Assisted cloning and orthogonal complementing of an unknown state

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We propose a protocol where one can exploit dual quantum and classical channels to achieve perfect ''cloning'' and ''orthogonal-complementing'' of an unknown state with minimal assistance from a state preparer (without revealing what the input state is). The first stage of the protocol requires usual teleportation, and in the second stage the preparer disentangles the leftover entangled states by a single-particle measurement process and communicates a number of classical bits (1-cbit per copy) to different parties so that perfect copies and complement copies are produced. Our protocol produces clones and complement clones of unknown qubit each with a probability $\frac{1}{2}$ and clones of real qubit with unit probability.

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Manipulation and extraction of quantum information are important tasks in the ongoing field of information theory. One of the interesting questions raised by Wootters and Zurek [1] and Dieks [2] is whether it is possible to copy a quantum state perfectly. It was found that linearity of quantum theory does not allow us to do so. Though exact cloning is not possible, in the literature various cloning machines have been proposed [3–9] which operate either in a deterministic or probabilistic way. Recently, we have proved that it is possible to build a novel cloning machine, which can produce linear superposition of multiple clones [10]. This would find a potential application in quantum information processing.

Recently, we found that it is not possible to produce a conjugate copy of an unknown state using either linearity or unitarity of the quantum process [11]. Independently, it was found that it is not possible to create an orthogonal-complement state of an unknown state [12]. In the case of a qubit, a complement state can be related to a conjugate state by a rotation operator. Hence, these problems are identical (up to a rotation operator). Also, in [13] it was proved that it is impossible to flip an arbitrary spin, as it involves antiunitary operations.

Although there has been an immense number of theoretical ideas about how well one can copy a quantum state, none of them seems to yield a perfect copy of the input state. Here, we mention that using the probabilistic cloning machine, one can produce perfect copies of linearly independent states with some probabilities [7]. The probabilities of success obey certain inequalities which depend on the inner product of the input states. However, in the present scheme the perfect copies are produced with a probability independent of the input state with unit fidelity (universal cloning). Moreover, there is no proof yet that by using the unitary and reduction operations one can produce an orthogonal complement copy of an unknown state. Also, there is no hope that one can test all the existing theoretical ideas experimentally where one can create a copy with some error.

The purpose of this paper is to investigate the possibility of copying and complementing an unknown state perfectly using resources such as entangled states, Bell-state measurement, single-particle von Neumann measurement, and classical communication. We go beyond the traditional cloning ideas that exist in the literature. The question we raise is, can we produce a perfect copy and complement copy of an unknown quantum state with ''minimal assistance'' (considered as an extra resource) from the state preparer (call him Victor). It turns out that with the help of Victor, our protocol can produce a perfect copy and a complement copy (anticlones). Ordinarily, if Victor sends his recipe to someone to prepare the state, this would require double infinity of bits of information [14] to be sent across a classical channel.

We show that instead of sending infinite bits of information, Victor can use the entangled state left after the teleportation process [15] and send *one classical bit* to Alice to create either a copy or an orthogonal-complement copy of the unknown state. As in the teleportation process, one qubit can be passed by sending two cbits and the remaining flows across the entanglement channel. Similarly here (in the second stage of protocol), the infinite amount of bits of information can be passed to a distant site by just sending one cbit and the remaining bits flow across the entanglement channel. This is a nontrivial observation in this context which must be remembered. This approach is important, where Victor is cut off (one way) from the rest of the world and he cannot send any quantum states but has a classical channel to send only cbits. An example could be as follows. Suppose Victor owned a private company (''qubit company'') which was producing qubits and sending them to interested parties. One day, his company crashes and he has no resources to start his company again nor has he any capital. Then, if the other interested parties can send him one-half of the particles from an entangled source, then they can benefit from getting a copy or a complement copy of an unknown state after receiving 1 cbit from Victor. This way, many parties can benefit from Victor, although he has lost his ''qubit company.'' But his knowledge and communication of 1 cbit per copy are quite helpful in such scenarios.

Suppose we have a pure input qubit state $|\Psi\rangle_1 \in \mathcal{H} = C^2$, represented as $|\Psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$, with α as real and β as a complex number, in general. Let Alice and Bob share one-half of the particles from an EPR source as in the quantum teleportation protocol [15]. The EPR state for the particles 2 and 3 is given by

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$$|\Psi^-\rangle_{23}=\frac{1}{\sqrt{2}}(|01\rangle_{23}-|10\rangle_{23}). \quad (1)$$

Alice is in possession of particle 2 and Bob is in possession of 3. The input state $|\Psi\rangle_1$ is unknown to both Alice and Bob. The combined state of the unknown state 1 and the EPR state 23 is

$$\begin{aligned}|\Psi\rangle_{123}&=|\Psi\rangle_1\otimes|\Psi^-\rangle_{23}\\&=-\frac{1}{2}[|\Psi^+\rangle_{12}(\sigma_z)|\Psi\rangle_3+|\Psi^-\rangle_{12}|\Psi\rangle_3\\&\quad+|\Phi^+\rangle_{12}(i\sigma_y)|\Psi\rangle_3+|\Phi^-\rangle_{12}(-\sigma_x)|\Psi\rangle_3],\end{aligned} \quad (2)$$

where the σ's are usual Pauli spin matrices. Once Alice performs a Bell-state measurement onto the two-particle states 1 and 2, and if the measurement outcome of Alice is $|\Psi^-\rangle_{12}$ (the probability of this outcome is only $\frac{1}{4}$), then the resulting three-particle state is given by

$$|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{123}=-\frac{1}{2}|\Psi^-\rangle_{12}\otimes|\Psi\rangle_3. \quad (3)$$

When Alice communicates two bits of classical information to Bob, then Bob knows that he has received the original state (or the state up to a rotation operator). Now can we create a copy of the original state at Alice's place? Since the teleportation process obeys the "no-cloning theorem," at first it might look impossible to have a copy at Alice's place. However, this impossibility becomes a possibility when we allow *one bit of classical information* from Victor to Alice.

Usually it is said that the Bell-state measurement destroys the particles 1 and 2, so there is no state available to Alice. The point to be noted is that, in general, the particles 1 and 2 need not be destroyed after the Bell-state measurement. It so happens that when one uses photons for Bell-state analysis as was done by Bouwmeester *et al.* [16], they are absorbed by the detectors and hence destroyed. For other particles this need not be so, because the projection postulate says that when we apply a Bell-basis projector onto the combined state $|\Psi\rangle_{123}$, indeed a Bell-state remains formally [see Eq. (3)]. Quantum theory does not say that the particles 1 and 2 need to be destroyed after a measurement. How to implement this in practice is another question. A possible way to do this is through Bell-state analysis using quantum-nondemolition measurement (QND) [17] in the case of photons. This would require a device which can perform photon number QND measurement at the single photon level. By sending particles 1 and 2 along with probe modes in a nonlinear Kerr medium, one may think of a QND measurement. This would be, indeed, a challenge for experimentalists in the future. For other entangled sources such as spin-$\frac{1}{2}$ particles or two-level atoms, one needs some suitable detector which can distinguish all four Bell states and allow them to propagate freely for further processing. Let there be particles 1 and 2 after Bell-state measurement, which are in a singlet state. Alice sends particle 1 to Victor and keeps particle 2 in her possession. Now Victor carries out another measurement on particle 1 by using a linear polarizer (in the case of a photon) or Stern-Gerlach apparatus (in the case of spin-$\frac{1}{2}$ particles) to measure the state in another basis $\{|x\rangle,|y\rangle\}$, where the new basis are given by

$$\begin{aligned}|0\rangle_1&=\alpha|x\rangle_1-\beta|y\rangle_1,\\|1\rangle_1&=\beta^*|x\rangle_1+\alpha|y\rangle_1.\end{aligned} \quad (4)$$

The normalization and orthogonality relation between these basis vectors are preserved under this transformation. Interestingly, we find that the basis $|x\rangle_1=|\Psi\rangle_1$ and the basis $|y\rangle_1=|\Psi_\perp\rangle_1$, where $|\Psi_\perp\rangle_1=(\alpha|1\rangle_1-\beta^*|0\rangle_1)$ is the orthogonal-complement state to $|\Psi\rangle_1$. However, we keep $|x\rangle_1,|y\rangle_1$ for Victor just to distinguish the fact that he knows the state. When we write $|\Psi\rangle$ and $|\Psi_\perp\rangle$ for other particles, we mean they are unknown to the parties concerned. Now writing the entangled state $|\Psi^-\rangle_{12}$ in the basis $|x\rangle_1,|y\rangle_1$ gives us

$$|\Psi^-\rangle_{12}=\frac{1}{\sqrt{2}}[|x\rangle_1(\alpha|1\rangle_2-\beta^*|0\rangle_2)-|y\rangle_1(\alpha|0\rangle_2+\beta|1\rangle_2)]. \quad (5)$$

If the outcome of Victor is $|y\rangle_1$, then he sends his measurement result (one bit of classical information) to Alice. Alice knows that her state of particle 2 has been found in the original state $(\alpha|0\rangle_2+\beta|1\rangle_2)$, which is just a copy.

More explicitly, the total state after a Bell-state measurement and a single-particle von-Neumann measurement is given by

$$|y\rangle_1\langle y|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{123}=-\frac{1}{2\sqrt{2}}|y\rangle_1\otimes|\Psi\rangle_2\otimes|\Psi\rangle_3. \quad (6)$$

If the outcome of Victor's measurement result is $|x\rangle_1$, then one cbit from Victor to Alice would yield a complement state given by

$$|x\rangle_1\langle x|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{123}=-\frac{1}{2\sqrt{2}}|x\rangle_1\otimes|\Psi_\perp\rangle_2\otimes|\Psi\rangle_3. \quad (7)$$

On the other hand, if the measurement outcome of Alice is other than $|\Psi^-\rangle_{12}$ in the first stage of the protocol, then the result of the second stage of the protocol can be worked out in detail. First, we note that although the singlet state is the same in any basis, the other Bell states are not the same. In the basis $\{|x\rangle_1,|y\rangle_1\}$ we can express the other three Bell states as

$$|\Psi^+\rangle_{12}=-\frac{1}{\sqrt{2}}[|x\rangle_1(\sigma_z)|\Psi_\perp\rangle_2+|y\rangle_1(\sigma_z)|\Psi\rangle_2],$$

$$|\Phi^+\rangle_{12}=\frac{1}{\sqrt{2}}[|x\rangle_1(i\sigma_y)|\Psi_\perp\rangle_2+|y\rangle_1(i\sigma_y)|\Psi\rangle_2], \quad (8)$$

$$|\Phi^-\rangle_{12}=\frac{1}{\sqrt{2}}[|x\rangle_1(\sigma_x)|\Psi_\perp\rangle_2+|y\rangle_1(\sigma_x)|\Psi\rangle_2].$$

When Alice's outcomes are $|\Psi^+\rangle_{12},|\Phi^\pm\rangle_{12}$ in the first stage of the protocol, then the resulting states can be calculated using Eqs. (2) and (8). Equations (6) and (7) are exact ones and one of the main results of this paper. The important observation is that if after the Bell-state measurement Victor finds $|y\rangle_1$ ($|x\rangle_1$) in a single-particle measurement, then one cbit from Victor to Alice will result in an exact copy (complement copy) or a copy up to a rotation operator (complement copy up to a rotation operator) at Alice's place. Interestingly, the rotation operators that Alice has to apply to get a copy are the same as those of Bob's case to get the original state.

In the special case, if the unknown state is real, i.e., $|\Psi\rangle=\cos\theta|0\rangle+\sin\theta|1\rangle$, then Alice just has to perform a rotation or do nothing after receiving the classical information from Victor. In both cases she gets a copy of the unknown state. This shows that our protocol produces clones of real qubits 100% of the time. For an arbitrary unknown state, our protocol produces an accurate copy of the original input state 50% of the time and an orthogonal-complement copy 50% of the time.

We generalize our protocol for producing more copies or complement copies using a multiparticle entangled state. At first it may seem that if we use three-particle entanglement we might be able to generate *one to three* copies at different sites with the help of Victor. But it turns out that with a three-particle entangled source one can again produce only one perfect copy or one complement copy. The useful resource for producing *one to three* copies or two copies and a complement copy is a four-particle entangled state of the type from Greenberger-Horne-Zeilinger (GHZ) [18] given by

$$|\Psi\rangle_{2345}=\frac{1}{\sqrt{2}}(|0011\rangle_{2345}+|1100\rangle_{2345}). \tag{9}$$

Here, Alice has particle 2, Bob has particles 3 and 4, and Carla has particle 5. Let Victor prepare a state which is unknown to Alice, Bob, and Carla and send it to Alice. Now the combined five-particle state is given by $|\Psi\rangle_1\otimes|\Psi\rangle_{2345}$. Let us express the basis of states of particles 1 and 2 in their respective Bell basis. Then the total state can be written as

$$\begin{aligned}|\Psi\rangle_{12345}=|\Psi\rangle_1\otimes|\Psi\rangle_{2345}=\frac{1}{2}[&|\Psi^+\rangle_{12}(\beta|011\rangle_{345}\\ &-\alpha|100\rangle_{345})-|\Psi^-\rangle_{12}(\beta|011\rangle_{345}+\alpha|100\rangle_{345})\\ &+|\Phi^+\rangle_{12}(\alpha|011\rangle_{345}-\beta|100\rangle_{345})\\ &+|\Phi^-\rangle_{12}(\alpha|011\rangle_{345}+\beta|100\rangle_{345})].\end{aligned} \tag{10}$$

Now Alice carries out a Bell-state measurement on particles 1 and 2. The two-particle projection would yield any one of four possible results. If the readout of Alice's measurement is $|\Psi^-\rangle_{12}$, then the resulting state will be

$$|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{12345}=-\frac{1}{2}|\Psi^-\rangle_{12}(\beta|011\rangle_{345}+\alpha|100\rangle_{345}). \tag{11}$$

Thus the Bell-state measurement leaves particles 1 and 2 in a maximally entangled state and particles 3, 4, and 5 in a three-particle entangled state. After the above measurement, Alice sends her results via a classical channel with two bits of information to both Bob and Carla. In the next step, Bob, who is in possession of particles 3 and 4, carries out another Bell-state measurement on them. Let us express the above state in terms of Bell states of 3 and 4. This is given by

$$\begin{aligned}|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{12345}=-\frac{1}{2}|\Psi^-\rangle_{12}[&|\Psi^+\rangle_{34}(\alpha|0\rangle_5+\beta|1\rangle_5)\\ &-|\Psi^-\rangle_{34}(\alpha|0\rangle_5-\beta|1\rangle_5)].\end{aligned} \tag{12}$$

After a Bell-state measurement onto particles 3 and 4 if the readout is (say) $|\Psi^+\rangle_{34}$, then the state of particle 5 is found to be in the original state. If the readout is $|\Psi^-\rangle_{34}$, then the state of particle 5 is merely the original state up to a rotation operator σ_z. Suppose Bob's measurement gives a result $|\Psi^+\rangle_{34}$. Then the classical information from Bob to Carla would yield the state

$$\begin{aligned}&|\Psi^+\rangle_{34}\langle\Psi^+|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{12345}\\ &\quad=-\frac{1}{2}|\Psi^-\rangle_{12}\otimes|\Psi^+\rangle_{34}\otimes|\Psi\rangle_5.\end{aligned} \tag{13}$$

In the second stage, Bob needs to send only one bit of classical information to Carla as he could get only two possible Bell-state measurement results. The resulting state for Carla is the teleported state of the original input state (this can be called *teleportation of an unknown state using a four-particle entangled state*, which has not been discussed in the literature). Recently, Karlsson and Bourennane [19] have discussed teleportation of an unknown state using a three-particle entangled state. After teleportation of the original state, the particles 1,2 and 3,4 are in a maximally entangled state. Now Alice and Bob send particle 1 and 3 to Victor one after the other. When Victor gets the particles (1 and 3), he chooses to measure the states in the basis $|x\rangle_i,|y\rangle_i$, ($i=1,3$), where these are given by $|0\rangle_i=\alpha|x\rangle_i-\beta|y\rangle_i$, $|1\rangle_i=\beta^*|x\rangle_i+\alpha|y\rangle_i$. In the new basis, the total state is given by

$$\begin{aligned}&|\Psi^+\rangle_{34}\langle\Psi^+|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{12345}\\ &\quad=\frac{1}{4}[|x\rangle_3(\alpha|1\rangle_4+\beta^*|0\rangle_4)-|y\rangle_3(\alpha|0\rangle_4-\beta|1\rangle_4)]\\ &\qquad\otimes[|x\rangle_1(\alpha|1\rangle_2-\beta^*|0\rangle_2)\\ &\qquad+|y\rangle_1(\alpha|0\rangle_2+\beta|1\rangle_2)]\otimes|\Psi\rangle_5.\end{aligned} \tag{14}$$

Suppose Victor first performs a von Neumann measurement on particle 1 and then on 3 and in both cases let the outcomes be $|y\rangle_1$ and $|y\rangle_3$. He can send the classical information (one cbit) to Alice and (one cbit) to Bob, who can in turn find their particles in the original state exactly or up to a rotation operator, respectively. Thus the final state after von Neumann measurements is given by

$$|y\rangle_1\langle y|y\rangle_3\langle y|\Psi^+\rangle_{34}\langle\Psi^+|\Psi^-\rangle_{12}\langle\Psi^-|\Psi\rangle_{12345}$$
$$=-\frac{1}{4}|y\rangle_1\otimes|\Psi\rangle_2\otimes|y\rangle_3\otimes(\sigma_z)|\Psi\rangle_4\otimes|\Psi\rangle_5. \quad (15)$$

It is clear from Eq. (15) that Alice, Bob, and Carla each acquire a perfect copy of the unknown state. If Victor's outcomes for particles 1 and 3 are $|x\rangle_1$ and $|x\rangle_3$ (after sending one bit to Alice and one to Bob), then Alice gets a complement copy, Bob gets a complement copy (up to a rotation operator), and Carla gets the original state. In general, when Victor finds both particles in the basis $|y\rangle$, then we have two copies with probability $\frac{1}{16}$, and when both particles are found in the basis $|x\rangle$, we have two complement copies with probability $\frac{1}{16}$. However, if Victor finds particle 1 in the basis $|x\rangle$ and 3 in $|y\rangle$ (or vice versa), then we have a copy and a complement copy (all are up to doing nothing or a rotation operation) with probability $\frac{1}{8}$. Thus the above protocol is able to produce two perfect copies or two complement copies or a copy and a complement copy. Since our protocol would work for all Bell-state outcomes, the probability of producing two clones (up to a rotation operator) is 25% and of producing two complement copies is 25% and one clone and one complement is 50%. The present results can be generalized to produce multiple copies and complement copies using a multiparticle entanglement source and multiparties. The useful resource to produce N copies or N complement copies (plus one original) is $n=2N$ particle entanglement source shared by $N+1$ parties. The entangled source has to be distributed in such a way that the first and the last person possess one particle each and all intermediate parties possess two particles from the entangled source.

We have proposed a protocol which can produce perfect copies and orthogonal-complement copies of an arbitrary unknown state with the help of dual quantum and classical channels and an extra resource of one cbit (per copy) from a state preparer. This protocol realizes perfect cloning and complementing of an unknown state, which works in a probabilistic manner. For a real unknown qubit our protocal works in a deterministic manner. We hope this will be a practical way of copying and complementing quantum states inside a quantum computer in the future. Also, the present work could have some application in quantum communication complexity [20] and quantum bit commitment protocols [21], which deserves further exploration. The present idea is useful in answering the question, what is the minimum number of classical bits required to simulate a quibit [22].

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