Nonlinear Landau-Zener tunneling

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A self-interacting two-level system depending on an external parameter is investigated. The most striking feature exhibited in this system is the presence of a nonzero tunneling probability in the adiabatic limit for large enough interaction strength. Possible experimental observation of this breakdown of adiabaticity using a Bose-Einstein condensate in an optical potential is suggested.

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Landau-Zener tunneling between energy levels is a basic process in quantum mechanics [1], and a vast amount of literature has been devoted to its application in various physical systems, such as current driven Josephson junctions [2], atoms in accelerating optical lattices [3], and field-driven superlattices [4].

In this paper, we study Landau-Zener transition in a nonlinear two-level system in which the level energies depend on the occupation of the levels, representing a mean-field-type of interaction between the particles. We show that the interactions tend to increase the tunneling probability, and that there exists a critical value of the interaction strength beyond which the transition probability becomes nonzero even in the adiabatic limit. As an application of this model, we consider the transition between Bloch bands for a Bose-Einstein condensate in an optical lattice [5–7], and work out the experimental conditions for the observation of the theoretical prediction. Other possible applications include the transition of a condensate in a double-well potential [8–10] and the motion of small polarons [11].

MODEL

The nonlinear two-level system is described by the dimensionless Schrödinger equation

$$i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = H(\gamma)\begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

with the Hamiltonian given by

$$H(\gamma)=\begin{pmatrix} \frac{\gamma}{2}+\frac{c}{2}(|b|^2-|a|^2) & \frac{v}{2} \\ \frac{v}{2} & -\frac{\gamma}{2}-\frac{c}{2}(|b|^2-|a|^2) \end{pmatrix}, \tag{2}$$

where γ is the level separation, v is the coupling constant between the two levels, and c is the nonlinear parameter describing the interaction. For convenience, we have left out the average of the diagonal elements because it does not affect the evolution of the probabilities. The total probability $|a|^2+|b|^2$ is conserved and is set to be 1. As in the linear Zener model, we will take v to be independent of time, and γ to change at a constant rate, i.e., $\gamma(t)=\alpha t$.

In the near adiabatic regime $\alpha \to 0$, the study of the Landau-Zener transition in the linear case is facilitated by solving for the eigenenergies and eigenstates of the Hamiltonian. The adiabatic levels are given by $\pm\frac{1}{2}(\gamma^2+v^2)^{1/2}$, which are plotted as dashed lines in Fig. 1. The nonadiabatic transition between these adiabatic levels is dominated by the branch point $t=iv/\alpha$ in the analytic dependence of these levels as functions of the time in its complex domain. The fact that the branch point is off the real axis leads to a transition probability vanishing exponentially in the adiabatic limit [1]:

$$r_0=\exp\left(-\frac{\pi v^2}{2\alpha}\right). \tag{3}$$

Similarly, we need to analyze the behavior of the adiabatic levels in the nonlinear model in order to understand the Landau-Zener transition for the new problem. We define these levels as the solution of the time-independent version

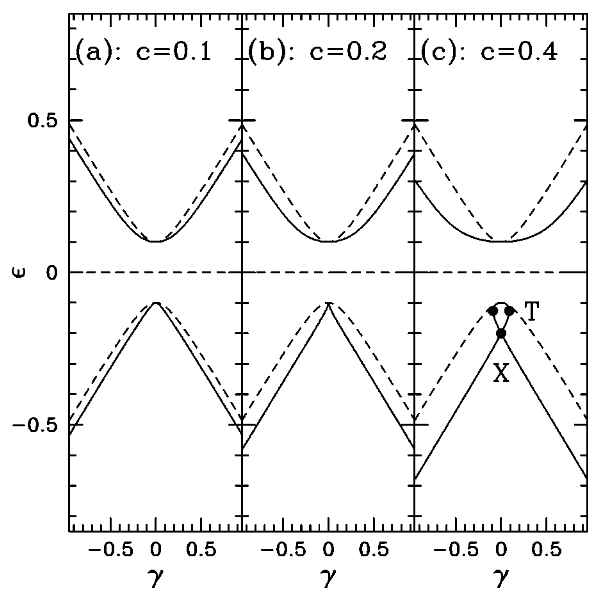

FIG. 1. Adiabatic energy levels for the linear case (dashed lines) and nonlinear cases (solid lines). The result is for $v=0.2$, and a proportional change of the energy scales of the two axes should be made for other values of v.

of Eq. (1) obtained by replacing $i(\partial/\partial t)$ with the energy ϵ. After some elaboration, we find the following quartic equation:

$$x^4+2gx^3+(g^2-h^2-1)x^2-2gx-g^2=0, \tag{4}$$

where $x=2\epsilon/v$, $g=c/v$, and $h=\gamma/v$. This quartic equation has two real roots when $c\leqslant v$ $(g\leqslant 1)$, while it can have four real roots when $c>v$ $(g>1)$. All the real solutions must lie outside the energy range $(-v/2,v/2)$. In Fig. 1(a), where $c<v$, we see that the energy levels are lower than the ones for $c=0$, except at $\gamma=0$, where they touch. The upper level becomes flatter at the tip while the lower level becomes sharper. In Fig. 1(b), where $c=v$, a quite drastic change appears with the lower tip becoming a sharp angle, a phenomenon related to the critical point of self-trapping in the context of small polaron theory [11]. In Fig. 1(c), where $c>v$, we have an even more dramatic change: a loop appears at the tip of the lower level, which reflects the hysteresis accompanying a phase transition.

The looping new feature in the adiabatic level brings up a very interesting phenomenon, the breakdown of adiabatic evolution even in the adiabatic limit. Suppose we start with a state on the lower branch of the adiabatic level in Fig. 1(c), and move it up along the branch by changing γ so slowly that little tunneling to the upper level is generated. After passing the crossing point *X*, the state remains in the course moving up in energy until hitting the terminal point *T*, where there is no way to go any further except to jump to the upper and lower levels. Because jumping to the lower level is as discontinuous as it is to the upper one from the terminal point, we expect that there will be a nonzero probability for going into the upper level (and remain in that level afterwards).

This breakdown of adiabatic evolution is confirmed by numerical calculation of tunneling probability directly from Eq. (1). The numerical results of the probability for different interaction strengths c are shown in Fig. 2 for $v=0.2$. The small wiggles appearing in the curves are because the numerical simulation must start at some finite time instead of the ideal limit: $t=-\infty$. We see that the tunneling is increased overall because of the interaction. Most strikingly, the tunneling probability for $c=0.4>v$ is not zero in the adiabatic limit $\alpha\rightarrow 0$, while it goes to zero for $c=0.1<v$. At the critical point $c=v=0.2$, the transition probability seems to vanish with α with a nonzero slope. On the other hand, for subcritical values of the interaction, $c<v$, the tunneling probability still vanishes exponentially with α (Fig. 3). From the slopes of the curves we find that for small α, $r\sim\exp[-q(\pi v^2/2\alpha)]$, where $q=1$, 0.7, 0.46, and 0.32 for $c=0$, 0.03, 0.07, and 0.1, respectively. The dependence of the exponent on c can be qualitatively understood from the fact that, for $c<v$, the nearest branching point that connects to the lower level occurs at $t=\pm i[1-(c/v)^{2/3}]^{3/2}v/\alpha$, whose distance to the real axis of time vanishes as c approaches its critical value v from below.

To achieve further insight into the behavior of the tunneling probability in the nonlinear model, we have obtained the dynamical energy levels by calculating the energy expectation values of two states evolved from $(a,b)=(1,0)$ and $(a,b)=(0,1)$ using Eq. (1). The results are shown in Fig. 4 for the case of $\alpha=0.001$ and $v=0.2$. In Fig. 4(a), where $c=0.1<v$, we see the excellent match between the dynamical levels and the adiabatic levels. In Fig. 4(b), where $c=0.2$, there is a small deviation between the two kinds of levels after passing the tip at $\gamma=0$. Beyond the critical interaction, as shown for the case of $c=0.4>v$ in Fig. 4(c), there is still almost perfect matching between the levels for the entire upper branch and part of the lower branch, but the loop is completely ignored by the dyamical level and there is a vio-

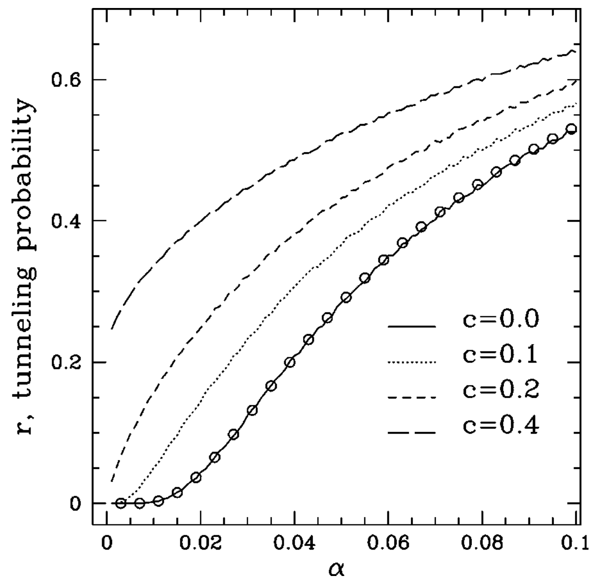

FIG. 2. Numerical results for the tunneling probability as a function of α for the linear and nonlinear cases with $v=0.2$. For comparison, we have included the result from Eq. (3) (open circles) for the linear case $c=0.0$.

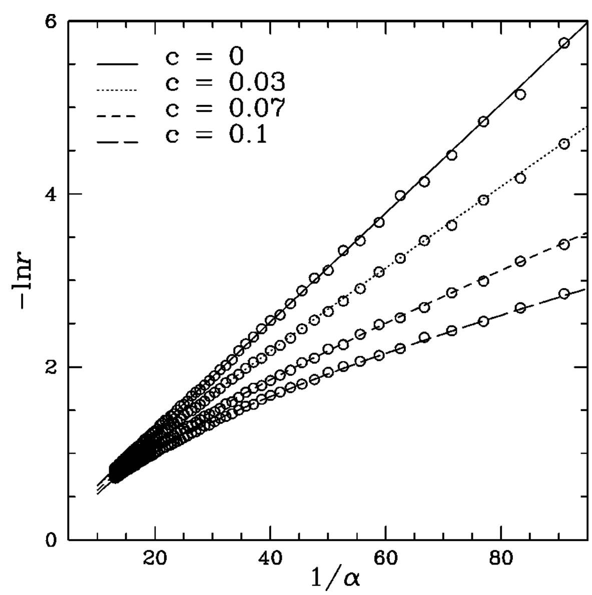

FIG. 3. Exponential dependence of the nonadiabatic transition probability on the speed of level crossing, α, for subcritical values of the interaction $c<v=0.2$.

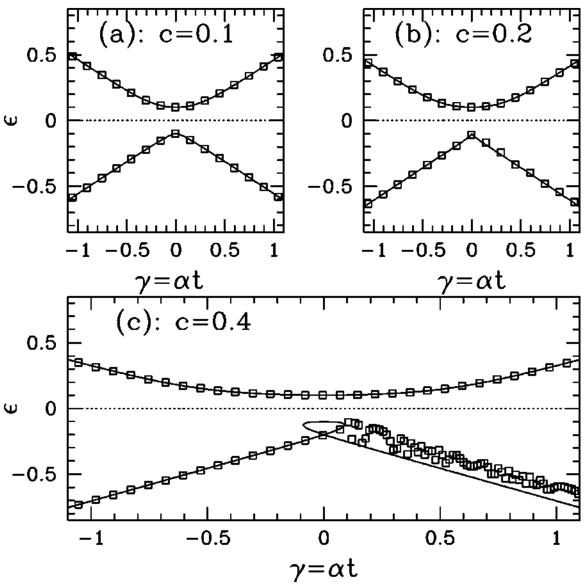

FIG. 4. Comparison of the dynamical levels (open squares) and the adiabatic levels (solid lines). The results are for $v=0.2$ and $\alpha=0.001$, and a proportional change of the energy scales of the two axes should be made for other values of v.

lent shaking above and about the lower branch of the adiabatic level after the terminal point. The shaking represents a quantum beating between the amplitude tunneled into the upper adiabatic level and that remaining in the lower adiabatic level.

APPLICATION

The nonlinear two-level model can be used to understand Landau-Zener tunneling of a Bose-Einstein condensate between Bloch bands in an optical lattice. In the low-density limit, where the interaction between the atoms can be neglected, the problem is essentially the same as that for a system of ultracold but noncondensate atoms [12,3], except with a much sharper initial condition. As we will show below, high enough densities of the atoms can be achieved such that the nonlinear effect discussed above should be readily detectable.

At sufficiently low temperatures, the motion of a Bose-Einstein condensate can be modeled by the one-dimensional nonlinear Schrödinger equation [6,12]

$$i\hbar\frac{\partial\phi}{\partial t}=-\frac{1}{2m}\left(\hbar\frac{\partial}{\partial x}-ima_l t\right)^2\phi+V_0\cos(2k_Lx)\phi+\frac{4\pi\hbar^2a_s}{m}|\phi|^2\phi, \tag{5}$$

where m is the mass of the atoms, k_L is the wave number of the laser light, a_s is the s-wave scattering length between atoms, and V_0 is the strength of the periodic potential that is proportional to the laser intensity. The absolute square of the wave function $\phi(x,t)$ is the number density of atoms at position x and time t. A force of ma_l is represented in the vector potential gauge, which may stand for either the inertial force in the comoving frame of an accelerating lattice or the gravity force. After the following change of variables

$$\tilde{x}=2k_Lx,\quad \tilde{\phi}=\frac{\phi}{\sqrt{n_0}},\quad \tilde{t}=\frac{4\hbar}{m}k_L^2t,$$

$$\tilde{v}=\frac{mV_0}{4\hbar^2k_L^2},\quad \tilde{\alpha}=\frac{m^2}{8\hbar^2k_L^3}a_l,\quad c=\frac{\pi n_0a_s}{k_L^2}, \tag{6}$$

where n_0 is the average density of the condensate, Eq. (5) is cast into a dimensionless form

$$i\frac{\partial\phi}{\partial t}=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}-i\alpha t\right)^2+v\cos(x)\phi+c|\phi|^2\phi. \tag{7}$$

We have replaced $\tilde{x}$ with x, etc, in the above equation without causing confusion. We assume that the nonlinear term does not break the periodic symmetry, so that the band structure remains. In the neighborhood of $k=1/2$, the Brillouin zone edge, the wave function can be approximated by

$$\phi(x,t)=a(t)e^{ikx}+b(t)e^{i(k-1)x}, \tag{8}$$

where $|a|^2+|b|^2=1$. Substituting this back into Eq. (7) and comparing the coefficients of e^{ikx} and $e^{i(k-1)x}$, we have

$$i\frac{\partial a}{\partial t}=\frac{1}{2}(k-\alpha t)^2a+\frac{v}{2}b+c(1+|b|^2)a,$$

$$i\frac{\partial b}{\partial t}=\frac{1}{2}(k-1-\alpha t)^2b+\frac{v}{2}a+c(1+|a|^2)b. \tag{9}$$

This equation is equivalent to Eq. (1) after the linearization of the quadratic kinetic terms around $k=1/2$ and dropping a constant energy of $c[1+(|a|^2+|b|^2)/2]$. Therefore, we see that the nonlinear two-level model does provide the basis for understanding the tunneling of the condensate in an optical lattice. To observe the breakdown of adiabaticity, we need the ratio

$$\frac{c}{v}=\frac{4\pi\hbar^2n_0a_s}{mV_0} \tag{10}$$

to be greater than unity. In typical experiments to date, we have $n_0=3\times10^{21}$ m^3, $a_s=2.75$ nm for sodium [13], which gives us $c/v=1.34$ for $V_0/h=34$ KHz, a result well within the experimental range [3,5].

Another system, a Bose-Einstein condensate in a double-well potential [8], is also very promising for observing the breakdown of adiabaticity. The mathematical model for this system [9] is very similar to our model described by Eq. (1). The only difference is that the external parameter γ should be replaced by the lowest energy levels in the two wells. Such a double well may be achieved by a laser sheet dividing a trap, and the energy levels may be moved by shifting the laser sheet. It must be noted, however, that one needs an attractive interaction between the atoms in this context in order to have a positive c in our model. For a repulsive interaction, the parameter c becomes negative, and the adiabatic spectrum is then inverted from that shown in Fig. 1. In order to see the adiabatic breakdown in this case, one has to trap the condensate initially in the well that is higher than the other, and lower this well relative to the other in time. The adiabatic state is then a metastable state, and one has the extra burden of distinguishing between the breakdown of adiabaticity and decay into the ground state [14] due to other degrees of freedom neglected in the nonlinear Schrödinger equation.

ACKNOWLEDGMENTS

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