of the nucleus B it has the form $\Delta_{min}=(\varepsilon_p^0+\varepsilon_q^0)/\tilde{\gamma}$, where $\varepsilon_{p,q}^0$ are the energies of the electron and positron, and $\tilde{\gamma}$ is the Lorentz factor of the nucleus A in this frame. In the laboratory frame, where the nuclei A and B have the Lorentz factors γ_A and γ_B, respectively, one has $\Delta_{min}=(p_+ + q_+)/\gamma_A$. Using this cutoff, we obtain

$$d\sigma_B^c = -\frac{4}{\pi m^2}(Z_A\alpha)^2(Z_B\alpha)^2 f(Z_B\alpha)\frac{d\varepsilon_p d\varepsilon_q}{(\varepsilon_p+\varepsilon_q)^2} \times\left(1-\frac{4\varepsilon_p\varepsilon_q}{3(\varepsilon_p+\varepsilon_q)^2}\right)\left[\ln\frac{m^2}{\Delta_{1min}^2}+\ln\frac{m^2}{\Delta_{2min}^2}\right]. \quad (12)$$

The sum of logarithms in this formula corresponds to the contributions of two kinematic regions: $p^z, q^z>0$ and $p^z, q^z<0$. In the first case $\Delta_{1min}=(\varepsilon_p+\varepsilon_q)/\gamma_A$, and the corresponding term in Eq. (12) is valid at $m\ll\varepsilon_{p,q}\ll m\gamma_A$. In the second case $\Delta_{2min}=m^2/(\varepsilon_p+\varepsilon_q)\gamma_A$, and the corresponding term is valid at $m\ll\varepsilon_{p,q}\ll m\gamma_B$. Performing the integration over $\varepsilon_{p,q}$ in the regions indicated, one has

$$\sigma_B^c = -\frac{28}{9\pi m^2}(Z_A\alpha)^2(Z_B\alpha)^2 f(Z_B\alpha)\ln^2(\gamma_A\gamma_B). \quad (13)$$

Formulas (12) and (13) can be easily obtained in the Weizsäcker-Williams approximation using the well-known result for the exact in $Z\alpha$ pair photoproduction cross section in the field of a nucleus. They coincide with the result of Ref. [5] (see also Ref. [8]). It also follows from the Weizsäcker-Williams method that the contribution of the terms, containing the higher orders of Z_A and Z_B simultaneously, can be neglected within our accuracy.

In Refs. [1–3] the amplitude of e^+e^- pair production was obtained at a fixed impact parameter between the nuclei. Using this amplitude, it is possible to represent the Coulomb corrections related to the nucleus B as the integral over the impact parameter:

$$d\sigma_B^c = \frac{m^2 d^3p\, d^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}\int d^2\rho \int\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2} \times\exp[i(\mathbf{k}_1-\mathbf{k}_2)\boldsymbol{\rho}]\mathcal{M}(\mathbf{k}_1)\mathcal{M}^*(\mathbf{k}_2) \times[\mathcal{F}_B(\mathbf{k}_1)\mathcal{F}_B^*(\mathbf{k}_2)-\mathcal{F}_B^0(\mathbf{k}_1)\mathcal{F}_B^{0\,*}(\mathbf{k}_2)] \times\mathcal{F}_A^0(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k}_1)\mathcal{F}_A^{0\,*}(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k}_2). \quad (14)$$

Again, changing the order of integration would lead to a result of zero. Indeed, taking the integral over $\boldsymbol{\rho}$ first, we obtain the factor $\delta(\mathbf{k}_1-\mathbf{k}_2)$ in the integrand, and, therefore, the integral over $\mathbf{k}_1$ vanishes due to the relation $|\mathcal{F}_B|^2=|\mathcal{F}_B^0|^2$. Let us demonstrate that, similar to the case of the calculation of integral (4), the change of the integration order in Eq. (14) is incorrect, and the result (11) also follows from Eq. (14). For this purpose, we restrict the region of integration over $\boldsymbol{\rho}$ by the condition $\rho<R$. After that it is possible to change the order of integration and take the integral over $\boldsymbol{\rho}$. Then the main contribution to the integral over $\mathbf{k}_{1,2}$ comes from the region $k_{1,2}\ll 1/R$. Since we are going to take the limit $R\to\infty$, we can replace $\mathcal{M}(\mathbf{k}_{1,2})$ with $\mathbf{k}_{1,2}\mathbf{L}$ and neglect $\mathbf{k}_{1,2}$ in $\mathcal{F}_A^0(\mathbf{q}_\perp+\mathbf{p}_\perp-\mathbf{k}_{1,2})$. Then, we have

$$d\sigma_B^c = \frac{m^2 d^3p\, d^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}|\mathcal{F}_A^0(\mathbf{q}_\perp+\mathbf{p}_\perp)|^2\frac{|\mathbf{L}|^2}{2}\tilde{G}_B, \quad (15)$$

$$\tilde{G}_B = 8\pi(Z_B\alpha)^2\int_0^\infty\int_0^\infty\frac{dk_1 dk_2}{k_1^2-k_2^2}\left\{\left(\frac{k_1}{k_2}\right)^{2iZ_B\alpha}-1\right\} \times[k_2 R J_0(k_2R)J_1(k_1R)-k_1RJ_0(k_1R)J_1(k_2R)].$$

Comparing the expression for the function $\tilde{G}_B$ with Eq. (8), we see that $\tilde{G}_B=G_B$. After the summation over the electron and positron polarizations, formulas (15) come into Eq. (11). Note that the expression (11) can be obtained directly from Eq. (14) by taking the integral over $\mathbf{k}_{1,2}$ in the region $k_{1,2}<k_0\ll|\mathbf{p}_\perp+\mathbf{q}_\perp|$, and then integrating over ρ in the infinite limits.

If the impact parameter ρ is restricted by the beam transverse size R_0, then it follows from the above consideration that the effect of the finite size appears when we cannot neglect $k\sim 1/R_0$ in the argument of $\mathcal{F}_A^0$ in comparison with $|\mathbf{q}_\perp+\mathbf{p}_\perp|$. This is equivalent to the condition $R_0\ll 1/\Delta_{min}\sim\gamma_A\gamma_B/m$.

Thus the method developed in Refs. [1–3] can be used for a calculation of the Coulomb corrections to the e^+e^- pair production cross section integrated over the direction of the positron (electron) momentum. Its careful application leads to the correct result.

We are grateful to V.M. Katkov and V.M. Strakhovenko for useful discussions.

[1] B. Segev and J. C. Wells, Phys. Rev. A **57**, 1849 (1998); Phys. Rev. C **59**, 2753 (1999).
[2] A.J. Baltz and L. McLerran, Phys. Rev. C **58**, 1679 (1998).
[3] U. Eichmann, J. Reinhardt, S. Schramm, and W. Greiner, Phys. Rev. A **59**, 1223 (1999).
[4] H. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954); H. Davies, H. Bethe, and L. C. Maximon, *ibid.* **93**, 788 (1954).
[5] D. Yu. Ivanov, A. Schiller, and V. G. Serbo, Phys. Lett. B **454**, 155 (1999).
[6] G. Moliére, Z. Naturforsch. Teil A **2a**, 133 (1947).
[7] V. B. Berestetskii, E. M. Lifshitz, L. B. Pitaevskii, Quantum Electrodynamics (Nauka, Moscow, 1989), p. 98.
[8] D. Yu. Ivanov, E. A. Kuraev, A. Schiller, and V. G. Serbo, Phys. Lett. B **442**, 453 (1998).