Manipulating spinor condensates with magnetic fields: Stochastization, metastability, and dynamical spin localization

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We study the dynamical response of a spinor Bose condensate under the influence of external magnetic fields. A rich set of phenomena are investigated such as stochastization in population evolution, metastability in spin composition, and dynamical localization in spin space.

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I. INTRODUCTION

Multicomponent Bose-Einstein condensates have been studied extensively for the past two years [1]. More recently experiments on ^{23}Na condensates confined in an optical dipole trap [2,3] have stimulated great interests in the study of spinor Bose condensates—condensates with spin degrees of freedom, represented by a vector order parameter. A variety of new phenomena have been predicted for these systems, such as spin textures, spin waves, superfluid flow, and matter wave phase conjugation [4–6]. In previous papers, we have studied the ground-state spin structure and nonlinear spin-mixing dynamics of such a system for the case of zero applied magnetic field [7,8]. However, as has been suggested [2,9], the properties of the spinor condensate can be conveniently manipulated with weak magnetic fields, and novel behavior may arise. In the present work, we study the evolution of the spinor condensate in the presence of external magnetic fields.

We describe our model in Sec. II. The main results are presented in Sec. III and Sec. IV, where the effects of longitudinal and transverse magnetic fields are studied, respectively. Some concluding remarks are given in Sec. V.

II. PHYSICAL MODEL

We begin by introducing our physical model as presented in Ref. [8]. We consider an $f=1$ spinor condensate trapped in a cigar-shaped harmonic potential with tight confinement in the transverse direction, i.e., $\omega_\perp \gg \omega_z$, with $\omega_\perp$, ω_z being the transverse and longitudinal trap frequencies, respectively (the trapping potential is assumed to be the same for all three spin states). The Hamiltonian of the system can be expressed as [4,5]

$$\mathcal{H}=\int d\mathbf{r}\,\hat{\mathbf{\Psi}}^\dagger(\mathbf{r})(\hat{K}+\hat{V})\hat{\mathbf{\Psi}}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\,\hat{\mathbf{\Psi}}^\dagger(\mathbf{r}_1)\hat{\mathbf{\Psi}}^\dagger(\mathbf{r}_2)U(\mathbf{r}_1,\mathbf{r}_2)\hat{\mathbf{\Psi}}(\mathbf{r}_2)\hat{\mathbf{\Psi}}(\mathbf{r}_1). \tag{1}$$

Here $\hat{K}$, $\hat{V}$ represent the kinetic energy and the external trapping potential, respectively, $\hat{\mathbf{\Psi}}=(\hat{\Psi}_{-1},\hat{\Psi}_0,\hat{\Psi}_1)^T$, where $\hat{\Psi}_j$ is the atomic field annihilation operator and is related to the wave function ψ_j via $\langle\hat{\Psi}_j\rangle=N_j\psi_j$, with N_j being the particle number with spin j. The two-body nonlinear interaction potential $U(\mathbf{r}_1,\mathbf{r}_2)\equiv\delta(\mathbf{r}_1-\mathbf{r}_2)\Sigma_{F=0}^2 g_F\mathcal{P}_F$, where $g_F\equiv 4\pi\hbar^2 a_F/m$ with a_F being the s-wave scattering length in the total spin F channel (for ^{23}Na, $a_2=52a_B$, and $a_0=46a_B$ [4], where a_B is the Bohr radius), $\mathcal{P}_F$ is the projection operator that projects the pair 1 and 2 into spin F state. For the trapping geometry considered here, we may approximate the wave function of spin state j $(j=0,\pm 1)$ as $\psi_j(x,y,z,t)=\phi_\perp(x,y)\phi_j(z,t)e^{-i\omega_\perp t}$, with $\phi_\perp(x,y)$ being the ground state of the two-dimensional harmonic potential $m\omega_\perp^2(x^2+y^2)/2$. At zero temperature and zero magnetic field, the equations of motion for the longitudinal wave function $\phi_j(z,t)$ may therefore be written as

$$\begin{aligned}
i\dot{\phi}_{-1}&=\mathcal{L}_z\phi_{-1}+\lambda_a N\eta(\phi_0^2\phi_1^*+|\phi_{-1}|^2\phi_{-1}+|\phi_0|^2\phi_{-1}-|\phi_1|^2\phi_{-1}),\\
i\dot{\phi}_0&=\mathcal{L}_z\phi_0+\lambda_a N\eta(2\phi_1\phi_{-1}\phi_0^*+|\phi_{-1}|^2\phi_0+|\phi_1|^2\phi_0),\\
i\dot{\phi}_1&=\mathcal{L}_z\phi_1+\lambda_a N\eta(\phi_0^2\phi_{-1}^*+|\phi_1|^2\phi_1+|\phi_0|^2\phi_1-|\phi_{-1}|^2\phi_1),
\end{aligned} \tag{2}$$

where N is the total particle number, $\lambda_a\equiv(g_2-g_0)/3$, $\mathcal{L}_z=-d^2/dz^2+z^2/4+\lambda_s N\eta(|\phi_{-1}|^2+|\phi_0|^2+|\phi_1|^2)$, $\lambda_s\equiv(2g_2+g_0)/3$, and $\eta=\int dxdy\,|\phi_\perp|^4/\int dxdy\,|\phi_\perp|^2=\omega_\perp/(4\pi\omega_z)$. The above equations have been written in dimensionless form and the units for length, energy, and time are $\sqrt{\hbar/(2m\omega_z)}$, $\hbar\omega_z$, and $1/\omega_z$, respectively. The first term inside the bracket on the right-hand side of Eqs. (2) originates from the nonlinear spin-exchange interaction.

Assuming the initial particle number in spin j is N_j, we study the dynamics of the system by taking the initial wave function $\phi_j(z,0)$ to be the ground-state solutions of

$$\mathcal{L}_z\,\phi_j(z,0)=\mu\,\phi_j(z,0).$$

Apart from a normalization constant, the three $\phi_j(z,0)$ have the same spatial profile and hence can be written as $\phi_j(z,0)=a_j(0)\phi(z)=\sqrt{P_j(0)}e^{i\theta_j(0)}\phi(z)$, where $P_j(0)$

$=N_j/N$ is the initial population in spin j, $\theta_j(0)$ the initial phase of wave function $\phi_j(z,0)$, and $\phi(z)$ is assumed to be real and normalized as $\int dz\, \phi^2(z)=1$.

Generally, the population in each spin component $P_j(t)=\int dz\,|\phi_j(z,t)|^2$ executes complex oscillatory behavior familiar in a nonlinear system. However, under certain conditions, we have shown that $\phi_j(z,t)$ does not evolve with time except for an overall phase change. Hence, $P_j(t)=P_j(0)$. This happens when the population and phase of the condensate satisfy either of the following two conditions:

$$\text{(A)}\ P_0=\tfrac{1}{2}[1-(P_{-1}-P_1)^2] \text{ and } \theta=0,$$

$$\text{(B)}\ P_{-1}=P_1 \text{ and } \theta=\pi,$$

where $\theta\equiv 2\theta_0-\theta_{-1}-\theta_1$. In fact, under these two special conditions [we will call them conditions (A) and (B), respectively], the system can be shown to be in one of the nonmixing eigenstates of the total Hamiltonian [8] in such a way that the contribution of the nonlinear spin-exchange interaction appears as a constant energy shift for all the spin components, and hence population transfer among different spin states does not occur.

III. EFFECTS OF LONGITUDINAL FIELDS

In this work, we study the effects of external magnetic fields using this same framework. First, let us consider only a longitudinal field $\mathbf{B}=B_l\hat{z}$. Such a field will lift the energy degeneracy of the spin states through Zeeman effect. The field introduces an extra term into the Hamiltonian:

$$\mathcal{H}_B=-\mu_B\sum_{j=-1}^{1}\int d^3r\,\hat{\Psi}_j^\dagger(jg_fB_l+j^2g_f^{(2)}B_l^2)\hat{\Psi}_j,$$

where μ_B is the Bohr magneton, and g_f, $g_f^{(2)}$ are the linear and quadratic Zeeman coefficients, respectively. For ^{23}Na, $\mu_Bg_f=0.7$ MHz/G and $\mu_Bg_f^{(2)}=-390$ Hz/G^2.

Figure 1 shows the population of spin-0 state as a function of time, evolving in the presence of a uniform longitudinal magnetic field, where initially all three spin states are equally populated. For small field strength [Fig. 1(a)], the dynamics are not very different from the zero-field case [8]: Populations in individual spin states oscillate with frequencies that are dependent on the condensate phase. For large field strength [Fig. 1(b)], however, a strong damping is observed in the population oscillation. Specifically, the spin-0 state becomes highly populated and the dynamics of the condensate are no longer sensitive to the phase. Such behavior can be intuitively understood in the following manner: In the presence of a uniform longitudinal magnetic field B_l, the quadratic Zeeman shift causes the energy of the two spin-0 atoms to be lower than that for a pair of spin-1 and spin-(-1) atoms by 2×390 Hz$\times B_l^2$/G^2 (note that this pair still has net spin zero in the z direction). This makes the spin-0 state energetically favorable. With a large B_l and substantially populated spin-(±1) states, the magnetic Hamiltonian $\mathcal{H}_B$ can contribute enough energy to the system so that as population is transferred from $P_{\pm1}$ to P_0, more highly excited spatial modes can be excited as the condensate evolves. Interference among these higher modes can cause an effective damping of the population oscillation, a process that has been referred to as stochastization. An analogous phenomenon was found in the two-component condensate system by Sinatra *et al.* [10], where the two components initially having the same density profile were set into motion by an abrupt displacement of their respective trap centers. For small displacement, a periodic oscillatory motion was found; whereas for large displacement, strong nonlinear mixing induced coupling to higher excitation modes, thus leading to damping of the relative motion of the two components.

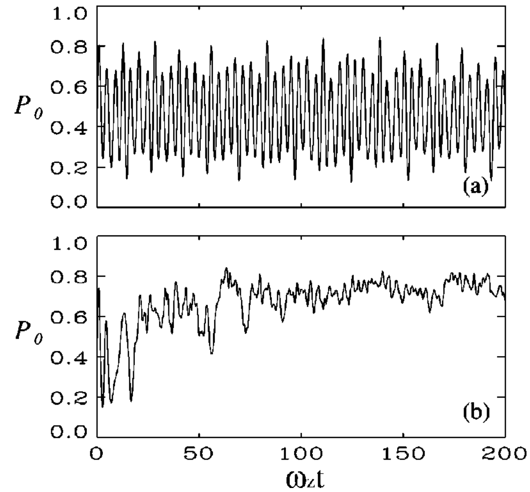

FIG. 1. Population of spin-0 component as a function of time, in the presence of a uniform longitudinal magnetic field (a) $B_l=0.1$ G and (b) $B_l=0.2$ G. Initially, $n_1=n_0=n_{-1}=1/3$, $N=20000$, $\theta=\pi/2$. Other parameters used in the calculation are: $\omega_z=2\pi\times40$ Hz, $\eta=1$.

In Fig. 2, we illustrate the effect of an inhomogeneous longitudinal field. Particularly, we have chosen a field linearly dependent on z: $B_l=B_0+B_1z$. Figure 2 shows the population of spin-0 component, starting from two different initial states: (1) the case where the entire condensate is placed in the spin-0 state; (2) the case where the condensate is in a 50-50 mixture of spin-1 and spin-(-1) components

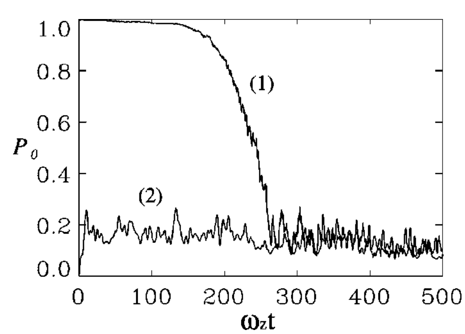

FIG. 2. Spin-0 population in the presence of a magnetic bias field $B_0=200$ mG, and gradient $B_1=200$ mG/cm. The initial state is (1) $n_0=0.998$, $n_1=n_{-1}=0.001$; (2) $n_0=0.001$, $n_1=n_{-1}=0.4995$. Other parameters are the same as in Fig. 1.

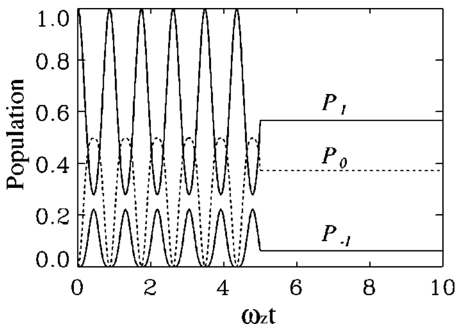

FIG. 3. Population as a function of time, in the presence of a uniform magnetic field with $B_z = 0.3$ mG and $B_x = 0.3$ mG. Initially, $n_1 = 1$, $n_0 = n_{-1} = 0$. The field is turned off at $t = 5/\omega_z$.

[11]. We assume that the inhomogeneous magnetic field is imposed after the initial state is prepared, such that for both cases, we can assume that the condensate has the same density profile at $t=0$. From Fig. 2, one can see that the population approaches an equilibrium, but the evolution to this equilibrium is quite different for different initial states. When one starts with a pure spin-0 state, the condensate remains in that state for a certain amount of time before spin mixing takes place, depleting P_0. On the other hand, starting from the 50-50 mixture of spin-1 and spin-(-1) states, the population for the spin-0 states grows immediately. As explained by Miesner *et al.* [3], such behavior can be understood as being caused by an interplay between a quadratic Zeeman effect (which makes the spin-0 state energetically favorable) and the effect of the magnetic field gradient [which makes the spin-(± 1) state energetically favorable by separating them into spin domains at the ends of the condensate]. Case (1) shown in Fig. 2 demonstrates the metastability of the spinor condensate in its spin composition.

IV. EFFECTS OF TRANSVERSE FIELDS

Next, let us consider the effects of transverse magnetic fields. Different from the longitudinal field, the transverse field appears as a coupling between different spin components. Without loss of generality, we choose the direction of the field to be along the x axis and the field strength B_x is assumed to be weak, then its contribution to the Hamiltonian may be written as

$$\mathcal{H}_B = -\mu_B g_f B_x \langle \hat{\boldsymbol{\Psi}} | \hat{L}_x | \hat{\boldsymbol{\Psi}} \rangle. \tag{3}$$

Figure 3 shows an example of the evolution of the population as functions of time in a weak uniform magnetic field with longitudinal and transverse field strength B_z and B_x, respectively. Initially, the system is spin polarized such that only the spin-1 state is populated. In this example, the field is on from $t=0-5$ and off after $t=5$. The population oscillates when this field is present and remains unchanged after it is turned off. Furthermore, we find that the condensate dynamics is independent of the total number of the atoms, and that the wave functions retain their initial spatial profile, i.e., $\phi_j(z,t) = a_j(t)\phi(z)$. A closer examination shows that at any given time t, the system satisfies condition (A) and as such remains in a nonmixing eigenstate in the absence of magnetic fields. This explains why after the field is turned off, the population in individual spin states ceases to evolve, becoming ''frozen.''

Such a behavior can be understood as follows. The presence of a weak magnetic field $\mathbf{B}$ introduces an extra term into the Hamiltonian $\mathcal{H}_B = \text{const} \times \mathbf{B} \cdot \hat{\mathbf{L}}$. Notice that $\hat{\mathbf{L}}^2$ is the total Hamiltonian in the absence of magnetic fields (apart from a constant part) [7,8]. Hence, if we start the condensate in an eigenstate of $\hat{\mathbf{L}}^2$ (for example, the state with complete spin polarization), the system will remain in an eigenstate of $\hat{\mathbf{L}}^2$ as long as the field is uniform, since $\hat{\mathbf{L}}^2$ commutes with $\mathcal{H}_B$. To describe the evolution of the condensate under these conditions, we seek an expression for the time evolution of the field amplitudes $a_j(t)$. We can derive the equations of motion of $a_j(t)$ from the Hamiltonian $\mathcal{H}_B$ (again, the contribution from the remaining part of the Hamiltonian is a constant energy shift, and can be neglected in dynamics)

$$\begin{aligned} i\dot{a}_{-1} &= b_z a_{-1} - b_x a_0, \\ i\dot{a}_0 &= -b_x(a_1 + a_{-1}), \\ i\dot{a}_1 &= -b_z a_1 - b_x a_0, \end{aligned} \tag{4}$$

where $b_z = \mu_B g_f B_0$ and $b_x = \mu_B g_f B_x/\sqrt{2}$. As we have assumed that the field is weak, the quadratic Zeeman effect has been neglected. With the initial condition $a_1(0)=1$, $a_{-1}(0) = a_0(0) = 0$, Eqs. (4) can be solved as

$$\begin{aligned} a_{-1}(t) &= -\frac{b_x^2}{\omega_b^2}(1-\cos\omega_b t), \\ a_0(t) &= -\frac{b_z b_x}{\omega_b^2}(1-\cos\omega_b t) + i\frac{b_x}{\omega_b}\sin\omega_b t, \\ a_1(t) &= \frac{b_x^2}{\omega_b^2} + \frac{b_z^2 + b_x^2}{\omega_b^2}\cos\omega_b t + i\frac{b_z}{\omega_b}\sin\omega_b t, \end{aligned} \tag{5}$$

where $\omega_b = \sqrt{b_z^2 + 2b_x^2}$. As a self-consistency check, using $a_j(t) = \sqrt{P_j(t)}e^{i\theta_j}$, it is easy to see that condition (A) is indeed satisfied with $a_j(t)$ given by Eqs. (5) [12].

From Eq. (5), we see that if $b_z \gg b_x$, then $P_{-1}(t) \approx P_0(t) \approx 0$ and $P_1(t) \approx 1$, i.e., the population becomes *localized* in the initially populated spin-1 state. Such a behavior is not unfamiliar in condensed matter systems. For example, in a system of tight-binding electrons moving in a lattice with nearest-neighbor transfer, if one of the lattice sites is initially occupied by the electron wave packet, spatial localization, known as Wannier-Stark localization, will occur under the action of a dc electric field [13]. However, a more interesting phenomenon found in that system is the effect of dynamical localization (DL) studied by Dunlap and Kenkre [14] and experimentally realized clearly, only recently, in optical lattices [15]. When the electric field is modulated as $E(t) = E_0\cos(\omega t)$, spatial localization will occur when

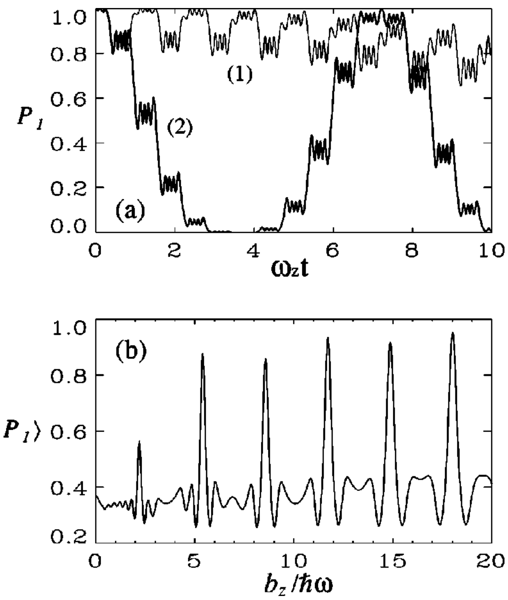

FIG. 4. (a) Population of spin-1 component as a function of time, in the presence of a uniform field with $B_x = 0.2$ mG and $B_z(t) = B_z\cos(\omega t)$, $\omega = 5$. $B_z = 3.4$ mG for curve (1) and $B_z = 3.7$ mG for curve (2). For curve (1), $J_0(b_z/\hbar\omega) = 0$. (b) Averaged population of spin-1 component (averaged over a time period of 10) as a function of $b_z/\hbar\omega$. $B_x = 0.2$ mG, $\omega = 5$, and B_z varies from 0 to 5.7 mG.

$J_0(eE_0a/\hbar\omega) = 0$, where a is the lattice constant and $J_0(x)$ is the zeroth-order ordinary Bessel function. It is interesting to investigate whether a similar effect can be found in the spinor condensate.

To study this possibility, we investigate the dynamics of the spinor condensate under a weak uniform magnetic field with a time-independent transverse component B_x (which plays a similar role to the intersite transfer that couples the neighboring states in the electron lattice system) and a time-dependent longitudinal component $B_z\cos(\omega t)$ (which is analogous to a time-dependent electric field in the electron-lattice system). Again, we assume that only the spin-1 state is initially populated. Figure 4(a) shows $P_1(t)$ for two situations: (1) $J_0(b_z/\hbar\omega) = 0$, and (2) $J_0(b_z/\hbar\omega) \neq 0$, where $b_z = \mu_B g_f B_z$. Figure 4(b) displays the averaged population in the spin-1 state as a function of $b_z/\hbar\omega$. These figures clearly demonstrate the effect of DL under a similar condition as found in the condensed matter system—population is localized in the initially occupied state when $b_z/\hbar\omega$ is a root of the Bessel function $J_0(x)$ provided that $B_z \gg B_x$.

A few explanatory remarks are in order. (1) The dynamical localization studied here occurs in *spin space* for the spinor condensate case as opposed to position space for the electron-lattice system considered by Dunlap and Kenkre. Furthermore, the spinor condensate has a finite number of states (three states for an $f = 1$ condensate), while the electron-lattice system contains an infinite chain. The DL effect nevertheless persists. One can show that this effect will persist for times $\propto 1/b_x$, much longer than the typical time scale of oscillations shown in Fig. 4. However, a finite-sized system will show a slow delocalization at long times even when the Bessel root condition is obeyed [see curve (1) in Fig. 4(a)]. A similar effect has also been observed in finite-sized tight-binding chains by Raghavan *et al.* [16]. (2) The condition for localization depends on the specific time dependence of the magnetic fields. In general, one can show that if the longitudinal field behavior is $B_z(t) = B_z\cos(\omega t)$ and the transverse field is $B_x(t) = B_x\cos(2n\omega t)$, localization occurs when $J_{2n}(b_z/\hbar\omega) = 0$. On the other hand, if $B_x(t) = B_x\sin[(2n+1)\omega t]$, we obtain localization when $J_{2n+1}(b_z/\hbar\omega) = 0$. (3) Curve (2) in Fig. 4(a) shows the system experiencing delocalization in steps. This happens because at every half-cycle of the longitudinal field zero crossing, the spin states become quasidegenerate permitting rapid tunneling of population induced by the transverse field. A simple Landau-Zener-type analysis [17] shows that the amount of probability transfer (or spin delocalization) will depend exponentially on the ratio $b_x^2/(\omega b_z)$ and that this transfer will be most pronounced when the energy levels of the spin states adiabatically cross. This also explains why we need to have $b_z \gg b_x$ in order to observe the DL effect—if this condition is not satisfied, at each zero crossing of the longitudinal field, substantial transfer to other spin states will take place and the discrimination of the Bessel root condition will be less effective.

V. CONCLUSIONS

In summary, we have studied in this paper the dynamical response of a spinor condensate under the influence of external magnetic fields. By manipulating the applied fields, a rich set of phenomena can be studied, and dynamical control of the collective spin states can be realized. We have shown that a relatively strong longitudinal field may populate a substantial amount of higher excitation modes. The interference among these modes drives the condensate into an equilibrium, a signature of stochastization. Also we have shown that the approach to equilibrium depends on the initial preparation of the system.

By applying a weak transverse field to a completely spin-polarized condensate, one can evolve the system into spin-non-mixing states as described by conditions (A) and (B). These states correspond to the eigenstates of the total Hamiltonian in the absence of magnetic fields. When the condensate is in such a state, the nonlinear atom-atom interaction does not affect the dynamics, and the condensate always occupies the lowest spatial mode. Furthermore, the condensate responds linearly to the applied fields such that its evolution can be described by a set of simple linear differential equations [Eqs. (4)]. With these interesting properties, a spinor condensate in such a state becomes very attractive for macroscopic coherent quantum control, a subject we are now studying using a fully quantum treatment.

Finally we have shown that dynamical localization in spin space can be realized using time-dependent magnetic fields. Dynamical localization in position space was first proposed in condensed matter systems such as tight-binding electrons moving in a lattice subject to a time-dependent electric field, but its experimental realization has been difficult. Our work shows that the spinor condensate provides an attractive system to study such a phenomenon.

In this study, we assume the radial wave function to be the ground state of the transverse trapping potential. Such an assumption is valid when $\hbar\omega_\perp$ is larger than the nonlinear interaction energy. For the parameters used in Figs. 1 and 2, these two quantities are comparable to each other. However, one can always increase the transverse trap frequency to make the assumption valid (changing $\omega_\perp$ will modify the value of parameter η). In future work, we plan to study the full dynamics of the three-dimensional spinor condensate, including the possibility of radial evolution.

ACKNOWLEDGMENTS

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