Quantum approach to coupling classical and quantum dynamics

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We present a consistent framework of coupled classical and quantum dynamics. Our result allows us to overcome severe limitations of previous phenomenological approaches, such as evolutions that do not preserve the positivity of quantum states or that allow one to activate quantum nonlocality for superluminal signaling. A ''hybrid'' quantum-classical density is introduced, and its evolution equation derived. The implications and applications of our result are numerous: it incorporates the back-reaction of quantum on classical variables, and it resolves fundamental problems encountered in standard mean-field theories, and remarkably, also in the quantum measurement process; i.e., the most controversial example of quantum-classical interaction is consistently described within our approach, leading to a theory of dynamical collapse.

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Opinions vary about the coexistence of and interaction between classical and quantum systems. In orthodox quantum theory, classical macrosystems and quantized microsystems coexist; their interaction is described asymmetrically. The influence of macrovariables upon microsystems is precisely taken into account as external forces. The *back-reaction* of quantized microsystems upon classical macrosystems is largely ignored, except for *detector variables* which are typically sensitive to certain microvariables. The theory of this specific back-reaction, called measurement theory [1], predicts the statistics of the final states after the interaction. However, the interpretation of quantized dynamics is exclusively based on this nondynamical model of back-reaction (cf. collapse of the quantum state); without it we could not test the validity of quantized dynamics at all. Possibly, quantization extends to macrosystems, indeed the criteria of being macroscopic or microscopic are loosely if ever defined. Contrary to quantized microvariables, quantized macrovariables may have significant back-reactions on generic classical macrovariables as well. This becomes apparent in the widely used mean-field approximation [2] which, however, has several fundamental drawbacks [3], as we shall recall in this paper. The measurement theory also describes back-reaction, but only for idealized detectors. In some attempts to define quantum gravity, matter and some fields are quantized, while other fields (gravity in particular) are treated as classical, thus requiring a definite *hybrid*—i.e., a coupled classical and quantum—dynamics. Thus a general model of the back-reaction is desirable. Such a theory would describe the ''collapse'' of the wave function dynamically [4], would replace mean-field approximations in a systematic way [5–7], and could have deep implications for quantum cosmology. The first conceptual attempts [4–6] were followed by ups and downs [8], until severe limitations were clarified [9]. In this paper we use a straightforward transformation of the problem which automatically leads to a consistent model of hybrid dynamics.

The difficulties to overcome in our paper are unrelated to the high complexity of the emblematic mean-field equation of quantum cosmology. Actually, they lie in foundational principles. To illustrate these difficulties, we assume a quantum Pauli spin $\hat{\sigma}$ interacting with a classical harmonic oscillator of Hamiltonian $H_{\mathcal{C}}(x,p)=\frac{1}{2}(p^2+x^2)$. The spin interacts with the ''magnetic'' field of the oscillator via the Hamiltonian

$$\hat{H}_I=\kappa\hat{\sigma}_3 p. \tag{1}$$

Regarding $\hat{H}=H_{\mathcal{C}}+\hat{H}_I$ as the total Hamiltonian, it is natural to prescribe the Heisenberg equation of motion $\partial_t\hat{\sigma}=i[\hat{H},\hat{\sigma}]=i\kappa p[\hat{\sigma}_3,\hat{\sigma}]$ to the spin. The classical oscillator momentum satisfies the Hamilton equation $\partial_t p=-\partial_x\hat{H}=-x$, but the coordinate cannot satisfy $\partial_t x=\partial_p\hat{H}=p+\kappa\hat{\sigma}_3$ since x, being a real number, should not evolve into a matrix. The obvious way out is to replace the operator $\hat{\sigma}_3$ by its quantum expectation value, i.e., to apply some mean-field approximation. Yet, if taken literally, this implies that quantum expectations can be deduced with arbitrary precision from a measurement of the classical variables x and p. Hence the message is that the classical oscillator should, in some way, inherit quantum fluctuations from the spin. It comes to one's mind that x and p should be random variables, but not arbitrary ones. As we shall demonstrate, mathematical consistency imposes that the classical variables x or p must never take sharp values: The consistent theory assumes an unremovable coarse-graining [10,11].

The mathematical issue is the following. The phase-space density $\rho_{\mathcal{C}}(x,p)$ of a classical canonical system $\mathcal{C}$ satisfies the Liouville equation of motion

$$\partial_t\rho_{\mathcal{C}}(x,p)=\{H_{\mathcal{C}}(x,p),\rho_{\mathcal{C}}(x,p)\}_P, \tag{2}$$

where $H_{\mathcal{C}}(x,p)$ is the Hamilton function and $\{f,g\}_P=\partial_x f\partial_p g-\partial_p f\partial_x g$ stand for the Poisson bracket. On the other hand, the density operator $\hat{\rho}_{\mathcal{Q}}$ of a (canonical, or maybe discrete) quantum system $\mathcal{Q}$ evolves according to the von Neumann equation

$$\partial_t \hat{\rho}_{\mathcal{Q}} = -i[\hat{H}_{\mathcal{Q}}, \hat{\rho}_{\mathcal{Q}}], \tag{3}$$

where $\hat{H}_{\mathcal{Q}}$ is the Hamilton operator. To introduce interaction between $\mathcal{Q}$ and $\mathcal{C}$, we assume a *hybrid* Hamiltonian in the form

$$\hat{H}(x,p) = \hat{H}_{\mathcal{Q}} + H_{\mathcal{C}}(x,p) + \hat{H}_I(x,p), \tag{4}$$

where the interaction term (and thus the total Hamiltonian too) is a Hermitian operator for $\mathcal{Q}$, depending on the phase-space coordinates of $\mathcal{C}$.

The mean-field approach assumes sharp classical coordinates x_t, and p_t at each time, and the current quantum expectation value $H_{MF}(x,p;t) = tr[\hat{H}(x,p)\hat{\rho}_{\mathcal{Q}}(t)]$ of the Hamiltonian is regarded as the effective Hamilton function for the classical subsystem $\mathcal{C}$. The coupled evolution equations then take this forms:

$$\partial_t \hat{\rho}_{\mathcal{Q}}(t) = -i[\hat{H}(x_t, p_t), \hat{\rho}_{\mathcal{Q}}(t)], \tag{5}$$

$$\partial_t x_t = \partial_p H_{MF}(x_t, p_t), \quad \partial_t p_t = -\partial_x H_{MF}(x_t, p_t). \tag{6}$$

This approach has well-known deficiencies. In particular, it gives no account of the indeterminacy of the classical states x and p inherited from the quantum uncertainties of $\hat{\rho}_{\mathcal{Q}}$. There is thus an essential nonlinearity in the mean-field von Neumann equation (5) which leads to fundamental conflicts with principles of locality [12]. Furthermore, the mean-field effective Hamilton function $H_{MF}(x,p)$ will never be the proper representative of the interaction when quantum uncertainties in $\hat{\rho}_{\mathcal{Q}}$ are large.

A promising *conceptual* approach [5] uses the *hybrid density* $\hat{\rho}(x,p)$ to represent the state of the composite system $\mathcal{C} \times \mathcal{Q}$. If the subsystems are uncorrelated, then the hybrid density simply factorizes as $\rho_{\mathcal{C}}(x,p)\hat{\rho}_{\mathcal{Q}}$. In the general case, the hybrid density $\hat{\rho}(x,p)$ should be an (x,p)-dependent nonnegative operator, satisfying an overall normalization condition $tr\int \hat{\rho}(x,p)dxdp = 1$. One interprets the marginal distribution $\rho_{\mathcal{C}}(x,p) \equiv tr\hat{\rho}(x,p)$ as the phase-space density of the classical subsystem $\mathcal{C}$, while the density operator $\hat{\rho}_{\mathcal{Q}} \equiv \int \hat{\rho}(x,p)dxdp$ represents the unconditional state of the quantum subsystem $\mathcal{Q}$. *Conditional quantum states* are natural to introduce at fixed canonical coordinates (x,p) of the classical subsystem [10]:

$$\hat{\rho}_{xp} \equiv \hat{\rho}(x,p)/\rho_C(x,p). \tag{7}$$

Aleksandrov [5] proposed the following evolution equation for the hybrid density:

$$\partial_t \hat{\rho}(x,p) = -i[\hat{H}(x,p), \hat{\rho}(x,p)] + \frac{1}{2}\{\hat{H}(x,p), \hat{\rho}(x,p)\}_P - \frac{1}{2}\{\hat{\rho}(x,p), \hat{H}(x,p)\}_P. \tag{8}$$

If $\hat{H}_I(x,p) = 0$, then this equation splits into the standard equations (2) and (3).

Let us test Aleksandrov's equation on the spin-oscillator system (1). The generic form of the hybrid state is

$$\hat{\rho}(x,p) = \tfrac{1}{2}[1 + \vec{s}(x,p)\hat{\vec{\sigma}}]\rho_{\mathcal{C}}(x,p), \quad |\vec{s}| \leq 1, \tag{9}$$

where $\vec{s}(x,p)$ is the spin vector correlated with the oscillator's state. We read off the conditional quantum state (7) of the spin: $\hat{\rho}_{xp} = \frac{1}{2}[1 + \vec{s}(x,p)\hat{\vec{\sigma}}]$. For the hybrid state (9) and interaction Hamiltonian (1), Aleksandrov's equation (8) reads

$$\partial_t \hat{\rho} = (x\partial_p - p\partial_x)\hat{\rho} - i\kappa p[\hat{\sigma}_3, \hat{\rho}] - \tfrac{1}{2}\kappa[\hat{\sigma}_3, \partial_x \hat{\rho}]_+. \tag{10}$$

This equation easily violates the positivity condition $|\vec{s}| \leq 1$ on $\hat{\rho}$. That is, the initial normalized polarization vector

$$\vec{s} = \frac{1}{x^2 + p^2 + 1}(2x, 2p, x^2 + p^2 - 1), \tag{11}$$

leads to $|\vec{s}| > 1$ for all $x > 0$ under the evolution [Eq. (10)]. So, the naive Eq. (8) is inconsistent since it does not guarantee the positivity of the hybrid density $\hat{\rho}(x,p)$ [6]. One sees that the mathematical textures of the classical ($\mathcal{C}$) and quantum ($\mathcal{Q}$) systems, though well understood separately, are not at all trivial to couple.

A royal road offers itself nonetheless. Let us quantize canonically $\mathcal{C}$ as well. We do so temporarily and, at the end of the day, we regard it classical again. The hybrid Hamiltonian (4) transforms into the total Hamilton operator of the fully quantized system $\mathcal{C} \otimes \mathcal{Q}$

$$:\hat{H}(\hat{x},\hat{p}) := \hat{H}_{\mathcal{Q}} + :H_{\mathcal{C}}(\hat{x},\hat{p}): + :\hat{H}_I(\hat{x},\hat{p}):, \tag{12}$$

where : ... : stand for normal ordering in terms of the usual annihilation and creation operators $(\hat{x} \pm i\hat{p})/\sqrt{2}$, respectively. Let the equation of motion for the total system's density operator $\hat{\rho}$ be the standard von Neumann one:

$$\partial_t \hat{\rho} = -i[:\hat{H}(\hat{x},\hat{p}):, \hat{\rho}]. \tag{13}$$

Our royal road is based on coherent states [13]. Coherent states $|x,p\rangle$ are eigenstates of the annihilation operator:

$$(\hat{x} + i\hat{p})|x,p\rangle = (x + ip)|x,p\rangle. \tag{14}$$

Using Bargmann's convention [13], the coherent states satisfy the following differential relation:

$$(\partial_x - i\partial_p)|x,p\rangle = (\hat{x} - i\hat{p})|x,p\rangle, \tag{15}$$

$$(\partial_x + i\partial_p)|x,p\rangle = 0. \tag{16}$$

The latter relation expresses the fact that the bras $|x,p\rangle$ are entire analytic functions of the complex canonical variable $x + ip$, a crucial fact as we shall see. The coherent states form an overcomplete basis, with normalization

$$I=\int |x,p\rangle\langle x,p|\frac{\exp(-\frac{1}{2}(x^2+p^2))}{2\pi}dxdp. \tag{17}$$

It follows from Eqs. (14) and (15) that

$$:f(\hat{x},\hat{p}):|x,p\rangle=:f\left(\frac{x+ip+\partial_x-i\partial_p}{2},\frac{p-ix+\partial_p+i\partial_x}{2}\right):|x,p\rangle \tag{18}$$

for an arbitrary normal ordered function of the quantized variables on the left-hand side. On the right-hand side, the same symbols : . . . : mean that all derivations must be done first.

We apply a projection to the density operator $\hat{\rho}$ of the fully quantized system $\mathcal{C}\otimes\mathcal{Q}$, and thus reintroduce the *hybrid density*

$$\hat{\rho}(x,p)\equiv\mathrm{tr}[(|x,p\rangle\langle x,p|\otimes I)\hat{\rho}]\frac{\exp[-\frac{1}{2}(x^2+p^2)]}{2\pi}. \tag{19}$$

Indeed, this can formally be considered the hybrid density of the composite system $\mathcal{C}\times\mathcal{Q}$ as if $\mathcal{C}$ were unquantized (i.e., classical) again. This is what we are going to do. We can thus *derive* the closed equation of motion of the hybrid density (19) from the exact von Neumann equation (13). Using the basic relation (18) and the identity (16), we obtain the desired evolution equation

$$\partial_t\hat{\rho}(x,p)=-i:\hat{H}\left(x+\frac{\partial_x+i\partial_p}{2},p+\frac{\partial_p-i\partial_x}{2}\right):\hat{\rho}(x,p)+\mathrm{H.c.} \tag{20}$$

This *hybrid dynamic equation* is our proposal to couple classical systems to quantum ones canonically. Note that the hybrid density $\hat{\rho}(x,p)$ incorporates our statistical knowledge of $\mathcal{Q}$'s quantum state, of $\mathcal{C}$'s classical state, and of their correlations. If $\hat{H}_I(x,p)=0$, then by integrating both sides of Eq. (20) over x and p we obtain the standard von Neumann equation (3), as it should be. The trace of both sides, however, does *not* lead to the standard classical dynamics [Eq. (2)]. Instead, we obtain (for obvious reasons) the evolution equation of a Husimi function [14]. To lowest order in the derivatives, however, this is the classical Liouville equation (2). Hence classical dynamics is recovered if both the Hamilton function and the state distribution change slowly with x and p (see, e.g., Ref. [15] and references therein).

Consistency of the hybrid equation of motion (20) is, contrary to the case of the naive Eq. (8), assured by construction. It preserves the positivity of the hybrid density $\hat{\rho}(x,p)$ along with a certain analyticity property. In fact, projection (19) always leads to hybrid densities of the form

$$\hat{\rho}(x,p)=\frac{\exp[-\frac{1}{2}(x^2+p^2)]}{2\pi}\sum_n \varphi_n(x-ip)\varphi_n^\dagger(x+ip), \tag{21}$$

where $\varphi_n(x-ip)$ are unnormalized nonorthogonal state vectors for $\mathcal{Q}$, being complex entire functions of the combinations $x-ip$ of $\mathcal{C}$'s classical variables. This positive form is then preserved by our hybrid equation (20). The analyticity condition (21) restricts the possible hybrid states: sharp values of x and p and, generally, characteristic phase-space dependences inside single Planck cells, are excluded.

In particular, the ''pure state'' form of Eq. (21), i.e. with a single $\varphi\varphi^\dagger$ dyadic term, is also preserved. The hybrid Schrödinger equation of the hybrid state vector $\varphi(x-ip)$ follows from Eq. (20) [16]. For completeness, we mention that the polarization (11) can be reproduced by $\varphi(x-ip)=[(x-ip)|+\rangle+|-\rangle]/\sqrt{3}$, where $\hat{\sigma}_3|\pm\rangle=\pm|\pm\rangle$.

A post-mean-field approximation, incorporating some of the fluctuations of system $\mathcal{Q}$ into its back-reaction on the system $\mathcal{C}$, is worth deriving. Consider the first-order expansion of Eq. (20) in the derivatives of $\hat{H}(x,p)$:

$$\begin{aligned}\partial_t\hat{\rho}(x,p)=&-i[\hat{H}(x,p),\hat{\rho}(x,p)]+\frac{1}{2}\{\hat{H}(x,p),\hat{\rho}(x,p)\}_P\\&-\frac{1}{2}\{\hat{\rho}(x,p),\hat{H}(x,p)\}_P\\&-\frac{i}{2}[\partial_x\hat{H}(x,p),\partial_x\hat{\rho}(x,p)]\\&-\frac{i}{2}[\partial_p\hat{H}(x,p),\partial_p\hat{\rho}(x,p)].\end{aligned} \tag{22}$$

This equation has two additional terms with respect to (mathematically inconsistent) Eq. (8). These two additional terms reduce the domain of inconsistency. More importantly, it can be shown that the above equation is equivalent to the exact Eq. (20) if $\mathcal{C}$ is harmonic and its coupling to $\mathcal{Q}$ is linear in x and p. In particular, Eq. (22) gives a mathematically consistent theory of fully quantized atomic systems ($\mathcal{Q}$) interacting with the fully developed *classical* radiation field ($\mathcal{C}$). Moreover, its physics is equivalent to the true fully quantized radiation theory [11].

By taking the trace of Eq. (22) one can show that the evolution of the classical states is a flow:

$$\partial_t x=\langle\partial_p\hat{H}(x,p)\rangle_{xp}, \quad \partial_t p=-\langle\partial_x\hat{H}(x,p)\rangle_{xp}, \tag{23}$$

where $\langle\cdots\rangle_{xp}$ stands for the expectation values in the current conditional quantum state $\hat{\rho}_{xp}$. Obviously this flow resembles locally the naive mean-field equation (6). Here, however, the classical state is randomly distributed: it inherits the quantum fluctuations of $\mathcal{Q}$. On the other hand, the deterministic evolution of the state's distribution is a remarkable fact with significant consequences being discussed elsewhere.

Dynamical collapse of the quantum state is encoded in our hybrid evolution equation (20). It is a most spectacular feature. The ''presence'' of standard collapse will be demonstrated on the spin-oscillator model. For the hybrid state [Eq. (9)], our evolution equation (20) reads

$$\partial_t\hat{\rho}=(x\partial_p-p\partial_x)\hat{\rho}-i\kappa p[\hat{\sigma}_3,\hat{\rho}]-\frac{1}{2}\kappa[\hat{\sigma}_3,\partial_x\hat{\rho}]_+-\frac{i}{2}\kappa[\hat{\sigma}_3,\partial_p\hat{\rho}], \tag{24}$$

which differs from the naive Eq. (10) by the presence of the fourth term on the right-hand side. This term guarantees the positivity of the hybrid state $\hat{\rho}(x,p)$ for all times. The oscillator plays the role of the Stern-Gerlach apparatus detecting the quantized spin-component $\hat{\sigma}_3$. The pointer variable of the apparatus is x, it is set initially to zero with precision $\Delta=1$. Accordingly, we choose $\rho_C^0(x,p)=(2\pi)^{-1}\exp(-\frac{1}{2}(x^2+p^2))$ for the oscillator's initial state. We shall switch on the interaction Hamiltonian at $t=0$ for a short time τ compared to the oscillator period, keeping the effective coupling as strong as $g=\kappa\tau\gg 1$. Actually, we replace κ by $g\delta(t)$ [1]. Let us assume that the spin's initial state is the superposition $|\psi\rangle=c_+|+\rangle+c_-|-\rangle$, and the initial hybrid state (9) is the uncorrelated $|\psi\rangle\langle\psi|\rho_C^0(x,p)$. Immediately after the interaction the hybrid state $\hat{\rho}(x,p)$ becomes

$$|c_+|^2|+\rangle\langle+|\rho_C^0(x+g,p)+|c_-|^2|-\rangle\langle-|\rho_C^0(x-g,p)+c_+c_-^\star|+\rangle\langle-|\exp\left(-\frac{1}{2}g^2-igp\right)\rho_C^0(x,p)+\text{H.c.} \tag{25}$$

The trace yields the pointer's state distribution

$$\rho_C(x,p)=|c_+|^2\rho_C^0(x-g,p)+|c_-|^2\rho_C^0(x+g,p). \tag{26}$$

Since $g\gg 1$, we shall ignore the overlap between the two terms on the right-hand side, so we can say that the pointer x has swung out to $g\pm\Delta$ or to $-(g\pm\Delta)$ with the probabilities predicted by the standard measurement theory. Invoking Eq. (7), we can easily read out the conditional quantum state of the spin from Eq. (25) [since the off-diagonal terms are damped by $\exp(-\frac{1}{2}g^2)$, we ignore them]:

$$\hat{\rho}_{xp}=|\pm\rangle\langle\pm|, \quad x\approx\pm g, \quad p\approx 0. \tag{27}$$

This shows the standard collapse of the spin's quantum state: the quantum state is correlated with the classical pointer's position.

Discussing this paper's results, we repeat that we are aware of the ambiguous contemporary views concerning the concept of genuine hybrid systems. Nevertheless, the old Copenhagen interpretation as well as recent quantum gravity and quantum cosmological models assume such hybrid systems. We made the necessary compromises to neutralize strict no-go theorems. Our Eq. (20) is an example of hybrid dynamics which is both mathematically consistent and physically relevant. The applications of our equation are numerous, for example as phenomenological models whenever the mean-field approximation is poor. Moreover, we derive post-mean-field equations which describe the back-reaction of quantum fluctuations to first order. On the foundational level, we point out that our hybrid dynamics [Eq. (20)] reproduces the ideal quantum measurement, including the collapse of the wave function *and* the motion of the classical pointer. Let us also stress the close connection to current phenomenological theories of dynamic collapse which *follow* from our hybrid dynamics [16]. Our hybrid theory is likely to be an integrating concept for treating quantum measurement dynamically and to overcome the inconsistent mean-field method in quantum cosmology.

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[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
[2] L. Rosenfeld, Nucl. Phys. **40**, 353 (1963).
[3] T. W. Kibble, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, New York, 1981).
[4] T. N. Sherry and E. C. G. Sudarshan, Phys. Rev. D **18**, 4580 (1978).
[5] I. V. Aleksandrov, Z. Naturforsch. **36A**, 902 (1981).
[6] W. Boucher and J. Traschen, Phys. Rev. D **37**, 3522 (1988).
[7] L. Diósi and J. J. Halliwell, Phys. Rev. Lett. **81**, 2846 (1998).
[8] J. Maddox, Nature (London) **373**, 469 (1995); A. Anderson, Phys. Rev. Lett. **74**, 621 (1995); **76**, 4090 (1996); K. R. W. Jones, *ibid.* **76**, 4087 (1996); L. Diósi, *ibid.* **76**, 4088 (1996); I. R. Senitzky, *ibid.* **76**, 4089 (1996); O. V. Prezhdo and V. V. Kisil, Phys. Rev. A **56**, 162 (1997).
[9] L. L. Salcedo, Phys. Rev. A **54**, 3657 (1996); J. Caro and L.L. Salcedo, *ibid.* **60**, 842 (1999).
[10] L. Diósi, e-print quant-ph/9503023; and in *New Developments on Fundamental Problems in Quantum Physics*, edited by M. Ferrero and A. van der Merwe (Kluwer, Denver, 1997).
[11] L. Diósi, Quantum Semiclassic. Opt. **8**, 309 (1996).
[12] N. Gisin, Helv. Phys. Acta **62**, 363 (1989); Phys. Lett. **143A**, 1 (1990).
[13] V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961).
[14] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).
[15] K. B. Moller *et al.*, J. Chem. Phys. **106**, 7228 (1997).
[16] For harmonic-oscillator classical systems, this is equivalent to the quantum state diffusion equation of open quantum systems, whose Markov limit is a most typical phenomenological theory of continuous wave function collapse; see N. Gisin and I. C. Percival, J. Phys. A **25**, 5677 (1992); I. C. Percival, *Quantum State Diffusion* (Cambridge University Press, Cambridge, 1998). Exact stochastic equations for the conditional state vector $\varphi/\|\varphi\|$ have been analyzed recently: L. Diósi, N. Gisin, and W. T. Strunz, Phys. Rev. A **58**, 1699 (1998); W. T. Strunz, L. Diósi, and N. Gisin, Phys. Rev. Lett. **82**, 1801 (1999).