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Non-Abelian Berry connections for quantum computation

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In the holonomic approach to quantum computation, information is encoded in a degenerate eigenspace of a parametric family of Hamiltonians and manipulated by the associated *holonomic gates*. These are realized in terms of the non-Abelian Berry connection and are obtained by driving the control parameters along adiabatic loops. We show how it is possible for a specific model to explicitly determine the loops generating any desired logical gate, thus producing a universal set of unitary transformations. In a multipartite system unitary transformations can be implemented efficiently by sequences of local holonomic gates. Moreover, a conceptual scheme for obtaining the required Hamiltonian family, based on frequently repeated pulses, is discussed, together with a possible process whereby the initial state can be prepared and the final one can be measured.

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The field of quantum information and computation (QC) [1] brings together ideas and techniques from very different areas ranging from fundamental quantum physics to solid-state engineering and computer science. QC synergetically benefits from all these contributions and conversely quite often offers fresh viewpoints on old subjects. Recently it has been suggested [2] that even tools related to gauge theories [3] might play a fruitful role in the arena of QC. Indeed, in Ref. [2] the possibility of realizing quantum information processing by using non-Abelian Berry holonomies [4] induced by moving along suitable loops in a control space $\mathcal{M}$ has been analyzed. The computational capability stems from the features of the global geometry of the bundle of eigenspaces associated with a family $\mathcal{F}$ of Hamiltonians parametrized by points of $\mathcal{M}$. The geometry is described by a nontrivial gauge potential A or *connection*, with values in the algebra $\mathrm{u}(n)$ of anti-Hermitian matrices (n is the dimension of the computational space). Since the unitary transformations realizing the computations are nothing but the *holonomies* associated with the connection A, this conceptual framework for QC is referred to as holonomic quantum computation (HQC). In a sense HQC can be considered as the (continuous) differential-geometric counterpart of the (discrete) topological QC with anyons described in Refs. [5,6].

In this paper we shall provide further analysis of this proposal. After concisely reviewing the conceptual basis of HQC, we shall show how, in a specific relevant model, one can explicitly determine the sequence of loops necessary for generating any given quantum gate. Then we shall introduce HQC models with a natural multipartite structure and discuss how this bears on the question of complexity. Finally we shall discuss how in principle one can implement HQC by repeated pulse control of a system with a degenerate spectrum.

Let us begin by recalling the basic ideas of HQC [2]. Quantum information is encoded in an n-fold degenerate eigenspace $\mathcal{C}$ of a Hamiltonian H_0, with eigenvalue ε_0. The operator H_0 belongs to a family $\mathcal{F}=\{H_\lambda\}_{\lambda\in\mathcal{M}}, H_0=H_{\lambda_0}$, in which no energy-level crossings occur as λ ranges over $\mathcal{M}$. In the following we shall satisfy this latter condition by assuming, for simplicity, that the Hamiltonians H_λ are isospectral $[H_\lambda=\mathcal{U}(\lambda)H_0\mathcal{U}(\lambda)^\dagger]$. The λ's represent the ''control'' parameters that one has to drive in order to manipulate the coding states $|\psi\rangle\in\mathcal{C}$. In general, the points of $\mathcal{M}$, from the physical point of view, can be thought of as describing external fields, such as electric or magnetic fields, or couplings between subsystems. Let C be a *loop* in the control manifold $\mathcal{M}$, with base point λ_0, $C:[0,1]\mapsto\mathcal{M}, C(0)=C(1)=\lambda_0$. We assume that C is traveled along slowly with respect to the longest dynamical time scale involved: in this case the evolution is adiabatic; i.e., no transitions among different energy levels are induced. If $|\psi\rangle_{in}\in\mathcal{C}$ is the initial state, at the end of this control process one gets $|\psi\rangle_{out}=e^{i\varepsilon_0 T}\,\Gamma_A(C)|\psi\rangle_{in}$. The first factor here is just an overall dynamical phase and in the following it will be omitted; let us just mention that such a decoupling of the fast dynamical evolution opens new possibilities for coherent and error avoiding encoding [6]. The second contribution, the holonomy $\Gamma_A(C)\in U(n)$, has a purely geometric origin and its appearance accounts for the nontriviality (curvature) of the *bundle* of eigenspaces over $\mathcal{M}$. By introducing the Wilczek-Zee connection [7]

$$A^{\lambda_\mu}_{\bar{\alpha}\alpha}:=\left\langle \psi^{\bar{\alpha}}(\lambda)\left|\frac{\partial}{\partial\lambda_\mu}\right|\psi^{\alpha}(\lambda)\right\rangle, \qquad (1)$$

one finds $\Gamma_A(C)=\mathbf{P}\exp\int_C A$ [4], where $\mathbf{P}$ denotes path ordering. The set $\mathrm{HOL}(A):=\{\Gamma_A(C)\}_C\subset\mathrm{U}(n)$ is known as the holonomy group [8]. In the case in which it coincides with the whole unitary group $\mathrm{U}(n)$ the connection A is called *irreducible*. In [2] it has been argued that for a large enough control manifold, the irreducible case is the *generic* one; therefore one can in principle implement any computation over the *code* $\mathcal{C}$ just resorting to this very special class of quantum evolutions.

Quantum gates. A workable HQC model that represents a natural non-Abelian generalization of the original Berry phase, for which explicit construction of the holonomic gates is possible, is now discussed. The model is worked out with some details in that it is extendable to the more general case when $\mathcal{M}$ is a coset space. The features of the construction presented are twofold. On the one hand, it fully exploits the loop composition structure at the basis of the holonomy group, showing a procedure whereby loops can be decomposed into two-dimensional components, which are simple to deal with. On the other, this topological construction overcomes the difficulties connected with the path ordering prescription.

Let us consider the Hamiltonian $H_0=\varepsilon_0|n+1\rangle\langle n+1|$ acting on the state-space $\mathcal{H}\cong\mathbb{C}^{n+1}=\mathrm{span}\{|\alpha\rangle\}_{\alpha=1}^{n+1}$. We shall take as the family $\mathcal{F}$ the whole orbit $\mathcal{O}(H_0):=\{\mathcal{U}H_0\mathcal{U}^\dagger/\mathcal{U}\in\mathrm{U}(n+1)\}$ of H_0 under the (adjoint) action of the unitary group $\mathrm{U}(n+1)$. This orbit is isomorphic to the n-dimensional complex projective space:

$$\mathcal{O}(H_0)\cong\frac{\mathrm{U}(n+1)}{\mathrm{U}(n)\times\mathrm{U}(1)}\cong\frac{\mathrm{SU}(n+1)}{\mathrm{U}(n)}\cong\mathbf{CP}^n.$$

The points of $\mathbf{CP}^n$ can be parametrized by the unitary matrices $\mathcal{U}(z)=U_1(z_1)U_2(z_2),\ldots,U_n(z_n)$ where $U_\alpha(z_\alpha)=\exp[G_\alpha(z_\alpha)]$ with $G_\alpha(z_\alpha)=z_\alpha|\alpha\rangle\langle n+1|-\overline{z}_\alpha|n+1\rangle\langle\alpha|$, $z_\alpha=\theta_\alpha e^{i\phi_\alpha}$. The eigenstates of the rotated Hamiltonians are

$$\begin{aligned}|\alpha(\theta,\phi)\rangle&:=\mathcal{U}(\theta,\phi)|\alpha\rangle\\&=\cos\theta_\alpha|\alpha\rangle-\exp(-i\phi_\alpha)\sin\theta_\alpha\\&\quad\times\sum_{j>\alpha}^{n+1}\exp(i\phi_j)\sin\theta_j\prod_{j>\gamma>\alpha}\cos\theta_\gamma|j\rangle\end{aligned}$$

and

$$\begin{aligned}|n+1(\theta,\phi)\rangle&:=\mathcal{U}(\theta,\phi)|n+1\rangle\\&=\sum_{j=1}^{n+1}\exp(i\phi_j)\sin\theta_j\prod_{\gamma<j}\cos\theta_\gamma|j\rangle,\end{aligned}$$

where $\theta_{n+1}:=\pi/2$ and $\phi_{n+1}:=0$. Notice that for $n=1$ the standard two-level model with (Abelian) Berry phase is recovered. By using Eq. (1) the components of the connection can now be explicitly computed. The only nonzero elements of the matrix A^{θ_β} $(\beta=1,\ldots,n)$ are $A^{\theta_\beta}_{\overline{\alpha}\beta}$ for $\overline{\alpha}=1,\ldots,\beta-1$, given by

$$A^{\theta_\beta}_{\overline{\alpha}\beta}=e^{i(\phi_{\overline{\alpha}}-\phi_\beta)}\sin\theta_{\overline{\alpha}}\prod_{\beta>\gamma>\overline{\alpha}}\cos\theta_\gamma,\qquad(2)$$

as well as $A^{\theta_\beta}_{\overline{\alpha}\beta}=-A^{\theta_\beta}_{\beta\overline{\alpha}}$. The anti-Hermitian matrix A^{ϕ_β} has nonzero elements for $\alpha=\beta$ and $\alpha\geqslant\overline{\alpha}$ given by

$$A^{\phi_\beta}_{\overline{\alpha}\beta}=-ie^{i(\phi_{\overline{\alpha}}-\phi_\beta)}\sin\theta_\beta\sin\theta_{\overline{\alpha}}\prod_{\beta\geqslant\gamma>\overline{\alpha}}\cos\theta_\gamma,$$

with $\Pi_{\beta\geqslant\gamma>\beta}\cos\theta_\gamma=1$; and, for $\beta>\alpha$ and $\alpha\geqslant\overline{\alpha}$, by

$$A^{\phi_\beta}_{\overline{\alpha}\alpha}=ie^{i(\phi_{\overline{\alpha}}-\phi_\alpha)}\sin\theta_\alpha\sin\theta_{\overline{\alpha}}\sin^2\theta_\beta\times\prod_{\beta>\gamma>\alpha}\cos\theta_\gamma\prod_{\beta>\gamma>\overline{\alpha}}\cos\theta_{\overline{\gamma}}.$$

The A^{θ_β}'s and A^{ϕ_β}'s are the $2n$ components of the $\mathrm{u}(n)$-valued connection over $\mathbf{CP}^n$.

For generating a given quantum gate $g\in U(n)$ one has to determine a loop C_g in $\mathcal{M}$ such that $\Gamma_A(C_g)=g$. Due to the non-Abelian character of the connection, such an inverse problem is in general hard to solve. To tackle it we choose specific families of loops $\{C_i\}$ that generate holonomies from which one can eventually construct any $\mathrm{U}(n)$ transformation. To this end we consider the two-dimensional submanifolds in the $2n$-dimensional space (θ,ϕ), spanned by two variables, $(\theta_\beta,\phi_{\overline{\beta}})$ or $(\theta_\beta,\theta_{\overline{\beta}})$, for specific values of β and $\overline{\beta}$. For these loops the line integral is given by $\oint_C A=\oint_C(A^{\theta_\beta}d\theta_\beta+A^{\lambda_{\overline{\beta}}}d\lambda_{\overline{\beta}})$, where $\lambda=\theta$ or ϕ. From Eq. (2) we see that we can always choose the parameters that define the position of the plane $(\theta_\beta,\lambda_{\overline{\beta}})$, where loop C lies, in such a way that the matrix A^{θ_β} is identically zero. If one takes $\theta_i=0,\forall i\neq\beta,\overline{\beta}$, matrices A^{θ_β} and $A^{\lambda_{\overline{\beta}}}$ commute, so that we can calculate the integral and exponentiate avoiding the path ordering problem.

In this framework, it is possible to identify first four families of loops in such a way as to produce the basis of four matrices (the Pauli matrices and the identity) of all possible two-by-two submatrices belonging to the algebra $\mathrm{u}(2)$. The first choice is $(\theta_\beta,\phi_\beta)$, where the nonzero component of the connection is $A^{\phi_\beta}_{\beta\beta}=-i\sin^2\theta_\beta$. The second choice is the loop on the submanifold $(\theta_\beta,\phi_{\overline{\beta}})$ for $\overline{\beta}>\beta$, with $\theta_{\overline{\beta}}=\pi/2$, giving a different connection with two nonzero elements, $A^{\phi_{\overline{\beta}}}_{\beta\beta}=i\sin^2\theta_\beta$ and $A^{\phi_{\overline{\beta}}}_{\overline{\beta}\overline{\beta}}=-i$. Of course the latter element will give zero when integrated along a loop. For $\overline{\beta}<\beta$ both matrices are identically zero, and give rise to trivial holonomy. With these two connections and for appropriate loops one can obtain all possible $\mathrm{U}(n)$ *diagonal* transformations. For loop $C_1\in(\theta_\beta,\phi_\beta)$, $\Gamma_A(C_1)=\exp[-i|\beta\rangle\langle\beta|\Sigma_1]$, Σ_1 denoting the area enclosed by C_1, on the S^2 sphere with coordinates $(2\theta_\beta,\phi_\beta)$. For $C_2\in(\theta_\beta,\phi_{\overline{\beta}})$, $\Gamma_A(C_2)=\exp[i|\overline{\beta}\rangle\langle\overline{\beta}|\Sigma_2]$. Recalling the constraint $\overline{\beta}>\beta$, we see that one can produce $n-1$ distinct holonomies from C_2.

To obtain the nondiagonal transformations one has to consider a loop on the $(\theta_\beta,\theta_{\overline{\beta}})$ plane, with $\theta_i=0$ for all $i\neq\beta,\overline{\beta}$. Then the only nonvanishing elements of the connection are $A^{\theta_{\overline{\beta}}}_{\beta\overline{\beta}}=e^{i(\phi_\beta-\phi_{\overline{\beta}})}\sin\theta_\beta=-\overline{A}^{\theta_{\overline{\beta}}}_{\overline{\beta}\beta}$. By choosing further the $(\theta_\beta,\theta_{\overline{\beta}})$ plane at $\phi_\beta=\phi_{\overline{\beta}}=0$ the holonomy becomes, for $C_3\in(\theta_\beta,\theta_{\overline{\beta}})_{\phi_\beta=\phi_{\overline{\beta}}=0}$,

$$\Gamma_A(C_3)=\exp[-(|\beta\rangle\langle\overline{\beta}|-|\overline{\beta}\rangle\langle\beta|)\Sigma_3],$$

while at $\phi_\beta=\pi/2$ and $\phi_{\overline{\beta}}=0$, for $C_4\in(\theta_\beta,\theta_{\overline{\beta}})_{\phi_\beta=\pi/2,\phi_{\overline{\beta}}=0}$,

$$\Gamma_A(C_4)=\exp[-i(|\beta\rangle\langle\overline{\beta}|+|\overline{\beta}\rangle\langle\beta|)\tilde{\Sigma}_4],$$

where $\tilde{\Sigma}$ is the area on the sphere with coordinates $(\pi/2-\theta_\beta,\theta_{\overline{\beta}})$. Note that any loop C on the $(\theta_\beta,\lambda_{\overline{\beta}})$ plane with the same enclosed area (when mapped on the appropriate sphere) Σ_C will give the same holonomy, independent of its position and shape. These four holonomies are restricted each time to a specific 2×2 submatrix, and generate all U(2) transformations. Finally it is easy to check that in this way one can indeed obtain $U=\exp[\mu_j T_j]$, where $T_j (j=1,\ldots,n^2)$ is a u(n) generator and μ_j an arbitrary real number. Therefore any element of U(n) can be obtained by controlling the $2n$ parameters labeling the points of $\mathbf{CP}^n$.

It is instructive to consider the form that the Hamiltonian family $\mathcal{F}$ takes when restricted to the particular 2 submanifolds. For the loop C_1 (similarly for C_2) one finds $H_1=-\varepsilon_0/2\vec{B}(2\theta_\beta,\phi_\beta)\cdot\vec{\sigma}$ for $\vec{B}(\theta_i,\phi_j)=(\sin\theta_i\cos\phi_j,\sin\theta_i\sin\phi_j,\cos\theta_i)^T$, where the only nonzero elements are on the βth and $(n+1)$th row and column. H_1 generates an Abelian $\mathbf{CP}^1$ phase between the states $|\beta\rangle$ and $|n+1\rangle$. On the other hand, for the path C_3 (similarly for C_4) we have $H_3=\varepsilon_0\vec{B}(\theta_\beta,\theta_{\overline{\beta}})\vec{B}(\theta_\beta,\theta_{\overline{\beta}})^T$, where the nonzero elements connect the states $|\beta\rangle$, $|\overline{\beta}\rangle$ and $|n+1\rangle$. In this Hamiltonian there is direct coupling among three states, giving rise to a non-Abelian interaction.

In order to represent a two-qubit system we have to consider the control manifold $\mathbf{CP}^4$. The holonomies in this case are 4×4 matrices, and we take as a representation basis of the unitary transformations the qubit basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. From the general scheme above it follows that by appropriate control of the parameters (θ,ϕ) for obtaining various loops C, we can generate all possible U(4) rotations, i.e., any logical gate, in particular, single-qubit rotations and two-qubit gates such as the controlled operations XOR and CROT. ($U_{XOR}:=|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\sigma_x, U_{CROT}:=|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\sigma_z$). For single qubit rotations we consider three unitaries $U_\alpha=\Gamma_A(C_\alpha)(\alpha=1,2,3)$ with $C_1\in(\theta_1,\phi_1)_{\theta_2=\phi_2=0}$, $C_2\in(\theta_1,\phi_2)_{\theta_2=\pi/2,\phi_1=0}$, $C_3\in(\theta_1,\theta_2)_{\phi_1=\phi_2=0}$. For areas $\Sigma_2=\Sigma_1$ let $U_{ph1}=(U_1U_2)U_3(U_1U_2)^{-1}$, equal to

$$U_{ph1}=\begin{bmatrix} \cos\Sigma_3 & -\sin\Sigma_3 e^{-2i\Sigma_1} & \\ & & \mathbf{0} \\ \sin\Sigma_3 e^{2i\Sigma_1} & \cos\Sigma_3 & \\ & \mathbf{0} & \mathbb{1} \end{bmatrix}. \tag{3}$$

With loops on the $(\theta_3,\phi_3)_{\theta_4=\phi_4=0}$, $(\theta_3,\phi_4)_{\theta_4=\pi/2,\phi_3=0}$, $(\theta_3,\theta_4)_{\phi_3=\phi_4=0}$ planes and spanning the same areas as before, one can produce the rotation U_{ph2} on the lower-right block, and with their product obtain the phase rotation of one qubit $\mathbb{1}\otimes U_q=U_{ph1}U_{ph2}$. In order to perform the phase rotation of the other qubit, all we need is the swapping operator S acting on the two qubits by $S|\psi\rangle\otimes|\phi\rangle=|\phi\rangle\otimes|\psi\rangle$. S is given in terms of holonomies by $S=\Gamma_A(C_4)|_{\tilde{\Sigma}_4=\pi/2}\Gamma_A(C_2)|_{\Sigma_2=\pi/2}$, where $C_4\in(\theta_2,\theta_3)_{\phi_2=\pi/2,\phi_3=0}$ and $C_2\in(\theta_1,\phi_2)_{\theta_2=\pi/2}$ and $(\theta_1,\phi_3)_{\theta_3=\pi/2}$. Hence, the phase rotation of the other qubit is given by $U_q\otimes\mathbb{1}=S(\mathbb{1}\otimes U_q)S$. The controlled rotation gate CROT, is given by $U_{CROT}=\Gamma_A(C_1)|_{\Sigma_1=\pi/2}\Gamma_A(C_2)|_{\Sigma_2=\pi/2}$, where $C_1\in(\theta_4,\phi_4)$ and $C_2\in(\theta_1,\phi_4)_{\theta_4=\pi/2}$. In addition the ''exclusive or'' gate XOR can be realized as $U_{XOR}=\Gamma_A(C_4)|_{\tilde{\Sigma}_4=\pi/2}\Gamma_A(C_2)|_{\Sigma_2=\pi/2}$, where $C_4\in(\theta_3,\theta_4)_{\phi_3=\pi/2,\phi_4=0}$ and $C_2\in(\theta_1,\phi_3)_{\theta_3=\pi/2}$ and $(\theta_1,\phi_4)_{\theta_4=\pi/2}$.

Complexity. So far the coding subspaces analyzed for HQC do not necessarily involve quantum entanglement; $\mathcal{H}$ could be the state space of a *single* quantum system. This is due to the fact that $\mathcal{C}$ does not have a built-in tensor product structure; thus, in general, it cannot be naturally interpreted as the state space of a multipartite system. This latter feature, however, is one of the essential ingredients that make QC more efficient than classical computation. Indeed, from the above construction for the $\mathbf{CP}^n$ model it is easy to realize that the number of elementary loops in 2 submanifolds necessary for implementing single and two-qubit operations scales exponentially as a function of the number $\log_2 n$ of encoded qubits. To overcome such a difficulty one has simply to consider Hamiltonian family, acting on (not just isomorphic to) the state space of a multipartite quantum system, with a special structure allowing for local QC's to be holonomically performed. Then global QC's involving nontrivial actions over many qubits can be efficiently realized in the standard way [9]. One possible formalization of this idea is contained in the following theorem.

Let $\mathcal{H}:=\otimes_{j=1}^n\mathcal{H}_j\otimes\mathcal{H}_a$, $\mathcal{H}_j\cong\mathbf{C}^d$, and $\mathcal{H}_a=\mathrm{span}\{|-\rangle,|+\rangle\}$ a single-qubit ancillary space. We set $H_a:=\varepsilon\sigma^z\in\mathrm{End}(\mathcal{H}_a)$ [10]. Moreover, let $H(\lambda):=\Sigma_{i<j}V_{ij}(\lambda_{ij})$, where the λ_{ij}'s belong to local control manifolds $\mathcal{M}_{ij}$. Suppose that $V_{ij}(\lambda)\in\mathrm{End}(\mathcal{H}_i\otimes\mathcal{H}_j\otimes\mathcal{H}_a)$, and $V_{ij}(0)=H_a+H_i+H_j$, where $H_j\in\mathrm{End}(\mathcal{H}_j)$ is such that $H_j|\alpha\rangle_j=0,(\alpha=0,1)$ and that the family $\{V_{ij}(\lambda)\}_{\lambda\in\mathcal{M}_{ij}}$ allows for universal HQC over the degenerate eigenspaces $\mathcal{C}_{ij}^{\pm}:=\mathrm{span}\{|\alpha\rangle_i\otimes|\beta\rangle_j\otimes|\pm\rangle:\alpha,\beta=0,1\}\cong\mathbf{C}^2\otimes\mathbf{C}^2$. Then the family $\{H(\lambda)/\lambda\in\Pi_{i<j}\mathcal{M}_{ij}\}$ allows for *efficient* universal HQC over the n-qubit codes $\mathcal{C}^{\pm}:=\mathrm{span}\{\otimes_{j=1}^n|\alpha_j\rangle\otimes|\pm\rangle/\alpha_j=0,1\}\cong(\mathbf{C}^2)^{\otimes n}$.

The proof of the latter proposition proceeds as follows. First one observes that the subspaces $\mathcal{C}_{ij}^{\pm}$ are degenerate eigenspaces of $V_{ij}(0)$ corresponding to the eigenvalues $\pm\varepsilon$. Hence the $\mathcal{C}^{\pm}$'s are 2^n-dimensional eigenspaces of $H(0)$ with eigenvalues $\pm n(n-1)\varepsilon/2$. By assumption for any pair (i,j) of subsystems [11] one can generate any unitary transformation over $\mathcal{C}_{ij}^{\pm}$ by adiabatic loops γ_{ij} in $\mathcal{M}_{ij}$. Keeping all the remaining λ's at 0, one has a trivial action over the other factors of $\mathcal{C}^{\pm}$, while $\Gamma_A(\gamma_{ij}):\otimes_k|\alpha_k\rangle\otimes|\pm\rangle\mapsto\Sigma_{\beta_i\beta_j}U_{\beta_i\beta_j,\alpha_i\alpha_j}(\gamma_{ij})\otimes_k|\tilde{\alpha}_k\rangle\otimes|\pm\rangle$, where $\tilde{\alpha}_k=\alpha_k$ for $k\neq i,j$ and $\tilde{\alpha}_k=\beta_k$ when $k=i,j$. In particular, one can obtain a universal set of gates, e.g., $\mathrm{XOR}_{ij}^{\pm}$, and single-qubit operations, by using a fixed amount of resources. The claim then follows by well-known universality results for QC [12]. The above scheme involves the use of an ancilla and requires controllability of three-body interactions, which is extremely difficult to achieve in practice. In this respect a simplification, involving just two-body interactions, can be obtained by considering N subsystems with d levels [13].

Implementation. Now we discuss how one could in principle implement the holonomic loops, even when the parametric Hamiltonian family $\mathcal{F}$ (or a part of it) is not available from the outset. We shall resort to ideas of quantum control theory in a way that is quite similar to the one adopted for symmetrization procedures [14] and decoherence control in open quantum systems [15]. Suppose that an experimenter has at his disposal the following resources: (i) a quantum system characterized by the Hamiltonian H_0 admitting an n-fold degenerate eigenspace $\mathcal{C}$; (ii) the way to turn on and off a set of interactions very quickly (with respect to the time scales associated with H_0) so that a set of unitary ''kicks'' $K := \{U_\lambda\}_{\lambda \in \mathcal{M}}$ can be realized. Let $T = N\Delta t$ and $t_0 = 0$, $t_{i+1} = t_i + \Delta t (i = 1, \ldots, N)$ be a partition of the time interval $[0,T]$. Now let the system evolution to be as follows: at any time t_i the experimenter kicks the system with the pulse $U_{i+1}^\dagger U_i$ where $U_i := U(\lambda_i)$ is a unitary chosen from the set $K(U_0 = U_{N+1} = \mathbb{1})$. Between the kicks the system evolution is unperturbed, $U(\Delta t) = e^{-i H_0 \Delta t}$. The global evolution is then given by

$$U_{N,\Delta t}(T) = \mathbf{T} \prod_{i=1}^{N} U_i U(\Delta t) U_i^\dagger, \tag{4}$$

where $\mathbf{T}$ denotes time-ordering. By considering the limit $\Delta t \mapsto 0, N \mapsto \infty$, with $N\Delta t = T$, one gets $U_{N,\Delta t}(T) \to \mathbf{T} \exp -i\int_0^T dt H(t)$, where $H(t) := U(\lambda(t)) H_0 U(\lambda(t))^\dagger$. In particular, by making the function $\lambda(t)$ vary adiabatically, one can obtain the desired holonomic evolution. This scheme is based on a strong separation between time scales: each U_λ has to be enacted impulsively, whereas the characteristic variation time of the control parameters λ has to be slow enough to satisfy the adiabaticity requirement. More precisely, if τ_k denotes the kicking time, ω the (highest) frequency associated with the dynamics generated by H_0, and τ_λ the time scale over which the function $\lambda(t)$ varies, one must have $\tau_k \leqslant \Delta t \ll \omega^{-1} \ll \tau_\lambda$. Notice that the pulses U_λ are not required to be a universal set of gates for QC; here they represent an extra resource needed for implementing HQC when $\mathcal{F}$ is missing.

Finally let us briefly consider the problem of code-word preparation and measurement. In order to encode the initial state into the degenerate subspace $\mathcal{C}$ or to make the measurement on the final state, it would be useful to lift the degeneracy. Indeed, in this way one would be able to distinguish *energetically* the different coding states. This characteristic is quite often desirable from the experimental point of view in that one can resort to procedures involving energy transitions with state-dependent frequencies. The idea is to lift the degeneracy between the coding states, for example, by switching on coherently an external (generic) perturbation. The basis states $|\psi_\alpha\rangle$ of $\mathcal{C}$ are mapped onto a set of states $|\psi'_\alpha\rangle$ that are no longer energy degenerate. Preparation and/or measurement is then performed and eventually degeneracy is coherently restored by switching off the perturbation.

Summary. In this paper we have provided further analysis of the proposal for holonomic quantum computation of Ref. [2]. We have explicitly designed control loops whose holonomies generate universal gates for a $\mathbf{CP}^n$ control parameter manifold. The basic idea is to associate with the U(n) generators computable transformations obtained by loops on two-dimensional subspaces of the control manifold. An explicit realization of two qubit gates has been given, with an indication of which particular loops the experimenter has to perform. In terms of such elementary holonomic gates we analyzed the complexity problem and we showed how to achieve efficient implementation of quantum computing by resorting to a HQC model involving only local interactions. Some implementative issues have been addressed, and we devised a scheme based on repeated pulses for realizing the parametric family of isospectral Hamiltonians required for HQC. Finally, we briefly indicated how to prepare the initial state and how to measure the final one by coherently switching on and off the energetic degeneracy of the computational subspace.

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