

Construction of quantum states with bound entanglement

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We introduce a family of bound-entangled quantum states in 3×3 dimensions. Their density matrix ρ depends on seven independent parameters and has four different nonvanishing eigenvalues.

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Entangled quantum states have been used since the very early days of quantum mechanics for computing the properties of atomic and molecular systems [1]. However, it is only in recent years that the existence of a hierarchy of entangled density matrices has become apparent, and it is not yet fully understood.

There are classical correlations, but there is no quantum entanglement in *separable* density matrices that can be written as convex sums,

$$\rho_{m\mu,n\nu} = \sum_K w_K (\rho'_K)_{mn} \otimes (\rho''_K)_{\mu\nu}, \quad (1)$$

where the density matrices $(\rho'_K)_{mn}$ and $(\rho''_K)_{\mu\nu}$ refer to two subsystems, possibly with different dimensions, and the coefficients w_K are positive and sum up to unity. If Eq. (1) holds, it readily follows that the partial transpose $\sigma_{m\mu,n\nu} = \rho_{n\mu,m\nu}$ is another separable density matrix and in particular has no negative eigenvalue. This property gives a very simple *necessary* condition for separability [2]. It also is a *sufficient* condition for systems of dimensions 2×2 and 2×3 , but not for higher dimensions [3]. The first counterexamples for dimensions 2×4 and 3×3 contained one free parameter [4]. Such states are called ‘‘bound-entangled’’ [5] because it is impossible to distill from them pure singlets by means of local operations and classical communication.

Recently, a new class of bound-entangled states was produced by means of unextendible product bases (UPB’s) [6]. In these states, the density matrix ρ depends on six parameters and is of rank 4 with equal eigenvalues 0.25. If all the matrix elements are real, there are only four free parameters, and $\sigma \equiv \rho$. In the complex case, σ and ρ have similar structures, but they correspond to different UPBs.

Here, we present a more general construction of bound-entangled states in 3×3 dimensions, depending on seven parameters (only five if ρ is real and $\sigma = \rho$), with four different nonvanishing eigenvalues. We hope that this explicit construction will be useful for elucidating properties of bound-entangled states, in particular for proving (or disproving) the conjecture that they satisfy all the Bell inequalities, and therefore are compatible with a local hidden-variable description [7,8]. Other open problems are mentioned in Ref. [9].

We write ρ in terms of four unnormalized eigenvectors,

$$\rho = N \sum_{j=1}^4 |V_j\rangle\langle V_j|, \quad (2)$$

where the coefficient $N = 1/\sum_j \langle V_j, V_j \rangle$ normalizes ρ to unit trace. The four eigenvalues, $N \langle V_j, V_j \rangle$, are in general different. Explicitly, we take

$$|V_1\rangle = |m, 0, s; 0, n, 0; 0, 0, 0\rangle,$$

$$|V_2\rangle = |0, a, 0; b, 0, c; 0, 0, 0\rangle,$$

$$|V_3\rangle = |n^*, 0, 0; 0, -m^*, 0; t, 0, 0\rangle,$$

$$|V_4\rangle = |0, b^*, 0; -a^*, 0, 0; 0, d, 0\rangle, \quad (3)$$

where the components of $|V_j\rangle$ are listed in the order 00, 10, 20; 01, It is easily seen that the ninth row and column of ρ vanish. The remaining 8×8 matrix is like a chessboard, with the odd-odd components depending on m, n, s, t , and the even-even components on a, b, c, d (still, some of these components are zeros).

In principle, all eight parameters on the right-hand side of Eqs. (3) can be complex (special values of these parameters yield results equivalent to those obtained by using UPBs). However, we still have the freedom of choosing the overall phase of each $|V_j\rangle$ (this obviously does not change ρ). Furthermore, we can define new phases for the basis vectors $|e'_k\rangle$ and $|e''_k\rangle$ used to describe the two subsystems. This corresponds to rewriting ρ in a different basis without changing its chessboard structure, nor the absolute values of the components of $|V_j\rangle$. This freedom can be used to make many of these components real, but not all of them, because the two combinations cm/bc and b^*t/n^*d are not affected by these changes of phase. This can be seen as follows: the parameters in cm/bc appear in $|V_1\rangle = |m, 0, s; \dots\rangle$ and $|V_2\rangle = |\dots; b, 0, c; \dots\rangle$. The ratios m/s and b/c are affected only by changes of the relative phase of $|e'_1\rangle$ and $|e'_3\rangle$, and cm/bc is not affected at all. Likewise, the parameters in b^*t/n^*d appear in $|V_3\rangle = |n^*, 0, 0; \dots; t, 0, 0\rangle$ and $|V_4\rangle = |0, b^*, 0; \dots; 0, d, 0\rangle$. The ratios n^*/t and b^*/d are affected, both in the same way, only by changes of the relative phase of $|e''_1\rangle$ and $|e''_3\rangle$. There are no other invariants of this type, and we can assume, without loss of generality, that s and t are complex, while the six other parameters are real.

We now prove that in the generic case (random parameters) ρ is inseparable: as shown in [4], a state ρ is inseparable if the range of ρ contains no product state. This is the case for our ρ , unless the parameters are chosen in a specific way. Indeed, assume that there is a product state such that

$$|p, q, r\rangle \otimes |x, y, z\rangle = \sum A_j |V_j\rangle. \quad (4)$$

Since the ninth components of all the $|V_j\rangle$ vanish, we have $rz=0$. Assume that $z=0$ (the same proof is valid for $r=0$, *mutatis mutandis*). Then the seventh and eighth components vanish, so that $A_3=A_4=0$. We then have $(px)(ry) = (A_1m)(A_2c)$ while $(py)(rx) = (A_2b)(A_1s)$, whence $mc = bs$, which does not hold in general for randomly chosen parameters.

Finally, we have to verify that σ is a positive matrix, so that the entanglement is bound. Namely, all the diagonal subdeterminants of σ have to be positive or zero. This gives a large number of inequalities. Here, we shall restrict ourselves to the study of two simple cases.

The simplest one is to assume $\sigma = \rho$. Owing to the chessboard structure of ρ , this leads to three nontrivial conditions, which can be written, with the parametrization of $|V_j\rangle$ in Eqs. (3),

$$\begin{aligned} \rho_{13} = \rho_{31} \quad \text{or} \quad m s^* = m^* s, \\ \rho_{26} = \rho_{35} \quad \text{or} \quad a c^* = s n^*, \\ \rho_{48} = \rho_{57} \quad \text{or} \quad a d = m t. \end{aligned} \quad (5)$$

With our choice of phases, these conditions mean that $s = ac/n$ and $t = ad/m$ are real. We thus have six free parameters in the vectors $|V_j\rangle$. These parameters can still be scaled by an arbitrary factor (that will be compensated by N), so that there are five independent parameters in our construction.

Another, more general way of constructing bound entangled states is to assume that σ is not the same as ρ , but still is spanned by two pairs of eigenvectors with a structure similar to those in Eqs. (3), with new parameters that will be called a', b', \dots . This assumption is obviously compatible with (but not required by) the chessboard structure of σ , which is the partial transpose of ρ . It is then easily seen that the various parameters in the eigenvectors of σ may differ only in their phases from those in the eigenvectors of ρ . We therefore write them as $a' = a e^{i\alpha}$, and so on. This gives eight additional arbitrary phases, besides those of s and t . The requirement that σ be the partial transpose of ρ imposes six conditions on these phases (apart from those on the absolute values). These are fewer conditions than phases at our disposal, so that the parameters s and t can now remain complex. Only their absolute values are restricted by

$$|s| = ac/n \quad \text{and} \quad |t| = ad/m. \quad (6)$$

We thus have seven independent free parameters for this case.

A natural question is whether this construction can be generalized to higher-dimensional spaces, with a larger number of pairs of eigenvectors $|V_j\rangle$, suitably structured. We have no definite answer: in such a generalization, the number of conditions grows much faster than the number of free parameters, and we think it unlikely that such a generalization is possible, but we have no formal proof.

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