Classical interventions in quantum systems. I. The measuring process

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The measuring process is an external intervention in the dynamics of a quantum system. It involves a unitary interaction of that system with a measuring apparatus, a further interaction of both with an unknown environment causing decoherence, and then the deletion of a subsystem. This description of the measuring process is a substantial generalization of current models in quantum measurement theory. In particular, no ancilla is needed. The final result is represented by a completely positive map of the quantum state ρ (possibly with a change of the dimensions of ρ). A continuous limit of the above process leads to Lindblad's equation for the quantum-dynamical semigroup [Commun. Math. Phys. **48**, 119 (1976)].

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I. INTRODUCTION

The measuring process [1,2] is the interface of the classical and quantum worlds. The classical world has a description which may be probabilistic, but in a way that is compatible with Boolean logic. In the quantum world, probabilities result from complex amplitudes that interfere in a nonclassical way. In this paper, the notion of measurement is extended to a more general one: an *intervention*. An intervention has two consequences. One is the acquisition of information by means of an apparatus that produces a record. This step is called a *measurement*. Its outcome, which is in general unpredictable, is the *output* of the intervention. The other consequence is a change of the environment in which the quantum system will evolve after completion of the intervention. For example, the intervening apparatus may generate a new Hamiltonian that depends on the recorded result. In particular, classical signals may be emitted for controlling the execution of further interventions. In the second paper of this series [3], these signals will be limited to the velocity of light, so as to obtain a relativistic version of quantum measurement theory.

Interventions are mathematically represented by completely positive maps. Their properties are discussed in Sec. II, where a detailed dynamical description is given of the measuring process: it involves unitary interactions with a measuring apparatus and with an unknown environment that causes decoherence, and then the optional deletion of a subsystem. The Hilbert space for the resulting quantum system may have a different number of dimensions than the initial one. Thus a quantum system whose description starts in a given Hilbert space may evolve into a set of Hilbert spaces with different dimensions. If one insists on keeping the same Hilbert space for the description of the entire experiment, with all its possible outcomes, this can still be achieved by defining it as a Fock space.

The term ''detector'' will frequently appear in this paper and in the following one. It means an elementary detecting element, such as a bubble in a bubble chamber, or a small segment of wire in a wire chamber. Note that in such a detector, the time required for the irreversible act of amplification (the formation of a microscopic bubble, or the initial stage of the electric discharge) is extremely brief, typically of the order of an atomic radius divided by the velocity of light. Once irreversibility has set in, the rest of the amplification process is essentially classical. It is noteworthy that the time and space needed for initiating the irreversible processes are incomparably smaller than the macroscopic resolution of the detecting equipment.

An intervention is described by a set of parameters that include the time at which the intervention occurs. Interventions of finite duration can also be considered [4], and will be briefly discussed. For a relativistic treatment (in the following paper), we shall need the location of the intervention in space-time, referred to an arbitrary coordinate system. In any case, we have to specify the speed and orientation of the apparatus in the coordinate system that we are using and various other *input* parameters that control the apparatus, such as the strength of a magnetic field, or that of an rf pulse used in the experiment, and so on. The input parameters are determined by classical information received from past interventions, or they may be chosen arbitrarily by the observer who prepares that intervention, or by a local random device acting in lieu of the observer.

A crucial physical assumption is that there exists an objective time ordering of the various interventions in an experiment. There are no closed causal loops. This time ordering defines the notions earlier and later. The input parameters of an intervention are deterministic (or possibly stochastic) functions of the parameters of earlier interventions, but not of the stochastic outcomes resulting from later interventions. In such a presentation, there is no ''delayed choice paradox'' [5] (there can be a delayed choice, of course, but no paradox is associated with it).

The word ''measurement'' is a bit misleading, because it suggests that there exists in the real world some unknown property that we are measuring [6]. This term was banned by Bell [7], though for a different reason: Bell pointed out that the notion of measurement, or observation, was logically inconsistent in a world whose description is purely quantum mechanical. However, the approach followed in the present paper does not comply with Bell's desiderata. It explicitly associates classical inputs and outputs with each intervention [8,9].

The probabilities of the various outcomes of an interven-

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tion can be predicted by using a suitable theory, such as quantum theory. Besides these outcomes, there may also be other output parameters: there may be modifications of the physical environment depending on which outcome arose, and the intervening apparatus may emit classical signals with instructions for setting up later interventions. As a concrete example, consider the quantum teleportation scenario [10]. The first intervention is performed by Alice: she has two spin-$\frac{1}{2}$ particles and she performs on them a test with four possible outcomes. When Alice gets the answer, she emits a corresponding signal, which becomes an input for Bob's intervention: the latter is one of four unitary transformations that can be performed on Bob's particle.

Quantum mechanics is fundamentally statistical [11]. In the laboratory, any experiment has to be repeated many times in order to infer a law; in a theoretical discussion, we may imagine an infinite number of replicas of our *gedanken-experiment*, so as to have a genuine statistical ensemble. The various experiments that we consider all start in the same way, with the same initial state ρ_0, and the first intervention is the same. However, later stages of the experiment may involve different types of interventions, possibly with different space-time locations, depending on the outcomes of the preceding events. Yet, assuming that each intervention has only a finite number of outcomes, there is for the entire experiment only a finite number of possible records. (Here the word ''record'' means the complete list of outcomes that occurred during the experiment. I do not want to use the word ''history,'' which has acquired a different meaning in the writings of some quantum theorists.)

Each one of these records has a definite probability in the statistical ensemble. In the laboratory, experimenters can observe its relative frequency among all the records that were obtained; when the number of records tends to infinity, this relative frequency is expected to tend to the true probability. The role of theoretical physics is to predict the probability of each record, given the inputs of the various interventions (both the inputs that are actually controlled by the local experimenter and those determined by the outputs of earlier interventions). Note that each record is objective: everyone agrees on what happened (e.g., which detectors clicked). Therefore, everyone agrees on what the various relative frequencies are, and the theoretical probabilities are also the same for everyone.

The ''detector clicks'' are the only real thing we have to consider. Their observed relative frequencies are objective data. On the other hand, wave functions and operators are nothing more than abstract symbols. They are convenient mathematical concepts, useful for computing quantum probabilities, but they have no real existence in Nature [12]. Note also that while interventions are localized in space-time, quantum systems are pervasive. In each experiment, irrespective of its history, there is only one quantum system. The latter typically consists of several particles or other subsystems, some of which may be created or annihilated at the various interventions.

Section II describes the quantum dynamics of the measuring process which is an essential part of each intervention. The role of decoherence, due to an inavoidable interaction with an unknown environment, is discussed in Sec. III. The final result [Eq. (20)], will be extensively used in the following paper. The right-hand side of that equation contains operators $A_{\mu m}$ which are typically represented by rectangular matrices. Some of their mathematical properties (in particular factorability) are discussed in Sec. IV.

Decoherence, whose role is essential in the measuring process, is a stochastic phenomenon similar to Brownian motion. However, when seen on a coarse time scale, it is possible to consider it as a continuous process. This continuous approximation leads to the Lindblad equation [13], which is derived in a simple way in Sec. V.

II. MEASURING PROCESS

The measuring process involves several participants: the physical system under study, a measuring apparatus whose states belong to macroscopically distinguishable subspaces, and the ''environment'' which includes unspecified degrees of freedom of the apparatus and the rest of the world. These unknown degrees of freedom interact with the relevant ones, but they are not under the control of the experimenter and cannot be explicitly described. Our partial ignorance is not a sign of weakness. It is fundamental. If everything were known, acquisition of information would be a meaningless concept.

In order to keep the discussion as general as possible, here I do not introduce any ''ancilla,'' contrary to current fashion. This omission is not an oversight. It is intentional and deserves a brief explanation. In the early years of quantum mechanics, von Neumann wrote a rigorous mathematical treatise [1] which had a lasting influence. According to von Neumann, the various outcomes of a measurement correspond to a complete set of orthogonal projection operators in the Hilbert space of the quantum system under study. It was later realized that von Neumann's approach was too restricted, because the measuring process may have more distinct outcomes than the number of dimensions of that Hilbert space. The appropriate formalism is that of a *positive operator valued measure* (POVM) [14,15]. That is, the various outcomes of the measurement correspond to positive operators E_μ, which sum up to the unit operator but need not commute.

This raised a new problem: the actual implementation of a given POVM. In the final section of his book, von Neumann formally showed how to construct a Hamiltonian that generated a dynamical evolution of the type required to obtain a projection-valued measure (PVM). This was a mathematical proof of existence, namely, quantum dynamics was compatible with the structure of a PVM. Is it compatible with a more general POVM? This question was answered by Helstrom [16], who converted the problem of implementation of a POVM into that of an ordinary von Neumann measurement, by introducing an auxiliary quantum system that he called *ancilla* (the Latin word for housemaid). By virtue of Neumark's theorem [17], any POVM can be obtained from a PVM applied to a composite system that consists of the original system and an ancilla having a sufficient number of dimensions. This provides a formal proof of existence, but in

real life this is usually not how measurements are actually performed. Even if an ancilla is used according to Helstrom's protocol, we may as well consider it as part of the measuring apparatus. Therefore, the following description of the measuring process will not involve any ancilla, and yet it will explicitly show how any POVM can be implemented by a unitary interaction of the quantum system with a suitable apparatus.

To simplify the notations, it will be assumed that finite-dimensional Hilbert spaces are sufficient for describing the quantum system under study, the apparatus, and even the environment. Moreover, the initial states ρ_i of the system and the apparatus are assumed to be pure. Initially mixed states would be a more realistic assumption, but since they can always be written as convex combinations of pure ρ_i, their use would not bring any essential change in the discussion below.

Let a set of basis vectors for the system under study be denoted as $\{|s\rangle\}$. The initial state of that system is a linear combination, $|\psi_0\rangle=\Sigma c_s|s\rangle$, with complex coefficients c_s. Let $|A\rangle$ be the initial state of the apparatus. In the first step of the measuring process, which may be called a ''premeasurement'' [18], the apparatus interacts unitarily with the quantum system and becomes entangled with it, so that they effectively become a single composite system $\mathcal{C}$:

$$\sum_s c_s|s\rangle\otimes|A\rangle\rightarrow\sum_{s,\lambda} c_s U_{s\lambda}|\lambda\rangle, \tag{1}$$

where $\{|\lambda\rangle\}$ is a complete basis for the states of $\mathcal{C}$. It is the choice of $U_{s\lambda}$ that determines which property of the system under study is correlated to the apparatus, and therefore is measured. Unitarity implies that

$$\sum_\lambda U_{s\lambda}U^*_{t\lambda}=\delta_{st}, \tag{2}$$

or $UU^\dagger=\mathbf{1}$, where $\mathbf{1}$ is the unit matrix in the Hilbert space of the original quantum system. The U matrix is not square: it has fewer rows than columns, because we have considered only a single initial state of the apparatus, namely, $|A\rangle$. If we had introduced a complete set of states for the apparatus, labeled $|B\rangle$, $|C\rangle$, etc., then U would have been a unitary matrix satisfying

$$\sum_\lambda U_{sA,\lambda}U^*_{tB,\lambda}=\delta_{st}\delta_{AB}. \tag{3}$$

Our freedom of choosing the required unitary matrix $U_{sA,\lambda}$ is equivalent to the freedom of choice of an interaction Hamiltonian in the von Neumann formalism.

The apparatus itself is an utterly complicated system, and some radical assumptions are needed in order to proceed with explicit calculations. The assumptions below are not as drastic as those commonly used in quantum measurement theory, yet they ought to be clearly spelled out. Let us assume that the composite system $\mathcal{C}$ can be fully described by the theory. Its complete description involves both ''macroscopic'' variables and ''microscopic'' ones. The difference between them is that the microscopic degrees of freedom can be considered as adequately isolated from the environment for the duration of the experiment, so that their evolution is in principle perfectly controlled, while the macroscopic ones cannot be isolated from the unknown environment and the dynamical evolution cannot be completely predicted. Statistical hypotheses are required in order to make plausible predictions, as explained below. Any other degrees of freedom of the apparatus, for which no explicit description is provided, are considered as part of the environment.

An essential property of the composite system $\mathcal{C}$, which is necessary to produce a meaningful measurement, is that its states form a finite number of orthogonal subspaces which are distinguishable by the observer. These subspaces are similar to, but more general than, Zurek's ''pointer basis'' [19] which is a preferred basis for the apparatus. Here we consider orthogonal *subspaces* of the composite system $\mathcal{C}$, which may have different numbers of dimensions. For example, a particle detector may have just two such subspaces: ''ready to fire'' and ''discharged'' (obviously, the latter has many more states than the former). Each macroscopically distinguishable subspace corresponds to one of the outcomes of the intervention and defines a POVM element E_μ, given explicitly by Eq. (7) below. The labels μ are completely arbitrary; for example they may be the labels printed on the various detectors. The initial state of $\mathcal{C}$, namely, $|\psi_0\rangle\otimes|A\rangle$, lies in the subspace that corresponds to the null outcome (no detection).

Obviously, the number of different outcomes μ is far smaller than the dimensionality of the composite system $\mathcal{C}$. Let us introduce a complete basis $\{|\mu,\xi\rangle\}$, where μ labels a macroscopic subspace, as explained above, and ξ labels microscopic states in that subspace. We thus have

$$\langle\mu,\xi|\nu,\eta\rangle=\delta_{\mu\nu}\delta_{\xi\eta}. \tag{4}$$

Note that the various subspaces labeled μ may have different dimensions, that is, the range of indices ξ may depend on the corresponding μ. We shall henceforth write $U_{s\mu\xi}$ instead of $U_{s\lambda}$.

After the premeasurement, given by Eq. (1), the state of $\mathcal{C}$ is given by

$$|\psi_1\rangle=\sum_{s,\mu,\xi} c_s U_{s\mu\xi}|\mu,\xi\rangle. \tag{5}$$

The probability of obtaining outcome μ is the contribution of subspace μ to the density matrix $|\psi_1\rangle\langle\psi_1|$. Explicitly, it is

$$\sum_{s,t,\xi} c_s c_t^* U_{s\mu\xi}U^*_{t\mu\xi}=\sum_{s,t}\rho_{st}(E_\mu)_{ts}, \tag{6}$$

where

$$(E_\mu)_{ts}=\sum_\xi U_{s\mu\xi}U^*_{t\mu\xi} \tag{7}$$

is a POVM element, defined in the Hilbert space of the system under study whose initial state was ρ_{st}. Note that the matrices E_μ satisfy

$$\sum_\mu (E_\mu)_{ts} = \sum_{\mu,\xi} U_{s\mu\xi} U^*_{t\mu\xi} = \delta_{ts}, \tag{8}$$

by virtue of the unitarity property in Eq. (2).

III. DECOHERENCE

Up to now, the quantum evolution has been well defined and in principle reversible. It would remain so if the macroscopic degrees of freedom of the apparatus could be perfectly isolated from their environment, and in particular from the ''irrelevant'' degrees of freedom of the apparatus itself. This demand is of course self-contradictory, since we have to read the result of the measurement if we wish to make any use of it.

Let $\{|e_\alpha\rangle\}$ denote a complete basis for the states of the environment, and let $|e_\omega\rangle$ be the state of the environment at the moment of the premeasurement. That state is of course unknown, but I temporarily assume that it is pure, and moreover that it is one of the basis states, in order to simplify the notations. This pure initial state will later be replaced by a density matrix

$$\rho_e = \sum_\omega p_\omega |e_\omega\rangle\langle e_\omega|, \tag{9}$$

with unknown random non-negative coefficients p_ω. There is no loss of generality in assuming that ρ_e is diagonal in the basis used for the states of the environment. This merely means that this basis was chosen in the appropriate way.

Recall that states $|\mu,\xi\rangle$ with different μ are macroscopically different, so that they interact with different environments. On the other hand, the labels ξ refer to microscopic degrees of freedom that are well protected from parasitic disturbances. This sharp dichotomous distinction between the two types of degrees of freedom is the only approximation that was made until now.

The unitary interaction of $\mathcal{C}$ with the environment thus generates an evolution which is not under the control of the experimenter:

$$|\mu,\xi\rangle \otimes |e_\omega\rangle \to |\mu,\xi\rangle \otimes \sum_\alpha b_{\mu\omega\alpha} |e_\alpha\rangle. \tag{10}$$

The coefficients $b_{\mu\omega\alpha}$ are unknown except for normalization,

$$\sum_\alpha |b_{\mu\omega\alpha}|^2 = 1; \tag{11}$$

they have no subscript ξ because the microscopic degrees of freedom do not interact with the environment, and they cannot mix different values of μ because the latter refer to macroscopic outcomes that are stable on the time scale of the experiment. One could also consider a more general evolution, where the right-hand side of Eq. (10) would involve different subspaces μ. This would mean that the measuring apparatus is actually disturbed by the environment. Such a process is called *noise*, and is essentially different from the phenomenon of *decoherence*, whose occurence is explained below. Here it is assumed that no noise affects the measuring process. It is only the environment, whose microscopic degrees of freedom are not robust, that is disturbed by the apparatus (this is the mechanism causing decoherence). Exactly how it is disturbed cannot be known; however, we do know that macroscopically different states of $\mathcal{C}$ lead to different disturbances of the environment, and hence to the appearance of an index μ in the coefficients $b_{\mu\omega\alpha}$.

The final state with all the participating subsystems is thus

$$|\psi_2\rangle = \sum_{s,\mu,\xi,\alpha} c_s U_{m\mu\xi} b_{\mu\omega\alpha} |\mu,\xi\rangle \otimes |e_\alpha\rangle. \tag{12}$$

The final density matrix (still a pure state, for the sake of simple notations) is $\rho = |\psi_2\rangle\langle\psi_2|$. Explicitly, we see from Eq. (12) that the expression for ρ contains, among other things, operators $|e_\alpha\rangle\langle e_\beta|$ which refer to states of the environment. They are unknown and are considered unknowable. The only operator acting on these states that we know to write is represented by the unit matrix $\delta_{\alpha\beta}$. Its meaning, in the laboratory, is that of complete ignorance. Therefore we can effectively replace the complete density matrix ρ by the reduced matrix obtained from it by ignoring the inaccessible degrees of freedom of the environment. That is, we replace the operators $|e_\alpha\rangle\langle e_\beta|$ which appear in ρ by $\langle e_\beta | e_\alpha\rangle = \delta_{\alpha\beta}$, and we perform a partial trace on the indices that refer to the environment.

The reduced density matrix thus contains expressions of the type $\Sigma_\alpha b_{\mu\omega\alpha} b^*_{\nu\omega\alpha}$. It will now be shown that, after a reasonably short time has elapsed, we have

$$\sum_\alpha b_{\mu\omega\alpha} b^*_{\nu\omega\alpha} \simeq \delta_{\mu\nu}. \tag{13}$$

The case $\mu = \nu$ is the normalization condition (11) due to unitarity. When $\mu \neq \nu$, the rationale for arguing that the left-hand side of Eq. (13) is very close to zero (and has a random phase) is that the environment has a huge number of states, say N, whose dynamics is chaotic. Therefore the scalar product of any two states such as $\Sigma b_{\mu\omega\alpha} |e_\alpha\rangle$ and $\Sigma b_{\nu\omega\beta} |e_\beta\rangle$ that may result from Eq. (10), at any random time, is of the order of $N^{-1/2}$, because the components of a random state, in a randomly chosen basis, are of the order of $N^{-1/2}$. The time that has to elapse to make Eq. (13) a good approximation is called the *decoherence time*, and it depends on how well the macroscopic degrees of freedom of the measuring apparatus are isolated from the environment. There may of course be large fluctuations on the left-hand side of Eq. (13), akin to Poincaré recurrences [20], but this expression is *almost always* very close to zero if $\mu \neq \nu$.

The approximation becomes even better if instead of an ideal pure initial state $|e_\omega\rangle$ for the environment, we take the more realistic density matrix given by Eq. (9). Instead of Eq. (13) we then have

$$\sum_{\omega} p_{\omega} \sum_{\alpha} b_{\mu\omega\alpha} b^{*}_{\nu\omega\alpha} \simeq \delta_{\mu\nu}, \tag{14}$$

where the off-diagonal terms on the right-hand side are now of order N^{-1}, rather than $N^{-1/2}$ as before. It is plausible that the above argument can be made mathematically rigorous in the thermodynamic limit $N \to \infty$.

It follows that states of the environment that are correlated to subspaces of $\mathcal{C}$ with different labels μ can safely be treated in our calculations as if they were exactly orthogonal. The resulting theoretical predictions will almost always be correct, and if any rare small deviation from them is ever observed, it will be considered as a statistical quirk, or an experimental error. The reduced density matrix thus is block-diagonal, and all our statistical predictions are identical to those obtained for an ordinary mixture of (unnormalized) pure states

$$|\psi_{\mu}\rangle = \sum_{s,\xi} c_s U_{s\mu\xi} |\mu, \xi\rangle, \tag{15}$$

where the statistical weight of each state is the square of its norm. This mixture replaces the pure state $|\psi_1\rangle$ in Eq. (5). This is the meaning of the term decoherence. From this moment on, the macroscopic degrees of freedom of $\mathcal{C}$ have entered into the classical domain [21,22]. We can safely observe them and ''lay on them our grubby hands'' [23]. In particular, they can be used to trigger amplification mechanisms (the so-called detector clicks) for the convenience of the experimenter.

Note that all these properties still hold if the measurement outcome happens to be the one labeled $\mu = 0$ (that is, if there is no detector click). It does not matter whether this is due to an imperfection of the detector or to a probability less than 1 that a perfect detector would be excited. The state of the quantum system does not remain unchanged. It has to change to respect unitarity. The mere presence of a detector that could have been excited implies that there has been an interaction between that detector and the quantum system. Even if the detector has a finite probability of remaining in its initial state, the quantum system correlated to the latter acquires a different state [24]. The absence of a click, when there could have been one, is also an event and is part of the historical record.

The final (optional) step of the intervention is to discard part of the composite system $\mathcal{C}$. In the case of a von Neumann measurement, the subsystem that is discarded and thereafter ignored is the measuring apparatus itself. In general, it is a different subsystem: the discarded part may depend on the outcome μ and in particular its dimensions may depend on μ. The remaining quantum system then also has different dimensions. In the subspace μ we therefore introduce two sets of basis vectors $|\mu, \sigma\rangle$ and $|\mu, m\rangle$ for the new system and the part that is discarded, respectively. They replace the original basis $|\mu, \xi\rangle$, and it is convenient to choose the latter in such a way that for each ξ we can write

$$|\mu, \xi\rangle = |\mu, \sigma\rangle \otimes |\mu, m\rangle \tag{16}$$

as a direct product, rather than a linear superposition of such products.

We are now ready to discard the subsystem whose basis vectors are denoted as $|\mu, m\rangle$. In the unnormalized density matrix $\rho_{\mu} = |\psi_{\mu}\rangle\langle\psi_{\mu}|$ (whose trace is the probability of observing outcome μ), we ignore the deleted subsystem. That is, we replace the operator $|\mu, m\rangle\langle\mu, n|$ that appears in ρ_{μ} by a unit matrix δ_{mn} and we perform a partial trace on the indices m and n, as we have done when we discarded the states of the environment. We thus obtain a reduced density matrix

$$\rho'_{\mu} = \sum_{s,t} c_s c^{*}_t \sum_{m,\sigma,\tau} U_{s\mu\sigma m} U^{*}_{t\mu\tau m} |\mu, \sigma\rangle\langle\mu, \tau|. \tag{17}$$

Its elements $\langle\mu, \sigma|\rho'_{\mu}|\mu, \tau\rangle$ can be written as

$$(\rho'_{\mu})_{\sigma\tau} = \sum_{m} \sum_{s,t} (A_{\mu m})_{\sigma s} \rho_{st} (A^{*}_{\mu m})_{\tau t}, \tag{18}$$

where $\rho_{st} \equiv c_s c^{*}_t$, and the notation

$$(A_{\mu m})_{\sigma s} \equiv U_{s\mu\sigma m} \tag{19}$$

was introduced for later convenience. Recall that the indices s and σ refer to the original system under study and to the final one, respectively. Omitting these indices, Eq. (18) takes the familiar form

$$\rho'_{\mu} = \sum_{m} A_{\mu m} \rho A^{\dagger}_{\mu m}, \tag{20}$$

which is the most general completely positive linear map [25]. This is sometimes written as $\rho'_{\mu} = S\rho$, where S is a linear *superoperator* which acts on density matrices (while an ordinary operator acts on quantum states). Note, however, that these superoperators have a very special structure, given by Eq. (20).

Clearly, the ''quantum jump'' $\rho \to \rho'_{\mu}$ is not a dynamical process that occurs in the quantum system by itself. It results from the introduction of an apparatus, followed by its deletion or that of another subsystem. In the quantum folklore, an important role is played by the ''irreversible act of amplification.'' The latter is irrelevant to the present issue. The amplification is solely needed to help the experimenter. A jump in the quantum state occurs even when there is no detector click or other macroscopic amplification, because we impose abrupt changes in our way of delimiting the object we consider as the quantum system under study. The precise location of the intervention, which is important in a relativistic discussion [3], is the point from which classical information is sent that may affect the input of other interventions. More precisely, it is the earliest space-time point from which classical information could have been sent. This is also true for interventions that gave no detection event. Such a passive intervention is located where the detection event would have occurred, if there had been one.

Is it possible to maintain a strict quantum formalism and treat the intervening apparatus as a quantum-mechanical sys-

tem, without ever converting it to a classical description? We could then imagine not only sets of apparatuses spread throughout space-time, but also truly delocalized apparatuses [26], akin to Schrödinger cats [27,28], so that interventions would not be localized in space-time as required in the present formalism. However, such a process would only be the creation of a correlation between two nonlocal quantum systems. This would not be a true measurement but rather a ''premeasurement'' [18]. A valid measuring apparatus must admit a classical description equivalent to its quantum description [22], and in particular it must have a positive Wigner function. Therefore a delocalized apparatus is a contradiction in terms. If a nonlocal system is used for the measurement, it must be described by quantum mechanics (no classical description is possible), and then it has to be measured by a valid apparatus that behaves quasiclassically and in particular is localized. It can indeed be localized as well as we wish, if it is massive enough.

Likewise, quantum measurements of finite duration, as discussed by Peres and Wootters [4], actually are only premeasurements. To obtain consistent results, these authors had to explicitly introduce a second apparatus that suddenly measures the first one. Their first apparatus has no classical description. In the language that I am using now, only the second apparatus performs a valid measurement.

In a purely quantum description of the apparatus, which is the one appropriate at the premeasurement stage, the new state is an incoherent mixture of various ρ_μ correlated to distinct outcomes μ of the apparatus. However, the description of the apparatus must ultimately be converted into a classical one [21,22] if we want it to yield a definite record. On the other hand, it is also possible to discard the apparatus *without* recording its result. We then have to describe the state of the quantum system by a mixture of mixtures, as in Eq. (37) below. The term ''compound'' [29] has been proposed for that kind of mixture which is solely due to our ignorance of the actual outcome and has no objective nature. Once we have a definite outcome μ, the new state is ρ_μ, given by Eq. (20).

IV. KRAUS MATRICES

A special case of Eq. (20) for square matrices $A_{\mu m}$ was obtained by Kraus [30], who sought to find the most general completely positive map for the density matrices of a given quantum system (no change of dimensions was allowed). Kraus's result obviously is a generalization of von Neumann's prescription for the state resulting from the μth outcome of a measurement, namely $\rho_\mu = P_\mu \rho P_\mu$, where P_μ is the projection operator associated with outcome μ. Recall that, even if the initial ρ is normalized to unit trace (as we always assume), the trace of ρ_μ in the above equations is not equal to 1. Rather, it is the probability of occurrence of outcome μ. It is quite convenient to keep ρ_μ unnormalized, with the above interpretation for its trace.

The results obtained here are more general than those of Kraus, because the matrices $A_{\mu m}$ may be rectangular. As Eq. (19) shows, these matrices are simply related to the unitary transformation $U_{s\mu\sigma m}$ that generates the premeasurement [31]. Superoperators that do not conserve the number of dimensions of the density matrix were also considered by other authors [32]. The present treatment is even more general, because it allows the number of rows in $A_{\mu m}$ (that is, the order of ρ'_μ) to depend on μ, since we may decide to discard different subsystems according to the outcome of the measurement.

From Eq. (19), which relates the matrix elements $(A_{\mu m})_{\sigma s}$ to the unitary transformation involved in the quantum intervention, it appears that if we multiply the order of ρ'_μ by the range of the indices m in $A_{\mu m}$, the product of these two numbers is the same for all μ, since it is equal to the number of dimensions of the composite system $\mathcal{C}$, namely, the original quantum system together with the measuring apparatus. However, the situation is more complicated, because it may happen that $A_{\mu m}=0$ for some values of μm. Moreover, if the matrices $A_{\mu m}$ and $A_{\mu n}$ are proportional to each other for some m and n, these matrices can be combined into a single one; and conversely, any $A_{\mu m}$ can be split into several which are proportional to each other. Therefore there is no simple rule saying how many terms appear in the sum in Eq. (20).

The probability of occurrence of outcome μ in a measurement is given by Eq. (6), and it can now be written as

$$p_\mu = \sum_m \mathrm{Tr}\,(A_{\mu m}\rho A_{\mu m}^\dagger) = \mathrm{Tr}\,(E_\mu \rho). \tag{21}$$

The positive (that is, non-negative) operators

$$E_\mu = \sum_m A_{\mu m}^\dagger A_{\mu m}, \tag{22}$$

whose dimensions are the same as those of the initial ρ, are elements of a POVM and satisfy $\Sigma_\mu E_\mu = \mathbf{1}$. Note that null outcomes (i.e., no detection) have to be included in that sum. They indeed are the most probable result in typical experimental setups. Yet, even if no detector is excited, the intervention may affect the quantum system [24], and the corresponding $A_{\mu m}$ are not trivial. There may even be several distinct $A_{\mu m}$ for ''no detection,'' depending on the cause of the failure.

Conversely, given E_μ (a non-negative square matrix of order k) it is always possible to split it in infinitely many ways as in Eq. (22). This is easily proved by taking a basis in which E_μ is diagonal. All the elements are non-negative, so that by taking their square roots we obtain a matrix $\sqrt{E_\mu}$ that satisfies the relation required for $A_{\mu m}$. Next let $\{S_{\mu m}\}$ be a set of complex rectangular matrices with k columns and any number of rows, satisfying $\Sigma_m S_{\mu m}^\dagger S_{\mu m} = \mathbf{1}$. It follows that $A_{\mu m} = S_{\mu m}\sqrt{E_\mu}$ satisfies Eq. (22).

Moreover, if a POVM is factorable, namely,

$$E_{\mu\nu} = E_\mu^{(1)} \otimes E_\nu^{(2)}, \tag{23}$$

where the indices (1) and (2) refer to two distinct subsystems, then the above construction provides factorable Kraus matrices:

$$A_{\mu\nu mn} = A_{\mu m}^{(1)} \otimes A_{\nu n}^{(2)}. \tag{24}$$

The operator sum in Eqs. (20) and (22) now becomes double sums, over the indices *m* and *n*. Such double sums are indeed needed. If we had simply written, instead of Eq. (23), $E_\mu = E_\mu^{(1)} \otimes E_\mu^{(2)}$, the corresponding Kraus matrices would in general not be factorable. Such a POVM, with a single index, is called *separable*, and it cannot in general be implemented by separate operations on the two subsystems with classical communication between them [32].

The factorization of a POVM as in Eq. (23) is not the most general one. It corresponds to two POVM's independently chosen by the two observers. However, the observers may also follow an adaptive strategy. After the first one (conventionally called Alice) executes the POVM $\{E_\mu^{(1)}\}$, she informs the second observer (Bob) of the result, μ say, and then Bob uses a POVM adapted to that result. This will be denoted as $\{E_{\nu\mu}^{(2)}\}$, with

$$\sum_\nu E_{\nu\mu}^{(2)} = \mathbf{1}^{(2)}, \quad \forall \ \mu. \tag{25}$$

Note that the chronological order of the Greek indices indicating the outcomes of consecutive measurements is from right to left, just as the order for consecutive applications of a product of linear operators. We then have, instead of Eq. (23),

$$E_{\nu\mu} = E_\mu^{(1)} \otimes E_{\nu\mu}^{(2)}, \tag{26}$$

and thus

$$A_{\nu\mu nm} = A_{\mu m}^{(1)} \otimes A_{\nu\mu n}^{(2)}. \tag{27}$$

A more complicated situation arises when the *same* system or subsytem is subjected to consecutive interventions that depend on the outcomes of preceding interventions. We have, subsequent to the map in Eq. (20),

$$\rho'_\mu \to \rho''_{\nu\mu} = \sum_n B_{\nu\mu n} \rho'_\mu B_{\nu\mu n}^\dagger. \tag{28}$$

The result of these two consecutive maps can be written

$$\rho \to \rho''_{\nu\mu} = \sum_{n,m} C_{\nu\mu nm} \rho C_{\nu\mu nm}^\dagger, \tag{29}$$

where

$$C_{\nu\mu nm} = B_{\nu\mu n} A_{\mu m}. \tag{30}$$

It follows that

$$E_{\nu\mu} = \sum_{n,m} C_{\nu\mu nm}^\dagger C_{\nu\mu nm} = \sum_m A_{\mu m}^\dagger \left(\sum_n B_{\nu\mu n}^\dagger B_{\nu\mu n} \right) A_{\mu m}. \tag{31}$$

There is no simple relationship between this expression and the antecedent POVM element E_μ, unless the Kraus matrices $B_{\nu\mu n}$ are chosen in such a way that

$$\sum_{\nu,n} B_{\nu\mu n}^\dagger B_{\nu\mu n} = \mathbf{1}. \tag{32}$$

We then have

$$\sum_\nu E_{\nu\mu} = E_\mu. \tag{33}$$

This splitting of E_μ into a sum of several parts is called a POVM *refinement* [33]. It may be repeated many times, until all the final POVM elements are matrices of rank 1. For example, this can be done by an apparatus that includes a multidimensional ancilla which is *not* discarded at intermediate stages but only after the completion of the measuring procedure. This is why it is convenient to consider that ancilla explicitly rather than as part of the measuring apparatus. In the resulting Neumark extension (that is, the joint Hilbert space of the quantum system and the ancilla), each POVM is implemented as an ordinary von Neumann PVM [16,17]. Initially, the latter is coarse grained: it distinguishes only multidimensional subspaces of the system and the ancilla. This PVM is then gradually ''refined'' by the observer who uses further PVM's to select smaller and smaller orthogonal subspaces. How these subspaces are explicitly defined depends on results obtained in preceding tests. This is an adaptive strategy which is particularly efficient for the optimal identification of an unknown bipartite quantum state [33]. The two observers take turns in performing local measurements and informing each other of the results they obtained. The final result again has the form of Eq. (26), but now the label μ stands for the entire sequence of intermediate outcomes that were obtained by the two observers, and the label ν indicates the result of the very last intervention.

For example, if each observer performs just two tests, with consecutive results μ (Alice), ν (Bob), σ (Alice), and τ (Bob), then Eq. (27) becomes

$$A_{\tau\sigma\nu\mu tsnm} = A_{\nu\mu\sigma s}^{(1)} A_{\mu m}^{(1)} \otimes A_{\tau\sigma\nu\mu t}^{(2)} A_{\nu\mu n}^{(2)}. \tag{34}$$

This relationship is valid for any two pairs of consecutive tests, not only for those of the ''refinement'' type.

Returning to the case of a single observer, let a complete set of POVM elements be given. It is then always possible to construct from their Kraus matrices (in infinitely many ways) unitary transformations that satisfy Eq. (2). Indeed, writing all indices explicitly, the relation $\Sigma_\mu E_\mu = \mathbf{1}$ becomes

$$\sum_{\mu,m,\sigma} (A_{\mu m})_{\sigma s}^* (A_{\mu m})_{\sigma t} = \delta_{st}. \tag{35}$$

This can be written, owing to Eq. (19), as

$$\sum_{\mu,m,\sigma} U_{s\mu\sigma m}^* U_{t\mu\sigma m} = \delta_{st}, \tag{36}$$

which is the same as Eq. (2) with the index λ (which refers to the composite system $\mathcal{C}$) replaced by the composite index $s\mu\sigma$. This explicitly shows how any POVM can be implemented by a unitary transformation and a suitable apparatus having the necessary dimensionality (usually much smaller

than the one required by the introduction of an ancilla).

Recall that the $A_{\mu m}$ matrices are in general rectangular. For example, in the teleportation scenario [10], Alice can discard her two quantum particles together with her measuring apparatus after she performs her four-way test. The Hilbert space of Alice's subsystem then becomes trivial: it has only one dimension (only one state). In that case, $A_{\mu m}$ has a single row, $A_{\mu m}^{\dagger}$ a single column, and ρ_{μ} is just a number, namely, the probability of obtaining outcome μ. Likewise, each stage of a quantum distillation [34,35] causes a reduction of the number of dimensions of the quantum system that is distilled. That system initially consists of a set of entangled subsystems. Successive interventions select suitable subsets that have higher degrees of entanglement. Ideally, the final result should be arbitrarily close to a pair of spin-$\frac{1}{2}$ particles in a singlet state.

Some authors consider only square matrices $A_{\mu m}$, and then it is mathematically permitted to sum all ρ_{μ} so as to obtain the average state of all the outgoing quantum systems. For example, if the outcomes of our interventions are not recorded, so that no subensembles are selected, we may write

$$\rho' = \sum_{\mu,m} A_{\mu m} \rho A_{\mu m}^{\dagger}, \tag{37}$$

where the trace of ρ' is 1, of course. Such sums are rarely needed in theoretical discussions. Different labels μ correspond to different world histories (that is, different samples in the ensemble of experiments). Summing over them is like saying that peas and peanuts contain on the average 42% of water, instead of saying that peas have 78% and peanuts 6% [36]. Still, there are some cases where this kind of averaging is justified. For example, when we compare the expected yields of various distillation methods [34,35], we are interested only in average results. Moreover, when the quantum system weakly interacts with an unknown environment (such as a heat bath), rather than with an apparatus that can neatly distinguish different outcomes of the intervention, the result is a continuous decoherence of the quantum state [37]. This issue is discussed quantitatively in Sec. V.

V. LINDBLAD EQUATION

For a complete treatment of the quantum system in its unknown environment, we write the Hamiltonian as

$$H = H_0 + H_{\rm env} + H_{\rm int}, \tag{38}$$

with obvious notations. The last two terms in H generate a stochastic, rapidly fluctuating motion. The exact evolution, taking everything into account, is a Brownian motion (a kind of random walk) superimposed on the ideal motion. Assume that the unperturbed evolution due to H_0 is very slow on the scale of $t_{\rm decoh}$, the time needed for Eq. (13) to be valid. One can thus write (combining the composite index μm into a single index j)

$$A_j = S_j + F_j, \tag{39}$$

where the matrices S_j correspond to the slow motion generated by H_0, and the matrices F_j to the fast fluctuations due to the environment. This neat decomposition into slow and fast variables involves of course some arbitrariness that will be reflected in the derivation of the Lindblad equation below.

It follows from Eq. (39) that $A_j \rho A_j^{\dagger}$ splits into three kinds of terms. Those terms quadratic in S_j represent the smooth evolution due to H_0. If we wish to write a differential equation for $d\rho/dt$, the other terms have to be smoothed out on a time scale much longer than $t_{\rm decoh}$. Since this is a random walk, the terms linear in F_j average out to zero on that time scale, and the terms that are quadratic in F_j grow linearly in time. Clearly, this smoothing out and the resulting linear growth involve some approximations whose validity have to be ascertained on a case by case basis.

Let us thus assume that there is a coarse time scale $\delta t \gg t_{\rm decoh}$, long enough so that the fluctuations are averaged out, and yet short enough so that the slow evolution due to H_0 is negligible beyond first order in δt. We can then write $F_j \simeq V_j \sqrt{\delta t}$, where the matrices V_j are finite. The result is the Lindblad equation [13]

$$d\rho/dt = i[\rho, H_0] + \sum_j \left(V_j \rho V_j^{\dagger} - \tfrac{1}{2} \rho V_j^{\dagger} V_j - \tfrac{1}{2} V_j^{\dagger} V_j \rho \right). \tag{40}$$

This equation is of course valid only in the future time direction ($dt > 0$), because the smoothing out of fluctuations entails an irreversible loss of information. Lindblad's original derivation used an abstract argument involving complete positivity and a semigroup structure (again $dt > 0$). An equivalent argument was given independently by Gorini *et al.* [38]. The present proof is based on an explicit dynamical model of interaction, and it may be easier to understand. A similar derivation was also obtained by Schumacher [39] and, after the present paper was submitted for publication, a more detailed discussion, also based on the Kraus formalism, was published by Bacon *et al.* [40]. Still more recently, Adler sharpened the above heuristic derivation of Eq. (40) by using Itô's stochastic calculus [41].

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