Comment on "Exact wave function of a harmonic plus an inverse harmonic potential with time-dependent mass and frequency"

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(Received 22 October 1998; published 13 January 2000)

The connection between wave functions of harmonic plus inverse harmonic potential with time-dependent mass and frequency and those of harmonic plus inverse harmonic potential with time-dependent frequency is investigated. Thus the correct wave function of the harmonic plus inverse harmonic potential with time-dependent mass and frequency is obtained.

PACS number(s): 03.65.Ca

A quantum time-dependent system described by the singular oscillator Hamiltonian

$$H=\frac{1}{2}\left(\frac{p^2}{M(t)}+\omega^2(t)M(t)q^2+\frac{1}{M(t)q^2}\right) \tag{1}$$

is one of the rare examples admitting exact solutions of the Schrödinger equation and have been studied intensively lately [1–9]. The distinguished role of the Hamiltonian (1) is explained by the fact that, in a sense, it belongs to a boundary between linear and nonlinear problems of classical and quantum mechanics. For this reason, it was used in many applications in different areas of physics. For example, it served as an initial point in constructing interesting exactly solvable models of interacting N-body systems [2,3]. It was also used for modeling diatomic and polyatomic molecules [4]. It can have some relation to the problem of the relative motion of ions in electromagnetic traps [9]. One of these recent papers is the one by Um *et al.* [1]. In this paper they have derived a Schrödinger wave function for such a system [formula (34) of Ref. [1]]. I want to call attention to an error in this paper which seems to be subtle and encountered in some other works. Indeed, the Schrödinger wave function (34) of Ref. [1] is not correct because (i) it includes the term $\dot{M}(t)/M(t)$ only once. Its time derivative in the Schrödinger equation produces the $\ddot{M}(t)$ term, but there is no other $\ddot{M}(t)$ term which cancels the first one, and (ii) it is not an eigenfunction of the invariant, Eq. (19), in [1]. This Comment addresses the source of an error in calculating the solution of the Schrödinger equation in the paper by Um *et al.* [1]. It is an interesting and important question that should be resolved. We assert that the error came about because in Ref. [1] Um *et al.* relate incorrectly the wave function in terms of the new variables to the wave function in terms of the original variables.

To clarify more, let us go back to some results of Ref. [1]. The authors have shown that the Hamiltonian (1) can be transformed to $H_N(t)$,

$$H_N(t)=\frac{1}{2}\left(P^2+\Omega^2(t)Q^2+\frac{1}{Q^2}\right), \tag{2}$$

where $\Omega^2=\omega^2-(\gamma^2/4+\dot{\gamma}/2)$ is the modified frequency, by means of the new canonical variables [Eqs. (6) and (7) of Ref. [1]]

$$Q=\sqrt{M}q, \tag{3a}$$

$$P=\frac{1}{\sqrt{M}}p+\sqrt{M}\frac{\gamma}{2}q, \quad \gamma=\left(\frac{d\ln M}{dt}\right). \tag{3b}$$

The invariant operator $I(t)$ satisfying $\partial I/\partial t=\iota[I,H_N]$ is obtained [expression (16) of Ref. [1]] (assume $\hbar=1$):

$$I(Q,P,\rho)=\frac{1}{2}\left(\frac{Q^2}{\rho^2}+(P\rho-\dot{\rho}Q)^2+\frac{\rho^2}{Q^2}\right), \tag{4}$$

where ρ is a c number satisfying Eq. (11) of Ref. [1]. According to Lewis-Riesenfeld theory [10], the spectral problem for the invariant operator $I(Q,P,\rho)$ is easily solved in the form

$$I\left(Q,-i\frac{\partial}{\partial Q},\rho\right)\phi_n(Q,\rho)=(2n+a+1)\phi_n(Q,\rho), \tag{5}$$

the eigenfunctions $\phi_n(Q,\rho)$ being explicitly given, in terms of Laguerre polynomials L_n^a, by the expression

$$\phi_n(Q,\rho)=(4)^{1/4}\left\{\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right\}^{1/2}\left(\frac{Q^2}{\rho^2}\right)^{(2a+1)/4}\times\exp\left\{\frac{\iota}{2}\left(\frac{\dot{\rho}}{\rho}+\frac{\iota}{\rho^2}\right)Q^2\right\}L_n^a\left(\frac{Q^2}{\rho^2}\right), \tag{6}$$

and those of the Schrödinger equation

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial Q^2}+\Omega^2(t)Q^2+\frac{1}{Q^2}\right]\psi_n(Q,t)=i\frac{\partial\psi_n(Q,t)}{\partial t}, \tag{7}$$

are related by

$$\psi_n(Q,t)=e^{i\alpha_n(t)}\phi_n(Q,\rho), \tag{8}$$

where the phases $\alpha_n(t)$ can be found as

$$\dot{\alpha}_n(t)=\int_R dQ\{\phi_n^*(Q,\rho)(i\partial_t-H_N)\phi_n(Q,\rho)\}$$
$$=-(2n+a+1)\frac{1}{\rho^2}. \qquad (9)$$

In Ref. [1], the authors introduced a time-dependent auxiliary transformation [formula (17) of Ref. [1]]

$$\rho(t)=\sqrt{M(t)}x(t), \qquad (10)$$

and used a time-dependent canonical transformation (3a) and (3b) to relate the wave function in terms of the new variables Q to the wave function in terms of the original variables q. This method leads them to obtain a wave function [Eq. (34) in Ref. [1]] in terms of the original variables q with an incorrect added factor $e^{i(M(t)\gamma(t)q^2)/4}$. To convince oneself, one has just to observe that, if one introduces $\rho(t)=\sqrt{M(t)}x(t)$ and expresses Eqs. (4) and (6) in terms of the auxiliary function $x(t)$ which satisfies Eq. (18) of Ref. [1], one obtains

$$\tilde{I}(Q,P,x)=\frac{1}{2}\left\{\frac{Q^2}{Mx^2}+\left[x\sqrt{M}\left(P-\frac{\gamma}{2}Q\right)-\sqrt{M}\dot{x}Q\right]^2+\frac{Mx^2}{Q^2}\right\} \qquad (11)$$

and

$$\tilde{\phi}_n(Q,x)=(4)^{1/4}\left\{\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right\}^{1/2}\left(\frac{Q^2}{Mx^2}\right)^{(2a+1)/4}\times\exp\left\{\frac{\iota}{4}\gamma(t)Q^2\right\}\times\exp\left\{\frac{\iota}{2}\left(\frac{\dot{x}}{x}+\frac{\iota}{Mx^2}\right)Q^2\right\}L_n^a\left(\frac{Q^2}{Mx^2}\right). \qquad (12)$$

Now some observations can be made

(i) The operator invariant $\tilde{I}(Q,P,x)$ is a particular invariant of the new Hamiltonian $H_N(t)$ verifying the invariance condition $\partial\tilde{I}/\partial t=\iota[\tilde{I},H_N]$.

(ii) The eigenfunctions $\tilde{\phi}_n(Q,x)$ of $\tilde{I}(Q,P,x)$ evolve according to the time-dependent Schrödinger equation (7) corresponding to the new Hamiltonian $H_N(t)$, when they are multiplied by the phase $\alpha_n(t)$, Eq. (9), that is expressed in terms of auxiliary function $x(t)$. But this is not the point. The point is to derive the wave function solution of the Schrödinger equation corresponding to the original system, i.e., the singular oscillator with time-dependent mass and frequency. It is in fact the objective of Um *et al.* [1] in their Brief Report as stated in their abstract, Introduction, and Conclusion.

In light of the above remarks and as we have said at the beginning of this Comment, the $\ddot{M}(t)$ terms cannot be canceled in the Schrödinger equation corresponding to the singular oscillator with time-dependent mass and frequency. Indeed, the wave functions $\tilde{\phi}_n(Q,x)$ are not the eigenfunctions of the invariant which is associated with the Hamiltonian of the singular oscillator with time-dependent mass and frequency.

In order to obtain the correct wave functions, we first note that

$$\tilde{I}(Q,P,x)=U^{-1}\left\{\frac{1}{2}\left(\frac{Q^2}{x^2}+(Px-M\dot{x}Q)^2+\frac{x^2}{Q^2}\right)\right\}U$$
$$=U^{-1}I(Q,P,x)U \qquad (13)$$

and

$$\tilde{\phi}_n(Q,x)=U^{-1}\left[(4M)^{1/4}\left\{\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right\}^{1/2}\left(\frac{Q^2}{x^2}\right)^{(2a+1)/4}\times\exp\left\{\frac{\iota}{2}\left(\frac{M\dot{x}}{x}+\frac{\iota}{x^2}\right)Q^2\right\}L_n^a\left(\frac{Q^2}{x^2}\right)\right]$$
$$=U^{-1}\phi_n(Q,x), \qquad (14)$$

where

$$U(t)=U_1(t)U_2(t)=\exp\left[\frac{\iota}{2}\ln\sqrt{M}(PQ+QP)\right]\times\exp\left[-\frac{\iota}{4}\gamma(t)Q^2\right] \qquad (15)$$

is a time-dependent unitary transformation which yields the desired transformations of the operators $U^{-1}QU=Q/\sqrt{M}=q$ and $U^{-1}PU=\sqrt{M}(P-\gamma Q/2)=p$ [like the time-dependent canonical transformations (3a) and (3b) and Eqs. (6) and (7) defined in Ref. [1]]. In Eq. (14) we used the formula

$$Uf(Q)=U_1\exp\left(-\frac{\iota}{4}\gamma(t)Q^2\right)f(Q)=\exp\left(\frac{\ln\sqrt{M}}{2}\right)\times\exp\left(-\frac{\iota}{4}\gamma(t)(Qe^{\ln\sqrt{M}})^2\right)f(Qe^{\ln\sqrt{M}}). \qquad (16)$$

We look for a time-dependent transformation $U(t)$ such that

$$\tilde{\psi}_n(t)=U^{-1}(t)\psi_n(t); \qquad (17)$$

i.e., $U(t)$ brings any solution of the equation $i\partial_t\tilde{\psi}_n=H_N\tilde{\psi}_n$ into a solution of the equation $i\partial_t\psi_n=H\psi_n$. Taking the time derivative of the relation (17), one sees that the transformed Hamiltonian H must satisfy the relation $H=UH_NU^{-1}-iU\partial_tU^{-1}$; this transformed Hamiltonian corresponds to the the singular oscillator Hamiltonian expressed in the new variables P and Q:

$$H=\frac{1}{2}\left(\frac{P^2}{M(t)}+\omega^2(t)M(t)Q^2+\frac{1}{M(t)Q^2}\right). \qquad (18)$$

Thus, the eigenvalue equation (5) is mapped onto

$$I\left(Q,-i\frac{\partial}{\partial Q},x\right)\phi_n(Q,x)=(2n+a+1)\phi_n(Q,x), \qquad (19)$$

which is cleary the eigenvalue equation corresponding to the invariant of the singular oscillator Hamiltonian with time-dependent mass and frequency. We can observe that the auxiliary transformation (10) induces a time-dependent unitary transformation, because we start with the eigenvalue problem of the singular oscillator with time-dependent frequency and we arrive at an eigenvalue problem of the singular oscillator with time-dependent mass and frequency. These two systems are different even though they are unitarily equivalent.

It must be highlighted that the wave functions $\phi_n(Q,x)$ multiplied by the phase $\alpha_n(t)$, Eq. (9), which is expressed in terms of auxiliary function $x(t)$, evolve according to the Schrödinger equation with the Hamiltonian (18). Therefore, we have obtained, in terms of the new coordinate Q, the correct wave function of the singular oscillator with time-dependent mass and frequency which coincides with those of Refs. [7], [8]. Thus the problem is completely solved. The alert reader can easily guess that the wave functions expressed in terms of the original variables q are those wave functions obtained by Eq. (19) where the substitution of Q by q is made.

In order to investigate the connection between wave functions in the original variables q and those of the new variables Q, let us go back to the wave function given by Eq. (12), which was obtained through substitution of the auxiliary transformation Eq. (10). In Ref. [1] the authors seek to express the wave function of the original system, i.e., the singular oscillator with time-dependent mass and frequency, by substituting Q by $\sqrt{M}q$ into Eq. (12); the result of the substitution is purported to be Eq. (34) in Ref. [1]. Unfortunately, this method leads the authors to conclude incorrectly that their wave function is an eigenfunction of the invariant of Eq. (19) of Ref. [1], and thus it satisfies the Schrödinger equation with the original Hamiltonian (1). However, Eq. (34) of Ref. [1] is incorrect because it includes an added factor $e^{i(M(t)\gamma(t)q^2)/4}$. We believe that this apparently natural procedure is ill founded. To transform back to the original variables q, we first note that since U performs the scale change $Q=q\sqrt{M}$, the states are related by [11–13]

$$\langle Q,t| = \langle q,t|U^{-1}(P(p,q,t),Q(p,q,t),t)$$
$$= \frac{1}{M^{1/4}} e^{[i(\gamma(t)M(t)q)^2]/4} \left\langle q=\frac{Q}{\sqrt{M}},t \right| . \qquad (20)$$

Hence the wave functions in the original variables and transformed coordinates are related by

$$\phi_n(q,x) = M^{1/4} \exp\left[-\frac{\iota}{4}\gamma(t)M(t)q^2 \right] \tilde{\phi}_n(q\sqrt{M},x), \qquad (21)$$

which is exactly the result given above and derived by an another approach in Refs. [7], [8]. We have clarified how to relate the wave function expressed in its original coordinate variables to that obtained in the new coordinate Q.

In conclusion, the quantum mechanical problem may be solved by making canonical transformations which are implemented by unitary operator U so that basis states and operators are transformed according to $\langle q,t| = \langle Q,t|U(t)$, $q=U^{-1}(t)QU(t)$, and $p=U^{-1}(t)PU(t)$.

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