Factorized representation for parity-projected Wigner $\mathbf{d}^j(\beta)$ matrices

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An alternative representation for the parity-projected Wigner $\mathbf{d}^j(\beta)$ rotation matrix is derived as the product of two triangular matrices composed of Gegenbauer polynomials with negative and positive upper indices, respectively. We relate this representation for $\mathbf{d}^j(\beta)$ to the one presented by Matveenko [Phys. Rev. A **59**, 1034 (1999)], which, in contrast with our result, requires for its evaluation a matrix inversion. In addition, identities for bilinear sums of Gegenbauer polynomials are derived. This work is based on our recently introduced invariant representations for finite rotation matrices [Phys. Rev. A **57**, 3233 (1998)].

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I. INTRODUCTION

Finite rotation matrices (FRM) $R^j_{km}(\Omega)$ and especially their parametrization by Euler angles $\Omega=(\alpha,\beta,\gamma)$ [in which case one speaks of Wigner functions, $D^j_{km}(\alpha,\beta,\gamma)$] are fundamental objects of the quantum theory of angular momentum [1]. They describe the transformation of an irreducible tensor of rank j (such as, e.g., the wave function of a quantum system having total angular momentum j) under a rotation Ω of the coordinate frame. These objects have an interdisciplinary interest and new results have an evident importance and usefulness in various applications. The nontrivial parts of the Wigner functions are the β-dependent $\mathbf{d}^j(\beta)$ matrices, defined by

$$D^j_{km}(\alpha,\beta,\gamma)=\exp(-ik\alpha)d^j_{km}(\beta)\exp(-im\gamma),$$

where $-j\leqslant k,m\leqslant j$. These have well-known representations in terms of trigonometric, hypergeometric, or Jacobi polynomial functions of β [1]. For physical problems having definite parity (or exchange symmetry, e.g., in the case of two-electron wave functions), the so-called parity-projected FRM's or Wigner functions are convenient. These functions are symmetrized with respect to the first (k) or second (m) lower index. The symmetrized combinations of Wigner functions were introduced for the first time by Fano [2] in his analysis of real representations for finite rotation matrices. The usefulness of parity-projected Wigner functions in many-body problems was demonstrated by Bhatia and Temkin [3], who analyzed the angular dependence of two-electron wave functions. Various authors have used slightly different phase and normalization conventions for such parity-projected functions (see, e.g., Refs. [3–6]). Recently, Matveenko [6] derived a new analytic expression for parity-projected combinations of Wigner $\mathbf{d}^j(\beta)$ matrices as a factorized product of two matrices, one of which is defined in terms of (renormalized) associated Legendre polynomials, and the other is the inverse of a similarly defined matrix. In this paper we employ the invariant representations for FRM's introduced in Ref. [5] to obtain a simple form for parity-projected Wigner functions as the product of two triangular matrices, each of which is given explicitly in terms of well-known classical (Gegenbauer) polynomials. In addition, we are able to give an alternative, simple derivation of Matveenko's result [6], thereby also establishing a relation between our result and his. As noted by Matveenko [6], factorized representations of the parity-projected Wigner rotation matrices should prove useful in the theoretical analysis and representation of interacting three-body states.

II. PARITY-PROJECTED ''MINIMAL'' BIPOLAR HARMONICS AND FRM'S

In order to present our results in the most compact way, it is useful to first rewrite some of the key results of Ref. [5] in more symmetric forms. Parity-projected FRM's, denoted by $R^{j\pm}_{km}(\Omega)$, were defined in Ref. [5] as follows:

$$R^{j+}_{km}(\Omega)=R^j_{-km}(\Omega)+(-1)^kR^j_{km}(\Omega), \tag{1}$$

$$R^{j-}_{km}(\Omega)=-i[R^j_{-km}(\Omega)-(-1)^kR^j_{km}(\Omega)],\quad k>0. \tag{2}$$

However, it is more convenient for our present purposes to define slightly different matrices, denoted by $R^{j\lambda_p}_{km}(\Omega)$, having a unified form for both + and − cases (corresponding, respectively, to $\lambda_p=0,1$):

$$R^{j\lambda_p}_{km}(\Omega)=\left(1-\frac{\delta_{0,k}}{2}\right)[R^j_{-km}(\Omega)+(-1)^{k+\lambda_p}R^j_{km}(\Omega)], \tag{3}$$

where $k\geqslant\lambda_p$. In Eqs. (47) and (48) of Ref. [5], the FRM's $R^{j\pm}_{km}(\Omega)$ are presented as expansions on ''minimal'' bipolar harmonics (MBH) depending on two noncolinear unit vectors, $\mathbf{n}$ and $\mathbf{n}'$:

$$\mathcal{Y}^s_{jm}(\mathbf{n},\mathbf{n}')=\{Y_{j-s}(\mathbf{n})\otimes Y_s(\mathbf{n}')\}_{jm}, \tag{4}$$

where $s=0,1,\dots,j$. $Y_{a\alpha}$ denotes the spherical harmonic, and standard definitions of angular-momentum techniques are used [1]. The unit vector $\mathbf{n}$ is directed along the z axis of the ''old'' (fixed) frame K; $\mathbf{n}'$ lies in the zx plane of K; and θ is the angle between $\mathbf{n}$ and $\mathbf{n}'(0<\theta<\pi)$. Thus, the three real parameters of the rotation, Ω (such as, e.g., the Euler angles for the Wigner representation of the FRM), in our approach are determined by angular coordinates of vectors $\mathbf{n}$ and $\mathbf{n}'$ in the ''new'' (rotated) frame K'. Moreover, in Ref. [5] different sets of MBH,

$$\mathcal{Y}^{s}_{jm}(\mathbf{n},\mathbf{n}') \quad \text{and} \quad \{[\mathbf{n}\times\mathbf{n}']\otimes\mathcal{Y}^{s}_{j-1}(\mathbf{n},\mathbf{n}')\}_{jm},$$

enter the expansion of $R^{j+}_{km}(\Omega)$ and $R^{j-}_{km}(\Omega)$, respectively. The identity [see Eq. (B.6) of Ref. [7]]

$$\{[\mathbf{n}\times\mathbf{n}']\otimes\mathcal{Y}^{s}_{j-1}(\mathbf{n},\mathbf{n}')\}_{jm}=i\sqrt{\frac{j+1}{(2s+3)(2j-2s+1)}}\times\{Y_{j-s}(\mathbf{n})\otimes Y_{s+1}(\mathbf{n}')\}_{jm}, \tag{5}$$

allows us to introduce a unified basis set of parity-projected MBH's,

$$\mathcal{Y}^{s\lambda_p}_{jm}(\mathbf{n};\mathbf{n}')=\{Y_{j-s}(\mathbf{n})\otimes Y_{s+\lambda_p}(\mathbf{n}')\}_{jm}=\sum_{\alpha\beta} C^{jm}_{j-s\,\alpha\, s+\lambda_p\beta}Y_{j-s\,\alpha}(\mathbf{n})Y_{s+\lambda_p\beta}(\mathbf{n}'), \tag{6}$$

for both cases, i.e., $\lambda_p=0$ (or+) and $\lambda_p=1$ (or−), where the $C^{c\gamma}_{a\alpha b\beta}$ denote Clebsch-Gordan coefficients.

Using definitions in Eqs. (3) and (6), Eqs. (47) and (48) of Ref. [5] for the FRM's $R^{j\pm}_{km}(\Omega)$ may be written in the following compact form in terms of $R^{j\lambda_p}_{km}(\Omega)$ and $\mathcal{Y}^{s\lambda_p}_{jm}(\mathbf{n};\mathbf{n}')$:

$$R^{j\lambda_p}_{km}(\Omega)=\frac{4\pi}{(\sin\theta)^k}\sum_{s=0}^{k-\lambda_p} A^{(\lambda_p)}_{ks}\mathcal{C}^{(1/2)-k}_{k-s-\lambda_p}(\cos\theta)\mathcal{Y}^{s\lambda_p}_{jm}(\mathbf{n};\mathbf{n}'), \tag{7}$$

where $k\geqslant\lambda_p$, and the factor A is given by

$$A^{(\lambda_p)}_{ks}=2(k)^{1-\lambda_p}\frac{(k+s+\lambda_p-1)!}{(2k-1)!!}\sqrt{\frac{(j+\lambda_p)!(j-s-\lambda_p)!(2j-1)!!}{s!(j+k)!(j-k)!(2j-2s+1)!!(2s+2\lambda_p+1)!!}},$$

where $A^{(\lambda_p)}_{00}\equiv\delta_{0,\lambda_p}$ and where the $\mathcal{C}^{(1/2)-k}_{k-s-\lambda_p}(\cos\theta)$ are Gegenbauer polynomials with negative upper indices [8].

It is important to note that the set of $(2j+1)$ MBH's in Eq. (6) with $\lambda_p=0,1$ and $s=0,\dots,j-\lambda_p$ form a basis set of irreducible tensors with integer rank j [5,7] and that Eq. (7) is an example of an expansion in this basis [9]. For $\lambda_p=0$, the MBH's in Eq. (6) are polar tensors, while for $\lambda_p=1$ they are axial tensors (pseudotensors). In the terminology of Ref. [6], $R^{j\lambda_p}_{km}(\Omega)$ with $\lambda_p=0$ $(\lambda_p=1)$ is said to have ''normal'' (''abnormal'') parity. Equation (7) is the simplest form of the invariant representations for FRM's derived in Ref. [5]. In what follows, we analyze the algebraic properties of the representation (7) and utilize it for various special choices of the parameter θ.

III. FACTORIZED FORM OF $\mathbf{d}^j$

The right-hand side of Eq. (7) is the product of two matrices: one of them, involving Gegenbauer polynomials, has (lower left) triangular form; the other one has matrix elements which are components of a MBH [cf. Eq. (6)] with indices s and m. Although for an arbitrary parameter θ the factorized form of the FRM is rather complicated, it is possible to find a simple factorized form for the Wigner $\mathbf{d}^j(\beta)$ matrix by considering the (auxiliary) rotation from K to K' described by the Euler angles $\alpha=\gamma=0$, $\beta=\theta$. For such a rotation, Eq. (7) reduces to

$$d^{j\lambda_p}_{km}(\theta)=\frac{4\pi}{(\sin\theta)^k}\sum_{s=0}^{s_{\max}} A^{(\lambda_p)}_{ks}\mathcal{C}^{(1/2)-k}_{k-s-\lambda_p}(\cos\theta)\times\mathcal{Y}^{s\lambda_p}_{jm}(-\theta,0;0,0), \tag{8}$$

where $s_{\max}=\min(k-\lambda_p,j-m)$ and the parity-projected combination of $\mathbf{d}^j$ matrices is defined by Eq. (3), taking into account the relation $R^j_{km}(0,\theta,0)=d^j_{km}(\theta)$. The MBH on the right-hand side of Eq. (8) may be calculated explicitly as

$$4\pi\mathcal{Y}^{s\lambda_p}_{jm}(-\theta,0;0,0)=\sqrt{4\pi(2s+2\lambda_p+1)}\times C^{jm}_{j-s\,m\,s+\lambda_p\,0}Y_{j-s\,m}(-\theta,0)=B^{(\lambda_p)}_{sm}\mathcal{P}^m_{j-s}(\cos\theta). \tag{9}$$

Here $\mathcal{P}^m_{j-s}(\cos\theta)$ is an associated Legendre polynomial [1], which is related to a Gegenbauer polynomial with positive upper index by the identity

$$\mathcal{P}^\alpha_a(\cos\theta)=(2\alpha-1)!!(-\sin\theta)^\alpha\mathcal{C}^{(1/2)+\alpha}_{a-\alpha}(\cos\theta),\qquad \alpha\geqslant0.$$

The coefficient $B^{(\lambda_p)}_{sm}$ in Eq. (9) is defined by

$$B_{sm}^{(\lambda_p)}=\frac{(-1)^m m^{\lambda_p}}{(j-s+m)!}\sqrt{\frac{(2s+2\lambda_p+1)!!(j-s-\lambda_p)!(2j-2s+1)!!(j-m)!(j+m)!}{s!(j+\lambda_p)!(2j-1)!!}}. \tag{10}$$

For the simplest presentation of results, it is convenient to introduce the renormalized matrix $\tilde{d}_{km}^{j\lambda_p}(\theta)$, where

$$d_{km}^{j\lambda_p}(\theta)=\sqrt{\frac{(j+m)!(j-m)!}{(j+k)!(j-k)!}}\left(\frac{\sin\theta}{2}\right)^{m-k}\tilde{d}_{km}^{j\lambda_p}(\theta). \tag{11}$$

Substituting Eq. (11) into Eq. (8), the matrix $\tilde{\mathbf{d}}$ is obtained as the product of two triangular matrices,

$$\tilde{\mathbf{d}}^{j\lambda_p}(\theta)=\mathbf{C}^{(j,\lambda_p)}(\theta)\cdot\mathbf{P}^{(j,\lambda_p)}(\theta), \tag{12}$$

where the matrix elements of the lower left triangular matrix $\mathbf{C}$ and the upper left triangular matrix $\mathbf{P}$ are defined by

$$C_{ks}^{(j,\lambda_p)}(\theta)=\frac{(k-\lambda_p)!(k+s-1)!}{(s-\lambda_p)!\,(2k-1)!}\mathcal{C}_{k-s}^{(1/2)-k}(\cos\theta), \tag{13}$$

$$\begin{aligned}P_{sm}^{(j,\lambda_p)}(\theta)&=\frac{(2m)!(j-s)!}{(m-\lambda_p)!(j-s+m+\lambda_p)!}\mathcal{C}_{j-s-m+\lambda_p}^{(1/2)+m}(\cos\theta)\\&=\left(\frac{-2}{\sin\theta}\right)^m\frac{m^{\lambda_p}(j-s)!}{(j-s+m+\lambda_p)!}\mathcal{P}_{j-s+\lambda_p}^{m}(\cos\theta),\end{aligned} \tag{14}$$

where in these equations $\lambda_p \leqslant k,m,s \leqslant j$, and $C_{00}^{(j,\lambda_p)}(\theta)=1$.

Equations. (11)–(14) are the principal result of this paper. We note the surprisingly simple form and the intrinsic beauty of this new factorized form for $\mathbf{d}^j$: matrix elements of both triangular matrices are well-known classical (Gegenbauer) polynomials in $\cos\theta$ with rational coefficients; diagonal elements of $\mathbf{C}^{(j,\lambda_p)}$ and $\mathbf{P}^{(j,\lambda_p)}$ are equal to unity, i.e.,

$$C_{kk}^{(j,\lambda_p)}(\theta)=P_{m\,j-m+\lambda_p}^{(j,\lambda_p)}(\theta)=1. \tag{15}$$

Note that matrix elements of $\mathbf{C}^{(j,\lambda_p)}$ and $\mathbf{P}^{(j,\lambda_p)}$ do not depend on the value of j; rather, only their dimension depends on j. Indeed, both indices j and s enter Eq. (14) only in the combination $(j-s)$, which determines the dimensions of the matrices entering Eq. (12). These dimensions are j and $j+1$ for $\lambda_p=1$ and $\lambda_p=0$, respectively. As a consequence of the independence of the matrix elements on j, the matrices $\mathbf{C}^{(j+1,\lambda_p)}$ and $\mathbf{P}^{(j+1,\lambda_p)}$ [and obviously the $\mathbf{d}^{j+1,\lambda_p}(\beta)$ functions] can be calculated by simply adding one additional (lowest) row to $\mathbf{C}^{(j,\lambda_p)}$ and one additional (highest) row to $\mathbf{P}^{(j,\lambda_p)}$, as we illustrate in the example below, in which we present explicit forms for the matrices $\mathbf{C}^{(j,\lambda_p)}$ and $\mathbf{P}^{(j,\lambda_p)}$ for $j\leqslant 3$. The results for $\lambda_p=1$ are

$$\mathbf{C}^{(2,1)}(\theta)=\begin{pmatrix}1 & 0\\ -\cos\theta & 1\end{pmatrix}, \tag{16}$$

$$\mathbf{C}^{(3,1)}(\theta)=\begin{pmatrix}1 & 0 & 0\\ -\cos\theta & 1 & 0\\ \frac{1}{4}(3\cos^2\theta+1) & -2\cos\theta & 1\end{pmatrix}, \tag{17}$$

$$\mathbf{P}^{(2,1)}(\theta)=\begin{pmatrix}\cos\theta & 1\\ 1 & 0\end{pmatrix}, \tag{18}$$

$$\mathbf{P}^{(3,1)}(\theta)=\begin{pmatrix}\frac{1}{4}(5\cos^2\theta-1) & 2\cos\theta & 1\\ \cos\theta & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \tag{19}$$

and we recall from Eq. (15) that the diagonal elements are unity, so that the 1×1 matrices $\mathbf{C}^{(1,1)}$ and $\mathbf{P}^{(1,1)}$ are trivially unity. As noted above, we see that $\mathbf{C}^{(2,1)}$ and $\mathbf{P}^{(2,1)}$ are the upper left and lower left parts of the matrices $\mathbf{C}^{(3,1)}$ and $\mathbf{P}^{(3,1)}$, respectively. The result for $j=3,\lambda_p=0$ is

$$\mathbf{C}^{(3,0)}(\theta)=\begin{pmatrix}1 & 0 & 0 & 0\\ -\cos\theta & 1 & 0 & 0\\ \frac{1}{2}(\cos^2\theta+1) & -2\cos\theta & 1 & 0\\ -\frac{\cos\theta}{4}(\cos^2\theta+3) & \frac{3}{4}(3\cos^2\theta+1) & -3\cos\theta & 1\end{pmatrix}, \tag{20}$$

$$\mathbf{P}^{(3,0)}(\theta)=\left(\begin{array}{cccc}\frac{\cos\theta}{2}(5\cos^2\theta-3) & \frac{3}{4}(5\cos^2\theta-1) & 3\cos\theta & 1\\ \frac{1}{2}(3\cos^2\theta-1) & 2\cos\theta & 1 & 0\\ \cos\theta & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right). \tag{21}$$

The marked internal 2×2 and 3×3 matrices in these equations are the results for $j=1$, $\lambda_p=0$ and $j=2$, $\lambda_p=0$, respectively. We observe that $\mathbf{C}^{(1,0)}=\mathbf{C}^{(2,1)}$ and $\mathbf{P}^{(1,0)}=\mathbf{P}^{(2,1)}$.

IV. RELATION TO THE RESULT OF MATVEENKO

Matveenko's derivation [6] of a factorized form for the parity-projected combinations of $\mathbf{d}^j(\beta)$ matrices exploits a nontrivial technique of hyperspherical harmonics in terms of Jacobi vectors $\mathbf{X}$ and $\mathbf{x}$ of the three-body problem. In order to establish the connection of these results [6] with our results presented above, let us consider the transformation of the MBH in Eq. (6) under the auxiliary rotation $\Omega=(0,\theta,0)$ from the frame K to K', i.e., the same type of rotation as considered in Eq. (8). Under this rotation, the irreducible tensor in Eq. (6) is transformed using the parity-projected $\mathbf{d}^j$ matrix according to the relation

$$\mathcal{Y}^{s\lambda_p}_{jm}(-\theta,0;0,0)=\sum_{k=\lambda_p}^{s+\lambda_p}\mathcal{Y}^{s\lambda_p}_{j-k}(0,0;\theta,0)d^{j\lambda_p}_{km}(\theta). \tag{22}$$

This equation may be written as a matrix identity

$$\mathbf{B}^{(j,\lambda_p)}(-\theta)=\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)\cdot\mathbf{d}^{j\lambda_p}(\theta), \tag{23}$$

where matrix elements of the upper left triangular matrix $\mathbf{B}^{(j,\lambda_p)}(-\theta)$ are defined above by Eq. (9) and coincide with matrix elements of $\mathbf{P}^{(j,\lambda_p)}(\theta)$ [which are defined by Eq. (14)] up to numerical coefficients:

$$\mathbf{B}^{(j,\lambda_p)}_{sm}(-\theta)=B^{(\lambda_p)}_{sm}\mathcal{P}^m_{j-s}(\cos\theta). \tag{24}$$

The lower left triangular matrix $\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)$ is defined by an interchange of the rows of $\mathbf{B}^{(j,\lambda_p)}(-\theta)$, namely

$$\underline{\mathbf{B}}^{(j,\lambda_p)}_{sm}(\theta)=\mathbf{B}^{(j,\lambda_p)}_{j-s-\lambda_p\, m}(-\theta). \tag{25}$$

Thus, both matrices $\mathbf{B}^{(j,\lambda_p)}(-\theta)$ and $\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)$ are composed of associated Legendre polynomials (or, equivalently, of Gegenbauer polynomials with positive half-integer upper indices). Equation (23) is equivalent to Eqs. (26) and (29) of Ref. [6]. The main difference between Eq. (23) and the result in Eq. (12) is that the calculation of the $\mathbf{d}^{j\lambda_p}$ matrix as the solution of the matrix equation (23) requires the inversion of the matrix $\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)$. Note that there are some evident differences in the notation between the results in this paper and those of Ref. [6]. These differences stem from the fact that we consider the rotation of the coordinate frame (because it provides the standard definition for rotation matrices [1]), while Ref. [6] considers the rotation of the body-fixed frame (because it is convenient for the three-body problem). The relation between our matrices $d^{j\lambda_p}_{m'm}(\theta)$ and the slightly different matrices $d^{jp}_{mm'}(\theta)$ of Ref. [6] is

$$d^{j\lambda_p}_{m'm}(\theta)=(-1)^{\lambda_p+m'}\sqrt{\frac{1+\delta_{0,m}}{1+\delta_{0,m'}}}d^{jp}_{m'm}(\theta). \tag{26}$$

Taking account of this relation, Eq. (23) for $\lambda_p=0$ and for $\lambda_p=1$ coincides with the basic Eqs. (26) and (29) of Ref. [6].

V. APPLICATION TO INVERSION OF A MATRIX COMPOSED OF CLASSICAL POLYNOMIALS

From Eqs. (12) and (23), it follows that there must exist appropriate algebraic formulas for explicitly inverting a triangular matrix composed of Gegenbauer (or associated Legendre) polynomials. Indeed, such formulas can be derived from Eq. (7) by considering a zero rotation, $\Omega=(0,0,0)$. On the one hand, we must have $R^j_{km}(0)=\delta_{k,m}$, but, on the other hand, the direct use of Eq. (7) gives

$$R^{j\lambda_p}_{km}(0)=\frac{4\pi}{(\sin\theta)^k}\sum_{s=0}^{s_{\max}}A^{(\lambda_p)}_{ks}\mathcal{C}^{(1/2)-k}_{k-s-\lambda_p}(\cos\theta)\mathcal{Y}^{s\lambda_p}_{jm}(0,0;\theta,0), \tag{27}$$

where $k\geqslant\lambda_p$. Calculating the MBH on the right-hand side of Eq. (27) explicitly [cf. Eq. (9)], we arrive at the identity (for both $\lambda_p=0$ and 1)

$$\delta_{k,m}=2k\sum_{s=m}^{k}\frac{(k+s-1)!}{(s+m)!}\mathcal{C}^{(1/2)-k}_{k-s}(\cos\theta)\mathcal{C}^{(1/2)+m}_{s-m}(\cos\theta). \tag{28}$$

We believe Eq. (28) to be a new result. Equation (28) can also be written in the matrix form

$$\mathbf{I}=\mathbf{C}^{(j,\lambda_p)}(\theta)\cdot\underline{\mathbf{P}}^{(j,\lambda_p)}(\theta), \tag{29}$$

where $\mathbf{I}$ is the diagonal unit matrix, and the matrix $\underline{\mathbf{P}}^{(j,\lambda_p)}(\theta)$ is defined by an interchange of the rows of $\mathbf{P}^{(j,\lambda_p)}(\theta)$,

$$\underline{P}_{sm}^{(j,\lambda_p)}(\theta)=P_{j-s+\lambda_p\, m}^{(j,\lambda_p)}(\theta).$$

Noting the fact that if the product of two matrices is the unit matrix **I**, then these matrices are commuting, we obtain from Eq. (29) one additional interesting relation [cf. Eq. (12)]:

$$\underline{\mathbf{I}}=\mathbf{P}^{(j,\lambda_p)}(\theta)\cdot\mathbf{C}^{(j,\lambda_p)}(\theta), \tag{30}$$

where $\underline{\mathbf{I}}$ is the ''quasidiagonal'' unit matrix

$$\underline{\mathbf{I}}=\begin{pmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Matrix identities (12), (29), and (30) allow us to write the chain of equalities

$$\tilde{\mathbf{d}}^{j\lambda_p}(\theta)\cdot\tilde{\mathbf{d}}^{j\lambda_p}(\theta)=\mathbf{C}^{(j,\lambda_p)}\cdot\mathbf{P}^{(j,\lambda_p)}\cdot\mathbf{C}^{(j,\lambda_p)}\cdot\mathbf{P}^{(j,\lambda_p)}$$
$$=\mathbf{C}^{(j,\lambda_p)}\underline{\mathbf{I}}\cdot\mathbf{P}^{(j,\lambda_p)}=\mathbf{C}^{(j,\lambda_p)}\cdot\underline{\mathbf{P}}^{(j,\lambda_p)}=\mathbf{I}. \tag{31}$$

Thus, we have proved that $\mathbf{d}^{j\lambda_p}(\theta)\cdot\mathbf{d}^{j\lambda_p}(\theta)=\mathbf{I}$. This identity can be proved also using unitarity properties of Wigner functions. It is important that Eq. (30) can be used to control the accuracy of numerical calculations of matrices $\mathbf{P}^{(j,\lambda_p)}(\theta)$ and $\mathbf{C}^{(j,\lambda_p)}(\theta)$. Namely, if the calculated matrix product on the right-hand side of Eq. (30) differs from the exact matrix $\underline{\mathbf{I}}$, then one loses precision in one's calculations.

The identity (29) demonstrates the unexpected fact that for a triangular matrix composed of Gegenbauer polynomials $\mathcal{C}_n^{(1/2)-k}(\cos\theta)$ or $\mathcal{C}_n^{(1/2)+k}(\cos\theta)$ (or of associated Legendre polynomials), the explicit form of the inverse matrix exists, again in terms of Gegenbauer polynomials whose upper indices have the opposite sign to those of the initial (noninverted) matrix. In particular, we have

$$[\underline{B}^{(j,\lambda_p)}(\theta)]_{ms}^{-1}=\frac{A_{ms}^{(\lambda_p)}}{(\sin\theta)^m}\mathcal{C}_{m-s-\lambda_p}^{(1/2)-m}(\cos\theta). \tag{32}$$

Thus, matrix elements of the matrix $[\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)]^{-1}$ coincide with matrix elements of **C** [cf. Eqs. (13) and (32)] up to some coefficients. Using Eqs. (24) and (32), the expression for $\mathbf{d}^{j\lambda_p}(\theta)$,

$$\mathbf{d}^{j\lambda_p}(\theta)=[\underline{\mathbf{B}}^{(j,\lambda_p)}(\theta)]^{-1}\cdot\mathbf{B}^{(j,\lambda_p)}(-\theta),$$

is easily transformed to the form in Eqs. (11)–(14). This result gives an independent proof of our basic result in Eq. (12) and establishes the connection of our result with that of Ref. [6].

VI. CONCLUSIONS

Invariant representations of FRM's [5] [such as, e.g., Eq. (7)] are powerful tools of angular-momentum algebra. Utilizing these results for special Euler rotations, we have found in this paper a simple form for the parity-projected Wigner $\mathbf{d}^{j\lambda_p}$ matrices as the product of two triangular matrices. Both matrices are presented here explicitly in terms of well-known classical polynomials. The use of Eqs. (12)–(14) in problems having definite parity λ_p is much more convenient than the standard representation for $\mathbf{d}^j$ matrices in terms of Jacobi polynomials or hypergeometric functions [1]. Indeed, for fixed j and λ_p, according to Eq. (12) one needs to calculate only $(j-\lambda_p)(j-\lambda_p+1)$ different Gegenbauer polynomials instead of $j(2j+1)$ different Jacobi or hypergeometric polynomials. Furthermore, Eq. (12) is especially convenient for numerical evaluation of $\mathbf{d}^{j\lambda_p}$ matrices having different ranks j, because, as mentioned above, matrix elements of **C** and **P** do not depend on j, and hence the matrices for $j'=j+1$ may employ results calculated for j without recalculation. Concerning applications of the present results in analyses of many-body problems, we note that the angle θ in Eq. (7) is an arbitrary parameter, which may be considered, e.g., as the angle between a Jacobi vector pair **X** and **x** of the three-body problem [6], choosing **n** and **n**$'$ as the corresponding unit vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{x}}$. In this case the set of parity-projected MBH's in Eq. (6), i.e., a basis set of irreducible tensors with integer rank j, may be useful in the construction of a convenient angular basis for the three-body problem similar to the three-body angular basis discussed in Ref. [6].

We note finally that both our results here and those in Ref. [6] are applicable for integer j, which is the most important case for a number of applications. Invariant representations for FRM's for half-integer j may also be derived using the formal similarity of vectors to spinors with rank 1/2, thereby generalizing the results of Ref. [5]. Based on such a generalization, the factorized form of $\mathbf{d}^j$ matrices can be derived for half-integer j also and will be published elsewhere.

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[1] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
[2] U. Fano, J. Math. Phys. **1**, 417 (1960).
[3] A.K. Bhatia and A. Temkin, Rev. Mod. Phys. **36**, 1050 (1964).
[4] V.N. Ostrovsky and S.I. Nikitin, J. Phys. B **18**, 4349 (1985).
[5] N.L. Manakov, A.V. Meremianin, and A.F. Starace, Phys. Rev. A **57**, 3233 (1998).
[6] A.V. Matveenko, Phys. Rev. A **59**, 1034 (1999).
[7] N.L. Manakov, S.I. Marmo, and A.V. Meremianin, J. Phys. B

29, 2711 (1996).
[8] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Trancendental Functions,* Bateman Manuscript Project (McGraw-Hill, New York, 1953), Vol. II, Chap. 10, Sec. 10.9.
[9] This basis was used implicitly for the first time in the analysis of higher angular momentum eigenfunctions for the three-body problem in Hylleraas-Breit coordinates by C. Schwartz, Phys. Rev. **123**, 1700 (1961).