Quantum equations of motion for a dissipative system

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We propose a method of determining quantum evolution equations for dissipative systems. Our approach is based on the observations that (1) the equations of quantum mechanics for a Hamiltonian system correspond to the classical equations describing the evolution of a special statistical ensemble of copies of the system, and (2) the quantum equations can be determined if the corresponding classical equations are known and the satisfaction of the uncertainty principle for particle position and momentum is postulated. Namely, if these statements are required to remain valid for a dissipative system, the quantum evolution equations for that system can be obtained.

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I. INTRODUCTION

Since the very beginning of quantum mechanics, this question has existed: Given a nonrelativistic physical system initially described by the laws of classical mechanics, how does one describe it in terms of the laws of quantum mechanics [1]? The most widely accepted answer to this question is given by the Dirac algebraic rules, the so-called canonical or standard quantization [1,2], that can be applied to Hamiltonian systems. Although there are crucial objections to the suitability of the standard quantization rules to connect the classical and quantum theories [1], this method has been extrapolated to the quantum-mechanical study of dissipative systems [3–6]. However, the difficulties and limitations in the use of the canonical quantization rules for dissipative systems make it impossible to obtain a quantum description of these systems from the Lagrangian and Hamiltonian formalisms [1,7–9]. Furthermore, if the dissipative system under consideration (system A) is coupled to a thermal reservoir (system B) so that system $A+B$ as a whole is considered to be conservative, and the canonical quantization rules are applied to system $A+B$ [10,11], physical inconsistencies appear [1].

In order to overcome the problems arising from the Dirac quantization, different quantization methods of dissipative systems have been proposed [1]. In particular, the nonlinear Schrödinger-Langevin equation has been obtained by applying the Schrödinger method of quantization [12] to the generalized Hamilton-Jacobi equation [8], by using the Nelson stochastic quantization procedure [13,14], and by applying the classical Wigner transformations to a Fokker-Planck equation [1]. Schrödinger-type equations for dissipative systems have also been derived by extensions of the Madelung model [15,16].

We propose another method of determining quantum evolution equations for dissipative systems. Namely, we first examine the classical limit of the hydrodynamic equations of quantum mechanics for a Hamiltonian system [15,17,18] and find that the quantum equations correspond to the classical equations describing the evolution of a special statistical ensemble of copies of the system. We show that such a special statistical ensemble of copies of the system can be called a semipure statistical ensemble, because of its properties. Next, we demonstrate that the quantum equations can be reproduced if the corresponding classical equations are known and the satisfaction of the uncertainty principle for particle position and momentum is postulated. Subsequently, we obtain the classical evolution equations for a semipure statistical ensemble of copies of a dissipative system in which the particle is acted on by a force proportional to its velocity. Then, the uncertainty principle for particle position and momentum is assumed to be satisfied and the required quantum evolution equations are determined. Finally, the Schrödinger-Langevin equation is derived from these equations.

For simplicity, we consider only one-dimensional systems, although the entire discussion may be repeated for a two- or three-dimensional system without any substantial modifications. Additionally, unless explicitly stated otherwise, all integrations are performed over the whole available space.

II. HYDRODYNAMIC EQUATIONS OF QUANTUM MECHANICS

In quantum mechanics the system, consisting of a particle acted on by a force $F(x)$, is described by means of a wave function $\psi(x,t)$ that satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+V(x)\,\psi(x,t), \tag{1}$$

where $V(x)$ is a potential of the force [17,19]. Alternatively, in order to describe the quantum system one can define the probability density for particle position $\rho(x,t)$ and the corresponding velocity field $v(x,t)$ so that they meet the conditions

$$\rho(x,t)=\psi^*(x,t)\,\psi(x,t), \tag{2}$$

$$\rho(x,t)\,v(x,t)=\frac{\hbar}{2\,i\,m}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)-\psi(x,t)\frac{\partial}{\partial x}\psi^*(x,t)\right]. \tag{3}$$

Then, it can be shown that the functions $\rho(x,t)$ and $v(x,t)$ satisfy the system of two evolution equations [15,17,18]

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}[\rho(x,t)\,v(x,t)]=0, \tag{4}$$

$$\frac{\partial}{\partial t}v(x,t)+v(x,t)\,\frac{\partial}{\partial x}v(x,t)=\frac{F(x)}{m}-\frac{1}{\rho(x,t)}\,\frac{\partial}{\partial x}T_q(x,t), \tag{5}$$

where the quantity $T_q(x,t)$ is given by

$$T_q(x,t)=\frac{\hbar^2}{4m^2}\left\{\frac{1}{\rho(x,t)}\left[\frac{\partial}{\partial x}\rho(x,t)\right]^2-\frac{\partial^2}{\partial x^2}\rho(x,t)\right\}. \tag{6}$$

The analogous equations for a three-dimensional system are called the hydrodynamic equations of quantum mechanics, since they are very similar to the equations of hydrodynamics [15,17]. Accordingly, Eqs. (4) and (5) can be referred to as the one-dimensional hydrodynamic equations of quantum mechanics.

We note that if the ratio of $\hbar$ to m is negligible, Eqs. (4) and (5) change into the classical equations

$$\frac{\partial}{\partial t}\rho_c(x,t)+\frac{\partial}{\partial x}[\rho_c(x,t)\,v_c(x,t)]=0, \tag{7}$$

$$\frac{\partial}{\partial t}v_c(x,t)+v_c(x,t)\,\frac{\partial}{\partial x}v_c(x,t)=\frac{F(x)}{m}, \tag{8}$$

where the functions $\rho(x,t)$ and $v(x,t)$ have been replaced by their classical counterparts $\rho_c(x,t)$ and $v_c(x,t)$, respectively. These equations describe the evolution of a statistical ensemble of copies of the corresponding classical system. However, the statistical ensemble of copies of the system must be prepared in a special manner so that it can be described by Eqs. (7) and (8), which is shown in the next section.

III. PROBABILITY LIQUID IN CLASSICAL MECHANICS

Having presented the hydrodynamic equations of quantum mechanics, we now show that the equations they become in the classical limit, Eqs. (7) and (8), are satisfied under additional conditions imposed on the related statistical ensemble of copies of the system.

For this purpose we note that in classical mechanics a statistical ensemble of copies of a single-particle system, in which the particle is acted on by a conservative force $F(x)$, is described by means of a probability density $f(x,p,t)$ satisfying the Liouville equation [1,17]

$$\frac{\partial}{\partial t}f(x,p,t)=-\frac{p}{m}\,\frac{\partial}{\partial x}f(x,p,t)-F(x)\,\frac{\partial}{\partial p}f(x,p,t), \tag{9}$$

and having the following marginal properties:

$$\rho_c(x,t)=\int dp\, f(x,p,t), \quad \tilde{\rho}_c(p,t)=\int dx\, f(x,p,t), \tag{10}$$

where $\rho_c(x,t)$ is a probability density for particle position and $\tilde{\rho}_c(p,t)$ is a probability density for particle momentum [1]. Therefore, if Eq. (9) is integrated over p, we find that

$$\frac{\partial}{\partial t}\rho_c(x,t)=-\frac{\partial}{\partial x}j^{(1)}(x,t), \tag{11}$$

where the current $j^{(1)}(x,t)$ is given by

$$j^{(1)}(x,t)=\frac{1}{m}\int dp\, p\, f(x,p,t). \tag{12}$$

Similarly, if Eq. (9) is multiplied by p^n and integrated over p, we get

$$\frac{\partial}{\partial t}j^{(n)}(x,t)=n\,\frac{F(x)}{m}\,j^{(n-1)}(x,t)-\frac{\partial}{\partial x}j^{(n+1)}(x,t), \tag{13}$$

where the functions $j^{(n)}(x,t)$ are defined in the following manner:

$$j^{(n)}(x,t)=\frac{1}{m^n}\int dp\, p^n\, f(x,p,t). \tag{14}$$

Of course, Eq. (13) is valid for $n=1$, since the function $j^{(0)}(x,t)$ equals $\rho_c(x,t)$, according to Eqs. (10) and (14).

We now see that the Liouville equation, Eq. (9), is equivalent to the infinite system of evolution equations for the functions $j^{(n)}(x,t)$. Therefore, the state of the classical statistical ensemble of copies of the system can be described by the probability density $f(x,p,t)$ or, equivalently, by the infinite set of functions $j^{(n)}(x,t)$. Of course, this means that the evolution of the statistical ensemble is determined if all the functions $j^{(n)}(x,t)$ are known.

It turns out that we are able to prepare a statistical ensemble of copies of the system so that Eq. (8) is valid and Eq. (13) is satisfied for any n greater than one. Namely, if the velocity field $v_c(x,t)$ is defined according to the formula

$$j^{(1)}(x,t)=\rho_c(x,t)\,v_c(x,t), \tag{15}$$

and the current $j^{(1)}(x,t)$ in Eqs. (11) and (13) is replaced by the product of $\rho_c(x,t)$ and $v_c(x,t)$, we find that

$$\frac{\partial}{\partial t}\rho_c(x,t)+\frac{\partial}{\partial x}[\rho_c(x,t)\,v_c(x,t)]=0, \tag{16}$$

$$\frac{\partial}{\partial t}v_c(x,t)+v_c(x,t)\,\frac{\partial}{\partial x}v_c(x,t)=\frac{F(x)}{m}-\frac{1}{\rho_c(x,t)}\,\frac{\partial}{\partial x}[j^{(2)}(x,t)-\rho_c(x,t)\,v_c^2(x,t)]. \tag{17}$$

Interestingly enough, Eq. (17) changes into Eq. (8) if $j^{(2)}(x,t)$ is equal to the product of $\rho_c(x,t)$ and $v_c^2(x,t)$. Moreover, if all the functions $j^{(n)}(x,t)$ are given by

$$j^{(n)}(x,t)=\rho_c(x,t)\, v_c^n(x,t), \tag{18}$$

Eq. (13) is satisfied for any n greater than one, which can be verified by differentiating Eq. (18) with respect to time and using Eqs. (7) and (8) for eliminating the time derivatives of $\rho_c(x,t)$ and $v_c(x,t)$, respectively, from the resultant equation. Then, according to Eqs. (14) and (18), the nth central moment $\mu_n(v)$ of the probability distribution of particle velocity is given by [20]

$$\mu_n(v)=\int dx[v_c(x,t)-\langle v\rangle_t]^n\, \rho_c(x,t), \tag{19}$$

where $\langle v\rangle_t$ denotes the average particle velocity at time t. Therefore, for even n, the moments $\mu_n(v)$ are nonnegative, which is correct. Thus we can state that Eqs. (7) and (8) describe the evolution of the relevant statistical ensemble if Eq. (18) is satisfied for any n at any time t.

It is shown in Appendix A that if Eq. (18) is satisfied for any n at the initial instant of the time, it is satisfied for any n at any time t. Hence, we see that Eqs. (7) and (8) are satisfied if the related statistical ensemble of copies of the system is prepared so that Eq. (18) is satisfied for any n at the initial instant of the time. (Of course, it could be very difficult to obtain such a statistical ensemble in a real experiment, but this is beyond the scope of our discussion.) We note that if Eqs. (7) and (8) are satisfied, the functions $\rho_c(x,t)$ and $v_c(x,t)$ can be treated as quantities describing the classical probability liquid. Then, the liquid of the density $\rho_c(x,t)$ is very simple, since it is a liquid without any internal forces [17]. Nonetheless, the statistical ensembles of copies of the system that can be described in terms of the classical probability liquid have some interesting properties, which are discussed in the next section.

IV. SEMIPURE ENSEMBLES OF COPIES OF THE SYSTEM

Having shown that statistical ensembles of copies of a classical single-particle Hamiltonian system must be prepared in a special manner so that they are described by Eqs. (7) and (8), we now discuss some important properties of such statistical ensembles.

We first observe that any of these statistical ensembles cannot be obtained as a mixture of two statistical ensembles of copies of the system which are not described by Eqs. (7) and (8). Namely, we recall that a statistical ensemble of copies of the system can be called a mixture of the statistical ensembles s^A and s^B if the corresponding probability density $f(x,p,t)$ is defined by

$$f(x,p,t)=\alpha\, f^A(x,p,t)+\beta\, f^B(x,p,t), \tag{20}$$

where α and β are nonnegative real numbers the sum of which equals one, and $f^A(x,p,t)$ and $f^B(x,p,t)$ are probability density functions that represent the statistical ensembles s^A and s^B, respectively [17]. Of course, in that case

$$\rho_c(x,t)=\alpha\, \rho_c^A(x,t)+\beta\, \rho_c^B(x,t), \tag{21}$$

according to Eq. (10). Similarly, for any integer n greater than zero

$$j^{(n)}(x,t)=\alpha\, j_A^{(n)}(x,t)+\beta\, j_B^{(n)}(x,t), \tag{22}$$

according to Eq. (14). Therefore, if a statistical ensemble of copies of the system described by Eqs. (7) and (8) is a mixture of the statistical ensembles s^A and s^B, these statistical ensembles are also described by Eqs. (7) and (8) and the corresponding velocity fields $v_c^A(x,t)$ and $v_c^B(x,t)$ are equal to the resultant velocity field $v_c(x,t)$, which is shown in Appendix B. Hence, the statistical ensembles of copies of the system that are described by Eqs. (7) and (8) can be referred to as semipure statistical ensembles, since none of them can be obtained as a mixture of statistical ensembles that are not semipure themselves.

Second, we note that semipure statistical ensembles of copies of a classical single-particle system minimize the average particle energy with respect to the probability density $\rho_c(x,t)$ and the velocity field $v_c(x,t)$. Namely, for any pair of functions $q(x)$ and $p(x)$ that can be a probability density for particle position and the corresponding velocity field and for an instant of time t_0, there is a family S_A of the statistical ensembles of copies of the system for which the related functions $\rho_c^A(x,t_0)$ and $v_c^A(x,t_0)$ are equal to $q(x)$ and $p(x)$, respectively. Then, in the family S_A there is exactly one semipure statistical ensemble of copies of the system s_0^A. On the other hand, in classical mechanics the average particle energy is given by

$$\begin{aligned}\langle E\rangle&=\frac{m}{2}\int dx\, dp\left(\frac{p}{m}\right)^2 f(x,p,t)+\int dx\, dp\, V(x)\, f(x,p,t)\\&=\frac{m}{2}\langle v^2\rangle_t+\langle V\rangle_t,\end{aligned} \tag{23}$$

where the time dependence of the average square of particle velocity and the average potential energy is explicitly denoted by the subscript t. Thus, according to Eqs. (10) and (14), this equation can be transformed in the following manner:

$$\begin{aligned}\langle E\rangle&=\frac{m}{2}\int dx\, v_c^2(x,t)\, \rho_c(x,t)+\int dx\, V(x)\, \rho_c(x,t)\\&\quad+\frac{m}{2}\int dx[j^{(2)}(x,t)-v_c^2(x,t)\, \rho_c(x,t)].\end{aligned} \tag{24}$$

Therefore, if the related statistical ensemble of copies of the system is semipure, the last term in Eq. (24) vanishes. Because this term is never less than zero, which is shown in Appendix C, the statistical ensemble s_0^A has the minimal average energy within the family S_A, and this means that the average energy is minimized with respect to the functions

$\rho_c(x,t)$ and $v_c(x,t)$ if the corresponding statistical ensemble of copies of the system is semipure.

We now see that semipure statistical ensembles of copies of the system are special. They are also important, since many statistical ensembles of copies of the system that are not semipure can be obtained as mixtures of semipure statistical ensembles. On the other hand, there are semipure statistical ensembles of copies of the system that violate the uncertainty principle for particle momentum and position.

In order to show this we first recall that, according to the uncertainty principle for particle momentum and position, the following inequality should be satisfied:

$$\sigma_x \sigma_v \geqslant \frac{\hbar}{2m}, \tag{25}$$

where σ_x and σ_v denote the standard deviations of particle position and velocity, respectively [17,19]. Next, we observe that for a semipure statistical ensemble of copies of a classical single-particle system the dispersion of particle velocity, being the second central moment of the probability distribution of particle velocity [20], satisfies the equation

$$\sigma_v^2 = \int dx\, v_c^2(x,t)\, \rho_c(x,t) - \left[\int dx\, v_c(x,t)\, \rho_c(x,t) \right]^2. \tag{26}$$

Of course, in general, the dispersion of particle velocity depends on time. Nonetheless, if the velocity field is independent of x at certain instant of time t', the dispersion of particle velocity vanishes at the time t', which follows from Eq. (26). Thus, because the standard deviation of particle velocity is a square root of σ_v^2 [20], the left-hand side of Eq. (25) is then equal to zero, since the probability density $\rho_c(x,t')$ is normalized to unity. Consequently, the uncertainty principle for particle momentum and position is violated, whatever the dispersion of particle position is.

We now see that there are semipure statistical ensembles of copies of the system having the dispersion of particle velocity that vanishes at certain time independent of the dispersion of particle position, in which case the uncertainty principle for particle momentum and position is violated. However, the formulas for mean values of various quantities relative to a semipure statistical ensemble of copies of the system can be modified so that the uncertainty principle is always satisfied. Moreover, the relevant modifications can be made in such a manner that they lead to the hydrodynamic equations of quantum mechanics. We show this in the next section.

V. EFFECTS OF THE UNCERTAINTY PRINCIPLE

In classical mechanics the uncertainty principle for particle momentum and position can be violated, because the dispersion of particle velocity may be arbitrarily small independent of the dispersion of particle position, especially if the related statistical ensemble of copies of the system is semipure. We now show that if the formula for the average square of particle velocity relative to a semipure statistical ensemble of copies of the system is modified so that the uncertainty principle is not violated, the well-known equations of quantum mechanics can be obtained.

We first observe that in order to satisfy the uncertainty principle one should add a term independent of the velocity field to the right-hand side of Eq. (26). Then, the right-hand side of the resultant equation may be required to depend only on the probability density for particle position and on the velocity field, as it does for a semipure statistical ensemble of copies of the system. In that case the additional term ought to depend only on the probability density for particle position. Hence, because the right-hand side of this equation must be the difference between $\langle v^2 \rangle_t$ and $\langle v \rangle_t^2$, one of these averages should be given by a different, nonclassical formula. Of course, the corresponding classical formula ought to be approached for $(\hbar/m)$ tending to zero, since the right-hand side of Eq. (25) is negligible in that limit.

We show in Appendix D that despite the uncertainty principle for particle momentum and position, the average particle velocity is given by

$$\langle v \rangle_t = \int dx\, v(x,t)\, \rho(x,t), \tag{27}$$

where $\rho(x,t)$ and $v(x,t)$ are counterparts of the classical functions $\rho_c(x,t)$ and $v_c(x,t)$, respectively, unless the interpretation of the continuity equation is changed. On the other hand, a modification of the formula for the average square of particle velocity does not affect the continuity equation. Therefore, it seems reasonable to attach the additional term to the formula for the average square of particle velocity rather than to the formula for the average particle velocity. Consequently, we postulate that the average square of particle velocity is given by

$$\langle v^2 \rangle_t = \int dx\, \rho(x,t)\, v^2(x,t) + \frac{\hbar^2}{4m^2} \int dx \frac{1}{\rho(x,t)} \left[\frac{\partial}{\partial x} \rho(x,t) \right]^2, \tag{28}$$

which is a sufficient condition for the satisfaction of the uncertainty principle for particle momentum and position, according to the discussion presented in Appendix E.

Having postulated that Eq. (28) is satisfied, we now observe that the formula for the average particle energy must be modified accordingly. Namely, if the right-hand side of Eq. (28) is substituted for $\langle v^2 \rangle_t$ in Eq. (23), we obtain the following formula:

$$\langle E \rangle = \frac{m}{2} \int dx\, v^2(x,t)\, \rho(x,t) + \int dx\, V(x)\, \rho(x,t) + U[\rho(x,t)], \tag{29}$$

where the functional $U[\rho(x,t)]$ is defined by

$$U[\rho(x,t)] = \frac{\hbar^2}{8m} \int dx \frac{1}{\rho(x,t)} \left[\frac{\partial}{\partial x} \rho(x,t) \right]^2. \tag{30}$$

Of course, the right-hand side of Eq. (29) cannot be a constant of motion as long as Eqs. (7) and (8) are satisfied by

$\rho(x,t)$ and $v(x,t)$. Therefore, if the average energy is required to be a constant of motion, the probability density for particle position and the velocity field should satisfy another system of evolution equations.

In order to find these equations we note that they should contain all the terms that appear in Eqs. (7) and (8) together with some additional terms vanishing for $(\hbar/m)$ tending to zero. The reason for this is that the required equations ought to become Eqs. (7) and (8) when the ratio of $\hbar$ to m is negligible. Moreover, any modifications of the continuity equation either lead to complications with the normalization of $\rho(x,t)$, or can be made in such a manner that the resultant equation remains a continuity equation, which is discussed in Appendix D. Therefore, we postulate that the equations satisfied by $\rho(x,t)$ and $v(x,t)$ be the following:

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}[\rho(x,t)\,v(x,t)]=0, \tag{31}$$

$$\frac{\partial}{\partial t}v(x,t)+v(x,t)\,\frac{\partial}{\partial x}v(x,t)=\frac{F(x)}{m}-\frac{1}{\rho(x,t)}\,\frac{\partial}{\partial x}T(x,t), \tag{32}$$

where $T(x,t)$ is an unknown quantity that vanishes for $(\hbar/m)$ tending to zero. We also require the quantity $T(x,t)$ to be a functional of $\rho(x,t)$ and $v(x,t)$, since we want Eqs. (31) and (32) to describe a statistical ensemble of copies of the system that corresponds to a classical semipure statistical ensemble of copies of the system. In other words, the evolution of $\rho(x,t)$ and $v(x,t)$ is assumed to depend only on their initial forms, on the force $F(x)$, and on the shape of the available space, which means that the right-hand side of Eq. (32), including $T(x,t)$, can be expressed by $\rho(x,t)$ and $v(x,t)$.

Next, we require that the average particle energy should be a constant of motion. Hence, because the average energy is given by Eq. (29), we find that

$$0=\frac{d}{dt}\langle E\rangle=\frac{d}{dt}U[\rho(x,t)]-m\int dx\,v(x,t)\,\frac{\partial}{\partial x}T(x,t), \tag{33}$$

which is obtained if the right-hand side of Eq. (29) is differentiated with respect to time and Eqs. (31) and (32) are used for eliminating the time derivatives of $\rho(x,t)$ and $v(x,t)$, respectively, from the resulting equation. Then, if the functional $U[\rho(x,t)]$ is differentiated with respect to time and the right-hand side of Eq. (31) is substituted for the time derivative of $\rho(x,t)$ in the resulting equation, we get

$$\frac{d}{dt}U[\rho(x,t)]=\frac{\hbar^2}{4m}\int dx\,v(x,t)\,\frac{\partial}{\partial x}\left\{\frac{1}{\rho(x,t)}\left[\frac{\partial}{\partial x}\rho(x,t)\right]^2-\frac{\partial^2}{\partial x^2}\rho(x,t)\right\}, \tag{34}$$

which is shown in Appendix F. This leads to the conclusion that

$$\int dx\,v(x,t)\,\frac{\partial}{\partial x}\left(T(x,t)-\frac{\hbar^2}{4m^2}\left\{\frac{1}{\rho(x,t)}\left[\frac{\partial}{\partial x}\rho(x,t)\right]^2-\frac{\partial^2}{\partial x^2}\rho(x,t)\right\}\right)=0. \tag{35}$$

Therefore, we introduce the function

$$w(x,t)=T(x,t)-T_q(x,t), \tag{36}$$

where $T_q(x,t)$ is given by Eq. (6), and find that

$$\int dx\,v(x,t)\,\frac{\partial}{\partial x}w(x,t)=0. \tag{37}$$

We observe that $w(x,t)$ is a functional of $\rho(x,t)$ and $v(x,t)$, since $T(x,t)$ and $T_q(x,t)$ are. Therefore, if t_0 is the initial instant of time and $w(x,t_0)$ depends on x, Eq. (37) leads to a condition for the functions $\rho(x,t_0)$ and $v(x,t_0)$. Moreover, if this equation is differentiated with respect to time, we obtain another condition for $\rho(x,t_0)$ and $v(x,t_0)$, because their time derivatives can be eliminated from the resulting equation with the help of Eqs. (31) and (32). Repeating this step we get an infinite set of conditions for $\rho(x,t_0)$ and $v(x,t_0)$, which means that these functions depend on one another. However, there is no physical reason why the initial functions $\rho(x,t_0)$ and $v(x,t_0)$ should not be independent. This leads to the conclusion that $w(x,t_0)$ does not depend on x. Consequently, $w(x,t)$ is independent of x for any t, since it must always be given by the same formula. Hence, Eqs. (31) and (32) turn out to be the hydrodynamic equations of quantum mechanics.

We now see that the well-known equations of quantum mechanics may result from the modifications which are made to the classical equations, describing a semipure statistical ensemble of copies of the system, in order to avoid violating the uncertainty principle. Namely, the equations of quantum mechanics follow from three basic assumptions. The first of these assumptions is that the average square of particle velocity should be given by a formula that avoids violating the uncertainty principle, the second is that the corresponding classical equations ought to be valid in the limit of $(\hbar/m)$ tending to zero, and the third is that the average particle energy should be a constant of motion. Thus, in order to find the quantum evolution equations one needs to determine the corresponding classical equations, modify the formula for the average square of particle velocity so that the uncertainty principle is not violated, and require that the average particle energy does not evolve. In the next section we apply a similar approach to a system in which the particle is acted on by a force proportional to its velocity.

VI. DISSIPATIVE SYSTEM

In the present section our previous results are used in order to obtain the quantum equations of motion for a single-particle dissipative system in which the particle is acted on by a force proportional to its velocity.

For this purpose we consider a statistical ensemble of

copies of a single-particle dissipative system satisfying the following Newton equation of motion:

$$\frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} + \frac{F(x)}{m}, \tag{38}$$

where $F(x)$ is a conservative force and γ is a positive constant. We assume that this statistical ensemble can be described by means of the probability density $f(x,p,t)$. Of course, in the present case the function $f(x,p,t)$ does not satisfy the Liouville equation. However, it is shown in Appendix G that for the dissipative system under consideration there are equations that can be treated as counterparts of Eqs. (11) and (13). Namely, if the function $\rho_c(x,t)$ is defined according to Eq. (10), it satisfies Eq. (11), where $j^{(1)}(x,t)$ is given by Eq. (12). Similarly, if the functions $j^{(n)}(x,t)$ are defined by Eq. (14), the equation

$$\frac{\partial}{\partial t} j^{(n)}(x,t) = n \frac{F(x)}{m} j^{(n-1)}(x,t) - \frac{\partial}{\partial x} j^{(n+1)}(x,t) - n\, \gamma\, j^{(n)}(x,t) \tag{39}$$

is satisfied for all integers n greater than zero. [Of course, the function $j^{(0)}(x,t)$ is identified with $\rho_c(x,t)$.] Thus, if the velocity field $v_c(x,t)$ is introduced according to Eq. (15), the following equations are satisfied:

$$\frac{\partial}{\partial t} \rho_c(x,t) + \frac{\partial}{\partial x}[\rho_c(x,t)\, v_c(x,t)] = 0, \tag{40}$$

$$\frac{\partial}{\partial t} v_c(x,t) + v_c(x,t) \frac{\partial}{\partial x} v_c(x,t) = -\gamma\, v_c(x,t) + \frac{F(x)}{m} - \frac{1}{\rho_c(x,t)} \frac{\partial}{\partial x} \times [j^{(2)}(x,t) - \rho_c(x,t)\, v_c^2(x,t)]. \tag{41}$$

Therefore, if the function $j^{(2)}(x,t)$ is equal to the product of $\rho_c(x,t)$ and $v_c^2(x,t)$, the last term in Eq. (40) vanishes and we find that

$$\frac{\partial}{\partial t} \rho_c(x,t) + \frac{\partial}{\partial x}[\rho_c(x,t)\, v_c(x,t)] = 0, \tag{42}$$

$$\frac{\partial}{\partial t} v_c(x,t) + v_c(x,t) \frac{\partial}{\partial x} v_c(x,t) = -\gamma\, v_c(x,t) + \frac{F(x)}{m}. \tag{43}$$

In that case, if Eq. (18) is satisfied for any n greater than one, Eq. (39) is also satisfied for any n greater than one. Moreover, as in the case of the analogous Hamiltonian system, Eq. (18) is satisfied for any n and for any instant of time t when it is satisfied for any n at the initial instant of the time, which is shown in Appendix H. Therefore, if a statistical ensemble of copies of the system is prepared in such a manner that Eq. (18) is satisfied for any n at the initial instant of the time, the corresponding functions $\rho_c(x,t)$ and $v_c(x,t)$ satisfy Eqs. (42) and (43). Accordingly, such a statistical ensemble of copies of the system can be called a semipure statistical ensemble, since the discussion presented in Sec. IV can be applied to it without any modifications. In particular, such a statistical ensemble of copies of the system can be constructed so that the uncertainty principle for particle momentum and position is violated at certain instant in time (e.g., initial).

The uncertainty principle can be prevented from being violated in the same manner as in the case of the analogous Hamiltonian system. Namely, it can be postulated that the average square of particle velocity be given by Eq. (28). Consequently, we must require that the average particle energy be given by Eq. (29). Of course, in the present case the average energy is not a constant of motion. However, if the evolution equation for the average particle energy is known, the equations of motion for the counterparts of $\rho_c(x,t)$ and $v_c(x,t)$ can be determined.

In order to find the evolution equation for the average particle energy, we note that in classical mechanics, if the underlying system is described by Eq. (38), it satisfies the equation

$$\frac{d}{dt}\langle E\rangle_t = -m\, \gamma \int dx\, j^{(2)}(x,t), \tag{44}$$

which is shown in Appendix I. Therefore, if the related statistical ensemble of copies of the system is semipure, we have

$$\frac{d}{dt}\langle E\rangle_t = -m\, \gamma \int dx\, v_c^2(x,t)\, \rho_c(x,t). \tag{45}$$

Of course, the right-hand side of Eq. (45) as well as the right-hand side of Eq. (44) are proportional to the average square of particle velocity. Consequently, Eqs. (44) and (45) mean that in classical mechanics the time derivative of the average particle energy is proportional to its kinetic part. On the other hand, it can be thought that each copy of the system in the related statistical ensemble contains a particle. Hence, one can state that in classical mechanics the time derivative of the average energy depends on the velocities of these particles. Specifically, if the particles do not move, the right-hand sides of Eqs. (44) and (45) vanish and the time derivative of the average energy is equal to zero. In other words, the particles must move so that the average energy can change.

If the uncertainty principle is satisfied and the particles are sufficiently small, we are unable to treat them as observable objects. Nonetheless, if the statistical ensemble under consideration is required to be a counterpart of a classical semipure statistical ensemble of copies of the system, we may assume that the particles do not move if the corresponding velocity field is equal to zero. Thus we may require that the time derivative of the average energy vanish in such a case. Consequently, the average square of particle velocity in Eq. (45) cannot be replaced with the right-hand side of Eq. (28) and it seems reasonable to leave it unmodified, since it must be valid in the classical limit. Moreover, if the average

square of particle velocity in Eq. (45) were replaced with the right-hand side of Eq. (28), we would find that the average energy might change because of the uncertainty principle itself, which would not be acceptable. Therefore, we postulate that the equation

$$\frac{d}{dt}\langle E\rangle_t = -m\,\gamma\int dx\, v^2(x,t)\,\rho(x,t), \tag{46}$$

where $\rho(x,t)$ and $v(x,t)$ are counterparts of $\rho_c(x,t)$ and $v_c(x,t)$, respectively, should be satisfied when the average particle energy is given by Eq. (29).

If Eq. (46) is satisfied and the average particle energy is given by Eq. (29), the functions $\rho(x,t)$ and $v(x,t)$ cannot satisfy Eqs. (42) and (43). Specifically, the continuity equation should be satisfied so that the normalization of $\rho(x,t)$ is conserved, but Eq. (43) must be modified. Moreover, it can be modified by introducing additional terms that vanish for $(\hbar/m)$ tending to zero, since the resultant equation should approach it in the classical limit. Therefore, we postulate the following evolution equations for $\rho(x,t)$ and $v(x,t)$:

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}[\rho(x,t)\,v(x,t)]=0, \tag{47}$$

$$\frac{\partial}{\partial t}v(x,t)+v(x,t)\,\frac{\partial}{\partial x}v(x,t)=-\gamma\,v(x,t)+\frac{F(x)}{m} - \frac{1}{\rho(x,t)}\,\frac{\partial}{\partial x}T(x,t), \tag{48}$$

where the unknown quantity $T(x,t)$ vanishes for $(\hbar/m)$ tending to zero. Consequently, because the average particle energy satisfies Eq. (46), we get

$$\begin{aligned} -m\,\gamma\int dx\, v^2(x,t)\,\rho(x,t) &= \frac{d}{dt}\langle E\rangle_t = -m\int dx\, v(x,t)\,\frac{\partial}{\partial x}T(x,t) \\ &\quad + \frac{d}{dt}U[\rho(x,t)] - m\,\gamma\int dx\, v^2(x,t)\,\rho(x,t), \end{aligned} \tag{49}$$

which is obtained by differentiating Eq. (29) with respect to time and using Eqs. (47) and (48) in order to eliminate the time derivatives of $\rho(x,t)$ and $v(x,t)$, respectively, from the resulting equation. Hence, we see that Eq. (35) is satisfied by $T(x,t)$. Therefore, we can introduce the function $w(x,t)$ according to Eq. (36) so that it satisfies Eq. (37). Then, the same arguments that have been applied in the case of the analogous Hamiltonian system lead to the conclusion that $w(x,t)$ should be independent of x. Finally, we find that

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}[\rho(x,t)\,v(x,t)]=0, \tag{50}$$

$$\frac{\partial}{\partial t}v(x,t)+v(x,t)\,\frac{\partial}{\partial x}v(x,t)=-\gamma\,v(x,t)+\frac{F(x)}{m} - \frac{1}{\rho(x,t)}\,\frac{\partial}{\partial x}T_q(x,t), \tag{51}$$

where the quantity $T_q(x,t)$ is given by Eq. (6).

We note that Eqs. (50) and (51) have been found by using the same arguments that can be used for obtaining Eqs. (4) and (5), which describe the analogous Hamiltonian system, except that the average particle energy is not a constant of motion in the present case. Accordingly, these equations can be treated as the hydrodynamic equations of quantum mechanics for a one-dimensional dissipative system, in which the particle is acted on by a force proportional to its velocity. Moreover, it turns out that Eqs. (50) and (51) lead to a known evolution equation for a complex-valued function being a counterpart of the Schrödinger wave function $\psi(x,t)$, which is shown in the next section.

VII. SCHRÖDINGER-LANGEVIN EQUATION

Having derived the quantum evolution equations for a dissipative system we now show that these equations lead to an equation of motion for a complex-valued function being a counterpart of the Schrödinger wave function $\psi(x,t)$.

For this purpose we assume that the relevant velocity field $v(x,t)$ is potential. Then, we are able to define the function $\psi(x,t)$ in the following manner:

$$\psi(x,t)=\sqrt{\rho(x,t)}\exp\left[\,i\,\frac{m}{\hbar}\int_{x_0}^{x}dq\;v(q,t)\right], \tag{52}$$

where x_0 is an arbitrary fixed spatial coordinate. We note that the function $\psi(x,t)$ is related to $\rho(x,t)$ and $v(x,t)$ via Eqs. (2) and (3). Consequently, if the right-hand side of Eq. (2) is substituted for the function $\rho(x,t)$ and the right-hand side of Eq. (3) for the product of $\rho(x,t)$ and $v(x,t)$ in Eq. (50), we find that

$$i\hbar\;\psi^*\frac{\partial\psi}{\partial t}+i\hbar\;\psi\frac{\partial\psi^*}{\partial t}=-\frac{\hbar^2}{2m}\left(\;\psi^*\frac{\partial^2\psi}{\partial x^2}-\psi\frac{\partial^2\psi^*}{\partial x^2}\right), \tag{53}$$

which is equivalent to the following system of two partial differential equations:

$$i\hbar\,\frac{\partial}{\partial t}\psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+\xi(x,t)\;\psi(x,t), \tag{54}$$

$$i\hbar\,\frac{\partial}{\partial t}\psi^*(x,t)=\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}\psi^*(x,t)-\xi(x,t)\;\psi^*(x,t), \tag{55}$$

where $\xi(x,t)$ is an unknown real-valued quantity. We observe that if $\xi(x,t)$ is a functional of $\psi(x,t)$ and $\psi^*(x,t)$,

and of the force $F(x)$, the evolution of $\psi(x,t)$ and $\psi^*(x,t)$ is determined by the above equations.

The unknown quantity $\xi(x,t)$ can be found with the help of Eq. (3). Namely, it follows from Eqs. (2), (52), (54), and (55) that the time derivative of the left-hand side of this equation is given by

$$\frac{\hbar}{2\,i\,m}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)$$
$$=\frac{1}{m}\rho\left(-\frac{\partial\xi}{\partial x}\right)+\frac{\partial}{\partial x}\left\{\frac{\hbar^2}{4m^2}\left[\frac{\partial^2\rho}{\partial x^2}-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\right]-\rho\,v^2\right\}. \tag{56}$$

Consequently, the right-hand side of Eq. (56) equals the time derivative of the product of $\rho(x,t)$ and $v(x,t)$. On the other hand, according to Eqs. (50) and (51), we find that

$$\frac{\partial}{\partial t}[\rho(x,t)\,v(x,t)]=-\gamma\,v(x,t)+\frac{F(x)}{m}\rho(x,t)$$
$$-\frac{\partial}{\partial x}[T_q(x,t)+\rho(x,t)\,v^2(x,t)], \tag{57}$$

where $T_q(x,t)$ is given by Eq. (6). This leads to the following result:

$$-\frac{\partial}{\partial x}\xi(x,t)=-\gamma\frac{\hbar}{2\,i}\left[\frac{1}{\psi(x,t)}\frac{\partial}{\partial x}\psi(x,t)\right.$$
$$\left.-\frac{1}{\psi^*(x,t)}\frac{\partial}{\partial x}\psi^*(x,t)\right]+F(x), \tag{58}$$

which is a consequence of Eqs. (3) and (56).

Now, we note that there is the function $\arg z$ returning the angle of the number z in the complex plane. Thus, for the complex-valued function $\psi(x,t)$ given by Eq. (52), we have

$$\arg[\psi(x,t)]=\frac{m}{\hbar}\int_{x_0}^{x}dq\;v(q,t)+2n\pi, \tag{59}$$

n being an integer. Hence, the definition of the function $\arg z$ implies the following identity:

$$\frac{\partial}{\partial x}\arg[\psi(x,t)]=\frac{1}{2\,i}\left[\frac{1}{\psi(x,t)}\frac{\partial}{\partial x}\psi(x,t)\right.$$
$$\left.-\frac{1}{\psi^*(x,t)}\frac{\partial}{\partial x}\psi^*(x,t)\right]. \tag{60}$$

Therefore, according to Eq. (58), the potential $V(x)$ can be redefined so that $\xi(x,t)$ is given by

$$\xi(x,t)=\gamma\hbar\,\arg[\psi(x,t)]+V(x). \tag{61}$$

Thus, if the right-hand side of this equation is substituted for the function $\xi(x,t)$ in Eq. (54), we find that

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+V(x)\,\psi(x,t)$$
$$+\gamma\hbar\,\arg[\psi(x,t)]\,\psi(x,t), \tag{62}$$

which is the required equation of motion for the function $\psi(x,t)$. This equation is apparently nonlinear. However, the nonlinear term appears in it because of the dissipation of energy. We note that Eq. (62) is very similar to the Schrödinger-Langevin equation originally derived by Kostin [11] from the Heisenberg-Langevin equation for a Brownian particle interacting with a thermal environment [1].

Some features of the Schrödinger-Langevin equation cause it to be considered as suitable for describing dissipative systems in quantum mechanics [1,8,12,14,21]. Most importantly, it causes the uncertainty principle for particle position and momentum to be satisfied and for a two-particle system it factorizes into two independent equations if there is no correlation between the motion of the two particles [21]. It turns out that the corresponding factorization occurs in classical mechanics for a semipure statistical ensemble of copies of the two-particle system, which is discussed in Appendix J.

VIII. CONCLUSIONS

It follows from the considerations presented that quantum evolution equations for a dissipative system can be formulated in a consistent manner. In particular, these equations agree with the equations for the corresponding Hamiltonian system in that they cause the normalization of the related probability density for particle position to be conserved. Moreover, they change into the equations for the corresponding Hamiltonian system if the energy dissipation is negligible.

The derivation of the quantum evolution equations for a dissipative system is based on the correspondence between classical and quantum mechanics. Namely, it can be observed that in the classical limit the hydrodynamic equations of quantum mechanics, resulting from the Schrödinger equation, lead to the equations that describe the evolution of a semipure statistical ensemble of copies of the system. On the other hand, the uncertainty principle for particle momentum and position can be violated by a classical semipure statistical ensemble of copies of the system. In order to assure that the uncertainty principle is always satisfied, the formulas for mean values of various physical quantities, including energy, should be modified. Then, if the average energy is required to be a constant of motion and the corresponding classical equations are required to be approached in the classical limit, the quantum evolution equations are found. Accordingly, in order to obtain the quantum evolution equations for a dissipative system one needs to derive the equations that describe the evolution of a classical semipure statistical ensemble of copies of that system. Next, if the classical formulas for mean values of various physical quantities are modified so that the uncertainty principle is not violated and a certain evolution equation for the average energy is postulated, the required equations can be determined.

It turns out that the quantum evolution equations for a dissipative system lead to the Schrödinger-Langevin equation in the same manner that the hydrodynamic equations of quantum mechanics lead to the Schrödinger equation [17]. Moreover, the Schrödinger-Langevin equation is nonlinear, with the nonlinearity arising directly from the terms describing the dissipation of energy. Therefore, it seems that the Schrödinger equation itself is linear because of the energy conservation principle. This leads to the general hypothesis that systems obeying the energy conservation principle may be expected to satisfy linear evolution equations in quantum mechanics. Be constrast, systems in which energy is not conserved should satisfy nonlinear equations of motion, with the nonlinear terms due to the dissipation or aggregation of energy.

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APPENDIX A

In the present appendix we show that in classical mechanics Eq. (18) is satisfied for any n and for any instant of time t if it is satisfied for any n at the initial instant of time t_0, provided that the corresponding probability density $f(x,p,t)$ satisfies Eq. (9).

For this purpose we note that in classical mechanics Eq. (13) is satisfied for any n greater than zero by the functions $j^{(n)}(x,t)$, $j^{(n-1)}(x,t)$, and $j^{(n+1)}(x,t)$ defined according to Eq. (14) if the corresponding probability density $f(x,p,t)$ satisfies Eq. (9). (This is shown in Sec. III.) Therefore, if the functions $w^{(n)}(x,t)$ are defined by

$$w^{(n)}(x,t)=j^{(n)}(x,t)-\rho_c(x,t)\,v_c^n(x,t), \tag{A1}$$

the following evolution equation is satisfied:

$$\frac{\partial}{\partial t}w^{(n)}(x,t)=n\,v_c^{n-1}(x,t)\,\frac{\partial}{\partial x}w^{(2)}(x,t) + n\,\frac{F(x)}{m}\,w^{(n-1)}(x,t)-\frac{\partial}{\partial x}w^{(n+1)}(x,t), \tag{A2}$$

for any integer n greater than one. Of course, the function $w^{(1)}(x,t)$ is always equal to zero, according to Eqs. (15) and (A1).

If Eq. (A2) is differentiated with respect to time, we find that

$$\frac{\partial^2}{\partial t^2}w^{(n)}(x,t)=n\,v_c^{n-1}(x,t) \times\frac{\partial}{\partial x}\,\frac{\partial}{\partial t}w^{(2)}(x,t)+n(n-1)\,v_c^{n-2}(x,t) \times\frac{\partial}{\partial t}v_c(x,t)\,\frac{\partial}{\partial x}w^{(2)}(x,t)+n\,\frac{F(x)}{m} \times\frac{\partial}{\partial t}w^{(n-1)}(x,t)-\frac{\partial}{\partial x}\,\frac{\partial}{\partial t}w^{(n+1)}(x,t), \tag{A3}$$

where the time derivatives of $w^{(2)}(x,t)$, $w^{(n-1)}(x,t)$, and $w^{(n+1)}(x,t)$ can be replaced with the right-hand side of Eq. (A2) evaluated for $n'=2$, $n'=n-1$, and $n'=n+1$, respectively. Therefore, if all the functions $w^{(n)}(x,t_0)$ are equal to zero, the right-hand side of Eq. (A3) vanishes and the second-order time derivatives of $w^{(n)}(x,t)$ are equal to zero at time t_0.

Repeating this step, we can show that the time derivatives of $w^{(n)}(x,t)$ of all orders are equal to zero at time t_0 if all $w^{(n)}(x,t_0)$ vanish, in which case all functions $w^{(n)}(x,t)$ must not evolve. On the other hand, all the functions $w^{(n)}(x,t_0)$ vanish if Eq. (18) is satisfied at time t_0 for any n. Hence, if Eq. (18) is satisfied at time t_0 for any n, all functions $w^{(n)}(x,t)$ vanish for any t, which means that Eq. (18) is satisfied for any n and for any t. Thus the proof is complete.

APPENDIX B

We have stated in Sec. IV that two statistical ensembles of copies of the system s^A and s^B must be semipure so that another semipure statistical ensemble can be obtained as a mixture of s^A and s^B. Moreover, the corresponding velocity fields $v_c^A(x,t)$ and $v_c^B(x,t)$ must be equal for this purpose.

In order to show this, we consider a statistical ensemble of copies of the system described by the probability density $f(x,p,t)$, which is given by Eq. (20), and calculate the quantity

$$W(x,t)=j^{(2)}(x,t)-\frac{[j^{(1)}(x,t)]^2}{\rho_c(x,t)}, \tag{B1}$$

where $\rho_c(x,t)$ is defined according to Eq. (21) and $j^{(1)}(x,t)$ and $j^{(2)}(x,t)$ are given by Eq. (22) for $n=1$ and $n=2$, respectively. We note that $W(x,t)$ must vanish so that the statistical ensemble under consideration is semipure.

We first observe that Eq. (B1) can be transformed in the following manner:

$$\rho_c(x,t)\,W(x,t)=[\alpha\,\rho_c^A(x,t)+\beta\,\rho_c^B(x,t)] \times[\alpha\,j_A^{(2)}(x,t)+\beta\,j_B^{(2)}(x,t)] -[\alpha\,j_A^{(1)}(x,t)+\beta\,j_B^{(1)}(x,t)]^2, \tag{B2}$$

with the help of Eqs. (21) and (22). Therefore, because $\rho_c(x,t)$ is nonnegative and normalized to unity, the right-hand side of Eq. (B2) must vanish so that $W(x,t)$ is equal to zero.

Next, we recalculate the right-hand side of Eq. (B2) in such a manner that

$$\begin{aligned}\rho_c(x,t)\, W(x,t) = &[\alpha\, \rho_c^A(x,t) + \beta\, \rho_c^B(x,t)] \\ &\times [\alpha\, W^A(x,t) + \beta\, W^B(x,t)] \\ &+ \alpha\,\beta \left[\sqrt{\frac{\rho_c^A(x,t)}{\rho_c^B(x,t)}}\, j_B^{(1)}(x,t) \right. \\ &\left. - \sqrt{\frac{\rho_c^B(x,t)}{\rho_c^A(x,t)}}\, j_A^{(1)}(x,t) \right]^2, \end{aligned} \tag{B3}$$

where the quantities $W^A(x,t)$ and $W^B(x,t)$ are defined in analogy with $W(x,t)$ for the statistical ensembles s^A and s^B, respectively.

Now, we note that the functions $W^A(x,t)$ and $W^B(x,t)$ are not less than zero, which is shown in Appendix C. Moreover, they vanish if and only if the corresponding statistical ensembles of copies of the system are semipure. Consequently, if any of the statistical ensembles s^A and s^B are not semipure, there is a spatial coordinate x for which the right-hand side of Eq. (B3) is greater than zero. Then, the quantity $W(x,t)$ is not equal to zero and the corresponding statistical ensemble of copies of the system is not semipure. Thus the statistical ensembles s^A and s^B must be semipure so that any mixture of them can be a semipure statistical ensemble of copies of the system.

We now assume that the statistical ensembles s^A and s^B are semipure. Then, according to Eq. (B3), we find that

$$\begin{aligned}\rho_c(x,t)\, W(x,t) = \alpha\,\beta &\left[\sqrt{\frac{\rho_c^A(x,t)}{\rho_c^B(x,t)}}\, j_B^{(1)}(x,t) \right. \\ &\left. - \sqrt{\frac{\rho_c^B(x,t)}{\rho_c^A(x,t)}}\, j_A^{(1)}(x,t) \right]^2. \end{aligned} \tag{B4}$$

Next, we observe that the right-hand side of Eq. (B4) vanishes if and only if

$$\frac{j_B^{(1)}(x,t)}{\rho_c^B(x,t)} = \frac{j_A^{(1)}(x,t)}{\rho_c^A(x,t)}, \tag{B5}$$

which means that the velocity fields $v_c^A(x,t)$ and $v_c^B(x,t)$ must be equal so that the right-hand side of Eq. (B4) can vanish. However, if $v_c^A(x,t)$ and $v_c^B(x,t)$ are equal, we have

$$\begin{aligned}\rho_c(x,t)\, v_c(x,t) &= j^{(1)}(x,t) \\ &= [\alpha\, \rho_c^A(x,t) + \beta\, \rho_c^B(x,t)] v_c^A(x,t) \\ &= \rho_c(x,t)\, v_c^A(x,t), \end{aligned} \tag{B6}$$

according to Eqs. (15), (21), and (22). Hence, the velocity field $v_c(x,t)$ is then equal to $v_c^A(x,t)$ and $v_c^B(x,t)$. Therefore, if the statistical ensembles s^A and s^B are semipure, the corresponding velocity fields $v_c^A(x,t)$ and $v_c^B(x,t)$ must be equal so that any mixture of s^A and s^B is a semipure statistical ensemble of copies of the system, in which case the resultant velocity field is equal to $v_c^A(x,t)$ and $v_c^B(x,t)$.

Finally, we can state that the statistical ensembles s^A and s^B must be semipure and the corresponding velocity fields $v_c^A(x,t)$ and $v_c^B(x,t)$ must be equal so that any mixture of s^A and s^B is a semipure statistical ensemble of copies of the system. Moreover, if a semipure statistical ensemble of copies of the system is a mixture of s^A and s^B, the corresponding velocity field is equal to $v_c^A(x,t)$ and $v_c^B(x,t)$. Thus the proof is complete.

APPENDIX C

In the present appendix we show that for any statistical ensemble of copies of a classical single-particle system, which is described by a probability density $f(x,p,t)$, there is a nonnegative quantity $W(x,t)$ defined by

$$W(x,t) = j^{(2)}(x,t) - \frac{[j^{(1)}(x,t)]^2}{\rho_c(x,t)}, \tag{C1}$$

where $\rho_c(x,t)$ is given by Eq. (10) and $j^{(1)}(x,t)$ and $j^{(2)}(x,t)$ are given by Eq. (14) for $n=1$ and $n=2$, respectively.

For this purpose we observe that

$$W(x,t) = \int dp \left[\frac{p}{m} - v_c(x,t) \right]^2 f(x,p,t), \tag{C2}$$

where the velocity field $v_c(x,t)$ is defined according to Eq. (15). Namely, we have

$$\begin{aligned}\int dp \left[\frac{p}{m} - v_c(x,t) \right]^2 f(x,p,t) &= \int dp\, \frac{p^2}{m^2} f(x,p,t) + v_c^2(x,t) \int dp f(x,p,t) \\ &\quad - 2\, v_c(x,t) \int dp\, \frac{p}{m} f(x,p,t), \end{aligned} \tag{C3}$$

which can be transformed so that

$$\int dp \left[\frac{p}{m} - v_c(x,t) \right]^2 f(x,p,t) = j^{(2)}(x,t) - v_c^2(x,t)\, \rho_c(x,t), \tag{C4}$$

with the help of Eqs. (10), (14), and (15). Of course, according to Eq. (15), the right-hand side of Eq. (C4) is equal to the right-hand side of Eq. (C1). Therefore, because the left-hand side of Eq. (C4) is never less than zero, the function $W(x,t)$ is nonnegative and the proof is complete.

APPENDIX D

We have stated in Sec. V that the equation describing the evolution of the probability density for particle position is a continuity equation in classical mechanics as well as in the case in which the uncertainty principle for particle position and momentum is satisfied. Moreover, in either case the related mean value of particle velocity is given by Eq. (27), where $\rho(x,t)$ and $v(x,t)$ denote the probability density for particle position and the corresponding velocity field, respectively.

In order to show this, we first observe that in classical mechanics Eq. (11) is satisfied by the probability density $\rho_c(x,t)$. Therefore, the average particle position and the average particle velocity satisfy the equation

$$\frac{d}{dt}\langle x\rangle_t=\langle v\rangle_t, \tag{D1}$$

which is verified by multiplying Eq. (11) by x and integrating over it with the help of Eq. (12). Of course, the mean values of particle position and velocity can be determined independently, and this means that the uncertainty principle should not have an effect on Eq. (D1). Consequently, Eq. (D1) ought to be satisfied when the uncertainty principle is.

We now suppose that the formula for the average particle velocity is modified in order to avoid violating the uncertainty principle. Then, we have

$$\frac{d}{dt}\langle x\rangle_t=\langle v\rangle_t^c-\int dx\, x\, Q(x,t), \tag{D2}$$

where the classical average particle velocity is denoted by $\langle v\rangle_t^c$ and the additional term is given by the spatial integral on the right-hand side of the equation. Hence, we find that

$$\frac{\partial}{\partial t}\rho(x,t)=-\frac{\partial}{\partial x}j^{(1)}(x,t)-Q(x,t), \tag{D3}$$

where $\rho(x,t)$ is a counterpart of the classical quantity $\rho_c(x,t)$ and $j^{(1)}(x,t)$ is defined by Eq. (15) with $\rho_c(x,t)$ substituted for by its counterpart $\rho(x,t)$.

We note that the spatial integral of $Q(x,t)$ must be equal to zero, since otherwise the normalization of $\rho(x,t)$ is not conserved, in which case $\rho(x,t)$ cannot be treated as a probability density function. Moreover, we can safely assume that $Q(x,t)$ is not singular. Therefore, the function

$$j_q(x,t)=\int_{x_0}^{x} dq\, Q(q,t), \tag{D4}$$

where x_0 is an arbitrary fixed spatial coordinate, is well defined for any spatial coordinate x. Consequently, we are able to define the current $j(x,t)$ in the following manner:

$$j(x,t)=j^{(1)}(x,t)+j_q(x,t), \tag{D5}$$

and the corresponding velocity field $v(x,t)$ can be defined according to Eq. (15). Then, if Eq. (D3) is multiplied by x and integrated over it and the resulting equation is compared with Eq. (D1), we state that the average particle velocity is given by Eq. (27). Of course, once the velocity field is redefined, the evolution equation for it must be modified accordingly. However, we can show that if the corresponding classical statistical ensemble of copies of the system is semipure, the relevant modification can be be made by adding a new term to the right-hand side of this equation.

For this purpose we observe that if $j_q(x,t)$ vanishes for $(\hbar/m)$ tending to zero, the following equation must be valid in the classical limit:

$$\frac{\partial}{\partial t}j(x,t)=\frac{F(x)}{m}-\frac{\partial}{\partial x}[\rho(x,t)\, v_c^2(x,t)], \tag{D6}$$

where $v_c(x,t)$ is defined according to Eq. (15) for the probability density $\rho(x,t)$ and the current $j^{(1)}(x,t)$. [Of course, in that case $v_c(x,t)$ satisfies Eq. (8).] Therefore, in general, the current $j(x,t)$ should satisfy the equation

$$\frac{\partial}{\partial t}j(x,t)=\frac{F(x)}{m}-\frac{\partial}{\partial x}[\rho(x,t)\, v_c^2(x,t)]+S(x,t), \tag{D7}$$

where $S(x,t)$ vanishes for $(\hbar/m)$ tending to zero. On the other hand, the right-hand side of Eq. (D7) is given by

$$\frac{\partial}{\partial t}j(x,t)=\rho(x,t)\frac{\partial}{\partial t}v(x,t)-v(x,t)\frac{\partial}{\partial x}[\rho(x,t)\, v(x,t)]. \tag{D8}$$

Thus, if the spatial derivative of the product of $\rho(x,t)$ and $v^2(x,t)$ is added to both sides of Eq. (D7), we obtain that

$$\begin{aligned}\rho(x,t)\frac{\partial}{\partial t}v(x,t)+\rho(x,t)\, v(x,t)\frac{\partial}{\partial x}v(x,t)&\\ =\frac{F(x)}{m}+S(x,t)-\frac{\partial}{\partial x}\{\rho(x,t)\, v_q(x,t)&\\ \times[v_q(x,t)+2\, v_c(x,t)]\},&\end{aligned} \tag{D9}$$

where $v_q(x,t)$ is defined according to Eq. (15) for the current $j_q(x,t)$ and the probability density $\rho(x,t)$. Consequently, if the function $T(x,t)$ is defined so that

$$\begin{aligned}\frac{\partial}{\partial x}T(x,t)=-S(x,t)+\frac{\partial}{\partial x}\{\rho(x,t)\, v_q(x,t)&\\ \times[v_q(x,t)+2\, v_c(x,t)]\},&\end{aligned} \tag{D10}$$

it vanishes in the classical limit, and the system of equations (D3) and (D9) can be represented as Eqs. (31) and (32).

We can conclude that any modification of the formula for the average particle velocity is irrelevant as far as the uncertainty principle is concerned, since the velocity field can always be changed so that Eq. (27) is satisfied. Moreover, any modification of the right-hand side of the continuity equation can be compensated by a change of the right-hand side of the evolution equation for the velocity field. Therefore, we can assume that the average particle velocity is given by Eq. (27) and that Eq. (31) is valid when the uncertainty principle for particle momentum and position is satisfied.

APPENDIX E

In the present appendix we show that Eq. (25) is satisfied if Eq. (28) is, which means that Eq. (28) can be treated as a sufficient condition for the satisfaction of the uncertainty principle for particle momentum and position.

For this purpose we first assume that Eq. (28) is satisfied and observe that, according to it, the square of the left-hand side of Eq. (25) is given by

$$\sigma_x^2\,\sigma_v^2=\sigma_x^2\left[\int dx\,v^2(x)\,\rho(x)-\langle v\rangle^2\right]+\sigma_x^2\frac{\hbar^2}{4m^2}\int dx\,\frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\rho(x)\right]^2, \tag{E1}$$

where the irrelevant time dependence of all quantities is neglected. We also note that the average particle velocity is given by Eq. (27), according to the discussion presented in Appendix D.

Next, we show that the first term on the right-hand side of Eq. (E1) is never less than zero. Namely, we observe that, according to Eq. (27), the following equation is satisfied:

$$\int dx\,[v(x)-\langle v\rangle]^2\rho(x)=\int dx\,v^2(x)\,\rho(x)-\langle v\rangle^2, \tag{E2}$$

since the probability density $\rho(x)$ is normalized to unity. Hence, because the first term on the right-hand side of Eq. (E1) is a product of σ_x^2 and of the right-hand side of Eq. (E2), it cannot be negative. This means, however, that

$$\sigma_x^2\,\sigma_v^2\geqslant\sigma_x^2\frac{\hbar^2}{4m^2}\int dx\,\frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\rho(x)\right]^2. \tag{E3}$$

Therefore, the proof will be complete if we show that the right-hand side of Eq. (E3) is not less than $(\hbar^2/4m^2)$.

In order to show this, we consider the quantity $Z(\xi)$ defined by [17]

$$Z(\xi)=\int dx\left\{\left[\frac{\partial}{\partial x}-\xi\,(x-\langle x\rangle)\right]\sqrt{\rho(x)}\right\}^2, \tag{E4}$$

and find that

$$Z(\xi)=\sigma_x^2\,\xi^2+\xi+\frac{1}{4}\int dx\,\frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\rho(x)\right]^2, \tag{E5}$$

since $\rho(x)$ vanishes on the system boundaries. Hence, because the quadratic form $Z(\xi)$ cannot be negative, according to Eq. (E4), we have

$$\Delta=1-\sigma_x^2\int dx\,\frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\rho(x)\right]^2\leqslant 0. \tag{E6}$$

Consequently, we can state that

$$\sigma_x^2\frac{\hbar^2}{4m^2}\int dx\,\frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\rho(x)\right]^2\geqslant\frac{\hbar^2}{4m^2}, \tag{E7}$$

which is the required result, since the left-hand side of this inequality is the right-hand side of Eq. (E3).

APPENDIX F

In the present appendix we provide the entire derivation of Eq. (34), which begins with differentiating Eq. (30) with respect to time.

For this purpose we first transform Eq. (30) in the following manner:

$$U[\rho(x,t)]=\frac{\hbar^2}{2m}\int dx\left[\frac{\partial}{\partial x}R(x,t)\right]^2, \tag{F1}$$

where $R(x,t)=\sqrt{\rho(x,t)}$. Then, if Eq. (F1) is differentiated with respect to time, we find that

$$\frac{d}{dt}U[\rho(x,t)]=\frac{\hbar^2}{m}\int dx\frac{\partial R}{\partial x}\,\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial t}\right)=-\frac{\hbar^2}{m}\int dx\,\frac{\partial^2 R}{\partial x^2}\,\frac{\partial R}{\partial t}, \tag{F2}$$

where the resulting equation is integrated by parts with the assumption that $\rho(x,t)$ as well as its spatial derivative vanish on the system boundaries.

Next, if the continuity equation, Eq. (4), is used for eliminating the time derivative of the function $R(x,t)$ from Eq. (F2) and the resulting equation is integrated by parts, we obtain that

$$\frac{d}{dt}U[\rho]=-\frac{\hbar^2}{2m}\int dx\,v\,\rho\,\frac{\partial}{\partial x}\left(\frac{1}{R}\,\frac{\partial^2 R}{\partial x^2}\right), \tag{F3}$$

which can be represented by means of the functions $\rho(x,t)$ and $v(x,t)$, i.e.,

$$\frac{d}{dt}U[\rho]=-\frac{\hbar^2}{2m}\int dx\,v\,\rho\,\frac{\partial}{\partial x}\left[\frac{1}{2\rho}\,\frac{\partial^2\rho}{\partial x^2}-\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2\right]. \tag{F4}$$

Finally, we observe that there is the following identity which holds for an arbitrary three-times-differentiable function $f(x)$:

$$f\,\frac{d}{dx}\left[\frac{1}{2f^2}\left(\frac{df}{dx}\right)^2-\frac{1}{f}\,\frac{d^2f}{dx^2}\right]=\frac{d}{dx}\left[\frac{1}{f}\left(\frac{df}{dx}\right)^2-\frac{d^2f}{dx^2}\right], \tag{F5}$$

and apply it to the probability density $\rho(x,t)$. Hence, with the help of Eq. (F4) we obtain the required result, Eq. (34).

APPENDIX G

In classical mechanics, if a system is described by Eq. (38) and there is a statistical ensemble of copies of this system that can be described by means of a probability density $f(x,p,t)$, the corresponding functions $\rho_c(x,t)$ and $j^{(n)}(x,t)$, defined by Eqs. (10) and (14), respectively, satisfy Eqs. (11) and (39), respectively.

In order to show this we consider a subset of N copies of the system, selected from the statistical ensemble described

by $f(x,p,t)$, and define the function

$$\rho_N(x,t)=\frac{1}{N}\sum_{j=1}^{N}\delta[x-x_j(t)], \tag{G1}$$

where $\delta(x)$ is the Dirac δ distribution and the functions $x_j(t)$ describe the time dependence of positions of particles in the corresponding copies of the system. Of course, the functions $x_j(t)$ satisfy Eq. (38).

Now, if the current $j_N(x,t)$ is defined in the following manner:

$$j_N(x,t)=\frac{1}{N}\sum_{j=1}^{N}v_j(t)\,\delta[x-x_j(t)], \tag{G2}$$

where $v_j(t)=(dx_j/dt)$, and $\rho_N(x,t)$ is differentiated with respect to time, we find that

$$\frac{\partial}{\partial t}\rho_N(x,t)=-\frac{\partial}{\partial x}j_N(x,t). \tag{G3}$$

Similarly, if the functions $j_N^{(n)}(x,t)$ are defined by

$$j_N^{(n)}(x,t)=\frac{1}{N}\sum_{j=1}^{N}v_j^n(t)\,\delta[x-x_j(t)], \tag{G4}$$

and differentiated with respect to time, we get

$$\frac{\partial}{\partial t}j_N^{(n)}(x,t)=\frac{n}{N}\sum_{j=1}^{N}a_j(t)\,v_j^{n-1}(t)\times\delta[x-x_j(t)]-\frac{\partial}{\partial x}j_N^{(n+1)}(x,t), \tag{G5}$$

where $a_j(t)=(d^2x_j/dt^2)$. Therefore, because the functions $x_j(t)$ satisfy Eq. (38), we obtain that

$$\frac{\partial}{\partial t}j_N^{(n)}(x,t)=n\frac{F(x)}{m}j_N^{(n-1)}(x,t)-n\,\gamma\,j_N^{(n)}(x,t)-\frac{\partial}{\partial x}j_N^{(n+1)}(x,t). \tag{G6}$$

We now note that the functions $\rho_c(x,t)$ and $j^{(n)}(x,t)$ are approximated by $\rho_N(x,t)$ and $j_N^{(n)}(x,t)$, respectively. Namely, one can define the quantity

$$P_N(x,t)=\int_{x_0}^{x}dq\,\rho_N(q,t), \tag{G7}$$

where x_0 is an arbitrary fixed spatial coordinate, which is a staircase function of x having jump discontinuities [20]. Then, the continuous function $P(x,t)$, defined by

$$P(x,t)=\int_{x_0}^{x}dq\,\rho_c(q,t), \tag{G8}$$

is approximated by $P_N(x,t)$. The greater the number N, the better the approximation. Therefore, the function $P(x,t)$ can be treated as a limit of a sequence of functions $P_N(x,t)$ for N tending to infinity, since any continuous function can be approximated by staircase functions with arbitrary precision [20]. Accordingly, the function $\rho_c(x,t)$ can be treated as a limit of a sequence of functions $\rho_N(x,t)$ for N tending to infinity. Of course, similar arguments can be applied to the functions $j_N^{(n)}(x,t)$ and $j^{(n)}(x,t)$.

Finally, we observe that Eqs. (G3) and (G6) are linear with respect to the functions $\rho_N(x,t)$ and $j_N^{(n)}(x,t)$. Therefore, they are satisfied in the limit of N tending to infinity. Consequently, Eqs. (11) and (39) are satisfied by $\rho_c(x,t)$ and $j^{(n)}(x,t)$ and the proof is complete.

APPENDIX H

In the present appendix we show that in classical mechanics Eq. (18) is satisfied for any n and for any instant of time t if it is satisfied for any n at the initial instant of time t_0, provided that the underlying single-particle system is described by Eq. (38).

For this purpose we observe that if the underlying system is described by Eq. (38), Eqs. (11) and (39) are satisfied by the functions $\rho_c(x,t)$ and $j^{(n)}(x,t)$, defined by Eqs. (10) and (14), respectively, which is shown in Appendix G. Therefore, if the functions $w^{(n)}(x,t)$ are introduced according to Eq. (A1), we find that

$$\frac{\partial}{\partial t}w^{(n)}(x,t)=n\frac{F(x)}{m}w^{(n-1)}(x,t)-n\,\gamma\,w^{(n)}(x,t)+n\,v_c^{n-1}(x,t)\frac{\partial}{\partial x}w^{(2)}(x,t)-\frac{\partial}{\partial x}w^{(n+1)}(x,t). \tag{H1}$$

Hence, if Eq. (H1) is differentiated with respect to time, we obtain an equation that is very similar to Eq. (A2). Accordingly, the right-hand side of this equation can be transformed by eliminating the time derivatives of $w^{(n)}(x,t)$ with the help of Eq. (H1). Consequently, if all the functions $w^{(n)}(x,t_0)$ are equal to zero, the right-hand side of this equation vanishes and the second-order time derivatives of $w^{(n)}(x,t)$ are equal to zero at the time t_0.

Repeating this step we can show that the time derivatives of $w^{(n)}(x,t)$ of all orders are equal to zero at time t_0 if all $w^{(n)}(x,t_0)$ vanish, in which case all functions $w^{(n)}(x,t)$ must not evolve. On the other hand, all the functions $w^{(n)}(x,t_0)$ vanish if Eq. (18) is satisfied at the time t_0 for any n. Hence, if Eq. (18) is satisfied at time t_0 for any n, all functions $w^{(n)}(x,t)$ vanish for any t, which means that Eq. (18) is satisfied for any n and for any t. Therefore, the proof is complete.

APPENDIX I

In classical mechanics, if a system is described by Eq. (38), the average particle energy relative to a statistical ensemble of copies of this system is given by Eq. (23). Therefore, the average energy satisfies Eq. (44).

In order to show this, we differentiate Eq. (23) with respect to time and obtain that

$$\frac{\partial}{\partial t}\langle E\rangle_t=\frac{m}{2}\int dx\,\frac{\partial}{\partial t}j^{(2)}(x,t)+\int dx\,V(x)\,\frac{\partial}{\partial t}\rho(x,t). \tag{I1}$$

Then, according to Eqs. (11) and (39), we get

$$\frac{\partial}{\partial t}\langle E\rangle_t=\int dx\,F(x)\,j^{(1)}(x,t)-m\,\gamma\int dx\,j^{(2)}(x,t) - \int dx\,V(x)\,\frac{\partial}{\partial x}j^{(1)}(x,t). \tag{I2}$$

Next, integrating by parts and using the observation that $j^{(1)}(x,t)$ vanishes on the system boundaries, we find that the last term on the right-hand side of Eq. (I2) is equal to the first term on the right-hand side of it with the opposite sign. Consequently, we have

$$\frac{\partial}{\partial t}\langle E\rangle_t=-m\,\gamma\int dx\,j^{(2)}(x,t), \tag{I3}$$

which is the required result, Eq. (23).

APPENDIX J

The Schrödinger-Langevin equation obtained in Sec. VII has the important property that for a two-particle system it can be factorized into two equations having the form of Eq. (62) if there is no correlation between the motion of the two particles [21]. Thus, for such a system, the wave function satisfying the Schrödinger-Langevin equation is separable. It turns out that the corresponding factorization occurs in classical mechanics for a semipure statistical ensemble of copies of the two-particle system.

In order to show this, we proceed in analogy with the discussion presented in Sec. VI. Namely, we first consider a statistical ensemble of copies of the two-particle system described by the following Newton equations of motion:

$$\frac{d^2x_1}{dt^2}=-\gamma\frac{dx_1}{dt}+\frac{F(x_1)}{m_1},\quad \frac{d^2x_2}{dt^2}=-\gamma\frac{dx_2}{dt}+\frac{F(x_2)}{m_2}, \tag{J1}$$

where $x_1(t)$, $x_2(t)$, and m_1, m_2 denote the positions and masses of the particles, respectively. We assume that the statistical ensemble of copies of the system can be described by means of the probability density function $f(x_1,p_1,x_2,p_2,t)$, where p_1, p_2 are the momenta of the two particles. Then we define the probability density $\rho(x_1,x_2,t)$ and the functions $j^{(n,m)}(x_1,x_2,t)$ in the following manner:

$$\rho(x_1,x_2,t)=\int dp_1\,dp_2\,f(x_1,p_1,x_2,p_2,t), \tag{J2}$$

$$j^{(n,m)}(x_1,x_2,t)=\frac{1}{m_1^n m_2^m}\int dp_1\,dp_2\,p_1^n p_2^m\,f(x_1,p_1,x_2,p_2,t). \tag{J3}$$

Next, we observe that these functions satisfy the equations

$$\frac{\partial}{\partial t}\rho(x_1,x_2,t)=-\frac{\partial}{\partial x_1}j^{(1,0)}(x_1,x_2,t)-\frac{\partial}{\partial x_2}j^{(0,1)}(x_1,x_2,t), \tag{J4}$$

$$\begin{aligned}\frac{\partial}{\partial t}j^{(n,m)}(x_1,x_2,t)&=n\frac{F(x_1)}{m_1}j^{(n-1,m)}(x_1,x_2,t)-n\,\gamma\,j^{(n,m)}(x_1,x_2,t)\\&+m\frac{F(x_2)}{m_2}j^{(n,m-1)}(x_1,x_2,t)-m\,\gamma\,j^{(n,m)}(x_1,x_2,t)\\&-\frac{\partial}{\partial x_1}j^{(n+1,m)}(x_1,x_2,t)-\frac{\partial}{\partial x_2}j^{(n,m+1)}(x_1,x_2,t),\end{aligned} \tag{J5}$$

which can be shown in analogy with the discussion presented in Appendix G.

We now note that, according to the discussion presented in Sec. IV, the statistical ensemble of copies of the system under consideration is semipure if the functions $j^{(n,m)}(x_1,x_2,t)$ are given by

$$j^{(n,m)}(x_1,x_2,t)=v_1^n(x_1,x_2,t)\,v_2^m(x_1,x_2,t)\,\rho(x_1,x_2,t), \tag{J6}$$

where the velocity fields $v_1(x_1,x_2,t)$ and $v_2(x_1,x_2,t)$ are defined in the manner of Eq. (15). Hence, if the considered statistical ensemble of copies of the system is semipure, we have

$$\frac{\partial}{\partial t}\rho(x_1,x_2,t)+\frac{\partial}{\partial x_1}[\rho(x_1,x_2,t)\,v_1(x_1,x_2,t)] +\frac{\partial}{\partial x_2}[\rho(x_1,x_2,t)\,v_2(x_1,x_2,t)]=0, \tag{J7}$$

$$\begin{aligned}\frac{\partial}{\partial t}v_1(x_1,x_2,t)&=-v_1(x_1,x_2,t)\frac{\partial}{\partial x_1}v_1(x_1,x_2,t)\\&-v_2(x_1,x_2,t)\frac{\partial}{\partial x_2}v_1(x_1,x_2,t)\\&-\gamma\,v_1(x_1,x_2,t)+\frac{F(x_1)}{m_1},\end{aligned} \tag{J8}$$

$$\frac{\partial}{\partial t}v_2(x_1,x_2,t) = -v_1(x_1,x_2,t)\frac{\partial}{\partial x_1}v_2(x_1,x_2,t) - v_2(x_1,x_2,t)\frac{\partial}{\partial x_2}v_2(x_1,x_2,t) - \gamma\, v_2(x_1,x_2,t) + \frac{F(x_2)}{m_2}. \tag{J9}$$

Finally, we observe that the two particles of which the underlying system consists are independent. Therefore, the probability density $f(x_1,p_1,x_2,p_2,t)$ can be factorized into two probability density functions:

$$f(x_1,p_1,x_2,p_2,t) = f_1(x_1,p_1,t)\, f_2(x_2,p_2,t), \tag{J10}$$

since the two pairs of random variables x_1, p_1 and x_2, p_2 are then statistically independent [20]. Consequently, according to Eqs. (J2) and (J3), we obtain that

$$\rho(x_1,x_2,t) = \rho_1(x_1,t)\,\rho_2(x_2,t), \tag{J11}$$

$$j^{(1,0)}(x_1,x_2,t) = j_1(x_1,t)\,\rho_2(x_2,t), \tag{J12}$$

$$j^{(0,1)}(x_1,x_2,t) = \rho_1(x_1,t)\, j_2(x_2,t), \tag{J13}$$

in which case the velocity field $v_1(x_1,x_2,t)$ is independent of x_2 and the velocity field $v_2(x_1,x_2,t)$ is independent of x_1. Thus, because Eq. (J7) can be transformed in the following manner:

$$\left\{\frac{\partial}{\partial t}\rho_2(x_2,t) + \frac{\partial}{\partial x_2}[\rho_2(x_2,t)\, v_2(x_2,t)]\right\} = -\frac{\rho_2(x_2,t)}{\rho_1(x_1,t)}\left\{\frac{\partial}{\partial t}\rho_1(x_1,t) + \frac{\partial}{\partial x_1}[\rho_1(x_1,t)\, v_1(x_1,t)]\right\}, \tag{J14}$$

we find that each of the two pairs of functions $\rho_1(x_1,t)$, $v_1(x_1,t)$ and $\rho_2(x_2,t)$, $v_2(x_2,t)$ satisfies Eqs. (42) and (43). Therefore, if the statistical ensemble of copies of the system under consideration is semipure, it can be split into two independent statistical ensembles of copies of single-particle systems, and this corresponds to the factorization of the related Schrödinger-Langevin equation.

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