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Reduction method for the linear quantum or classical oscillator with time-dependent frequency, damping, and driving

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We elucidate the relationship between the linear oscillator with time-dependent frequency, damping, and driving, and the *autonomous* oscillator with unit frequency and no damping or driving. Such a relationship can be derived from a canonical transformation and a redefinition of the time. A study of the scalars of the transformation makes it possible to write down the evolving quantum states for the nonautonomous Hamiltonian given the evolving states for the autonomous harmonic oscillator.

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Interest in the Hamiltonian

$$\overline{H}(p,q,t)=\frac{\beta(t)p^2}{2}+\beta^{-1}(t)\left[\frac{\overline{\Omega}^2(t)q^2}{2}-\overline{f}(t)q\right], \quad (1)$$

with time-dependent frequency $\overline{\Omega}(t)$, damping $-\dot{\beta}(t)/\beta(t)$, and driving $\overline{f}(t)$, is twofold. On the one hand, it provides an example of a completely soluble nonautonomous system, so that the adiabatic limit or the sudden limit can be studied beyond a perturbative approach. On the other hand, this problem has practical applications in the dynamics of charged particles in a time-dependent magnetic field [1,2] and for the coherent production of phonons in solids under the action of a time-dependent pressure field [3]. To the authors' knowledge, the earliest approach to the most general form of Eq. (1) was suggested by Kolsrud [4] in an attempt to construct the evolution operator in the quantum case.

We first proceed to a simplification, by introducing a new time T,

$$T=\int_{t_0}^{t}dt'\beta(t'), \quad \frac{d}{dt}=\beta\frac{d}{dT}, \quad (2)$$

in terms of which the damping can be eliminated from the equations of motion. The classical Hamilton equations and the Heisenberg equations of motion become

$$\frac{dq}{dT}\equiv\dot{q}=p,$$
$$\frac{dp}{dT}\equiv\dot{p}=-\frac{1}{\beta^2}[\overline{\Omega}^2q-\overline{f}]=f(T)-\Omega^2(T)q, \quad (3)$$

where

$$\Omega^2(T)\equiv\frac{\overline{\Omega}^2(t)}{\beta^2(t)}, \quad f(T)\equiv\frac{\overline{f}(t)}{\beta^2(t)}. \quad (4)$$

Equations (3) are a Hamiltonian system with respect to the new time T and Hamiltonian

$$H(p,q,T)=\frac{p^2+\Omega^2(T)q^2}{2}-f(T)q. \quad (5)$$

In the classical case, the equations of motion corresponding to Eq. (5) can be solved in terms of solutions for the harmonic oscillator. The quantum evolution with Hamiltonian (5) has been addressed in the special case of no driving $[f(T)=0]$ in [1,5–10]. The case with time-dependent driving is covered by the general quadratic Hamiltonian form studied in Ref. [11]. From Refs. [5,7,8,10,11], it is now definitely clear that a basis $\{|n,T\rangle\}$, evolving under the nonautonomous Schrödinger equation, can be given (for any quadratic Hamiltonian) in terms of the eigenfunctions of the *autonomous* harmonic oscillator. This makes one suspect that some first-principle relationship between nonautonomous and autonomous quadratic Hamiltonians should exist, leading to a simple and more straightforward approach. This is the aim of the present Rapid Communication, in which we deal with the Schrödinger equation

$$i\frac{\partial\overline{\Psi}(q,T)}{\partial T}=\left[-\frac{1}{2}\frac{\partial^2}{\partial q^2}+\frac{\Omega^2(T)q^2}{2}-f(T)q\right]\overline{\Psi}(q,T) \quad (6)$$

that follows from Eq. (5). By means of what we call the reduction method we get a general explicit relationship between the solutions of Eq. (6) and the solutions of the time-dependent Schrödinger equation for an *autonomous* oscillator with unit frequency and no driving. The method is based upon a canonical transformation $(p,q)\rightarrow(P,Q)$, followed by a suitable redefinition of the time $T\rightarrow\tau(T)$, such that the new Hamilton and Heisenberg equations in the time τ correspond to a harmonic autonomous Hamiltonian with unit fre-

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quency and no damping or driving. The quantum treatment of the problem is completed by considering the general quantum scalars of the space-time transformation that is used in the reduction method.

Define the classical or quantum canonical transformation $(p,q)\rightarrow(P,Q)$ [12] by

$$Q=\frac{q-\alpha}{\rho},\quad P=\rho(p-\dot{\alpha})-\dot{\rho}(q-\alpha), \tag{7}$$

where the dotted quantities are derivatives with respect to T. Let the two *real* functions $\rho(T)$, $\alpha(T)$ satisfy the equations

$$\ddot{\rho}+\Omega^2(T)\rho-\rho^{-3}=0,\quad \ddot{\alpha}+\Omega^2(T)\alpha=f(T), \tag{8}$$

with arbitary initial conditions. With respect to *another* new time τ, the new Hamiltonian K associated with H [Eq. (5)] and with the canonical transformation equation (7) is

$$K(P,Q)=\frac{P^2+Q^2}{2}, \tag{9}$$

where

$$\tau=\int_{T_0}^{T}dT'\rho^{-2}(T'),\quad \frac{d}{dT}=\rho^{-2}(T)\frac{d}{d\tau}. \tag{10}$$

The reduction to K is the reduction to the *autonomous* harmonic oscillator with unit frequency and no driving. By direct differentiation with respect to T of Q and P as defined by Eq. (7), and using the equations of motion (3), we can check that the Hamilton and Heisenberg equations associated with K, with respect to the new time τ, are equivalent to Eqs. (3). The result of the differentiation is

$$\dot{Q}=\rho^{-2}[-\dot{\rho}(q-\alpha)+\rho(p-\dot{\alpha})]=\rho^{-2}P, \tag{11}$$

$$\dot{P}=-\rho^{-2}\left(\frac{q-\alpha}{\rho}\right)=-\rho^{-2}Q. \tag{12}$$

Therefore, from Eq. (10), Eqs. (11) and (12) written in terms of τ, are

$$\frac{dQ}{d\tau}=P,\quad \frac{dP}{d\tau}=-Q. \tag{13}$$

Thus, the classical or quantum problem can be addressed explicitly with a unified treatment by using results for the autonomous harmonic oscillator with unit frequency and no driving.

It is perhaps useful to mention that Newton's equation of motion for a one-dimensional classical (linear) oscillator with arbitrarily time-dependent frequency is mathematically the same as the Schrödinger equation for a particle in an arbitrarily space-dependent one-dimensional static potential (apart from boundary conditions) [3,13]. Therefore, any treatment of that classical oscillator problem (time-dependent frequency, no damping or driving) is applicable to the quantum eigenvalue problem in terms of the function ρ of Eq. (8), and the new time τ in Eq. (10) is the same as in the earlier Milne–Young–Wheeler treatment of the quantum eigenvalue problem [14–17]. However, it should be borne in mind that the quantum and classical problem being addressed in the present paper is that of a linear oscillator with arbitrarily time-dependent frequency, damping and driving, *not* that of a particle in a static arbitrarily space-dependent potential.

The quantum treatment can be completed by using arguments of covariance, requiring the *same form* for the Schrödinger equation in terms of the transformed and original variables. More precisely, for the Schrödinger equation to be covariant with respect to the generalized space-time transformation, the wave function itself must change in a suitable way. For example, let $\Psi(Q,\tau)$ satisfy the Schrödinger equation for the transformed Hamiltonian Eq. (9):

$$i\frac{\partial\Psi(Q,\tau)}{\partial\tau}=K(P,Q)\Psi(Q,\tau)=\frac{1}{2}\left[-\frac{\partial^2}{\partial Q^2}+Q^2\right]\Psi(Q,\tau). \tag{14}$$

The function $\Psi(Q(q,T),\tau(T))$, obtained from Eqs. (7) and (10) as a function of q, T, does *not* satisfy Eq. (6) in general. Denote the true solution of Eq. (6) by $\overline{\Psi}(q,T)\equiv\psi(Q(q,T),\tau(T))$. Equations (7) and (10) represent a canonical transformation. Therefore, the particle number must be *locally* conserved:

$$dQ|\Psi(Q,\tau)|^2=dq|\overline{\Psi}(q,T)|^2\Rightarrow\psi(Q,\tau)=\frac{\Psi(Q,\tau)}{\sqrt{\rho}}\exp[-i\varphi(Q,\tau)], \tag{15}$$

with $\varphi(Q,\tau)$ a real phase to be determined. Another condition on $\overline{\Psi}(q,T)$ is the continuity equation for probability in q,T space. This yields, with the aid of the second of Eqs. (15),

$$\begin{aligned}\frac{\partial|\overline{\Psi}|^2}{\partial T}&=-\frac{i}{2}\left[\overline{\Psi}\left(\frac{\partial^2\overline{\Psi}}{\partial q^2}\right)^*-\overline{\Psi}^*\left(\frac{\partial^2\overline{\Psi}}{\partial q^2}\right)\right]\\&\Rightarrow-\frac{\dot{\rho}}{\rho}|\Psi|^2+\frac{1}{\rho}\left(\frac{\partial|\Psi|^2}{\partial Q}\frac{\partial Q}{\partial T}\right)+\frac{1}{\rho^3}\frac{\partial|\Psi|^2}{\partial\tau}\\&=-\frac{i}{2\rho^2}\left[\psi\left(\frac{\partial^2\psi}{\partial Q^2}\right)^*-\left(\frac{\partial^2\psi}{\partial Q^2}\right)\psi^*\right],\end{aligned} \tag{16}$$

where all dotted symbols are derivatives with respect to T, and $\partial Q/\partial T=-\rho^{-1}(\dot{\rho}Q+\dot{\alpha})$ [see Eq. (7)]. Even the right-hand side of the second of Eqs. (16) can be expressed in terms of Ψ, by means of the second of Eqs. (15). It is an easy matter to show that the terms containing $\partial^2\Psi/\partial Q^2$ are exactly canceled out by the last term in the left-hand side due to the continuity equation in Q,τ space. The resulting equation for the *real* phase $\varphi(Q,\tau)$ is, therefore,

$$-\left(\rho\dot{\rho}+\frac{\partial^2\varphi}{\partial Q^2}\right)|\Psi|^2=\left(\rho\dot{\rho}Q+\rho\dot{\alpha}+\frac{\partial\varphi}{\partial Q}\right)\frac{\partial|\Psi|^2}{\partial Q}. \tag{17}$$

According to Wigner's theorem [18], the transformation $\Psi \rightarrow \overline{\Psi}$ must be linear, which requires that $\varphi(Q,\tau)$ be independent of the wave function. Therefore, each side of each side of Eq. (17) must vanish, with the result

$$\varphi(Q,\tau) = -\frac{\rho\dot{\rho}}{2}Q^2 - \rho\dot{\alpha}Q + B(\tau),$$
$$\psi(Q,\tau) = \frac{\Psi(Q,\tau)}{\sqrt{\rho}}\exp\left[i\left(\frac{\rho\dot{\rho}}{2}Q^2 + \rho\dot{\alpha}Q - B(\tau)\right)\right], \tag{18}$$

with $B(\tau)$ a real function of τ alone, to be determined. The last step of the calculation can be implemented by using $\overline{\Psi}(q,T) = \psi(Q(q,T),\tau(T))$ in the Schrödinger equation (6), with $q = \rho Q + \alpha$, and $\partial^2/\partial Q^2 = \rho^{-2}\partial^2/\partial Q^2$. However, one does not need to calculate all the terms explicitly. In fact, the equation for $B(\tau)$ must appear in the form $\dot{B}\overline{\Psi} = \rho^{-2}(\partial B/\partial\tau)\overline{\Psi} = G(T)\overline{\Psi}$, with G *real* and depending only on T. All terms containing Q, $\partial/\partial Q$, or imaginary functions of T alone will cancel one another. On applying this rule, the terms contributing to G can be easily identified, and the resulting equation for B is

$$\dot{B} = \frac{\Omega^2(T)\alpha^2(T)}{2} - f(T)\alpha(T) - \frac{\dot{\alpha}(T)}{2}. \tag{19}$$

At this stage, the first Eq. (18) and the first Eq. (19) yield the phase as a function of q, T. Moreover, one can use Eqs. (8) to eliminate $f(T)$, $\Omega(T)$, which leads to a functional expression depending only on α and ρ:

$$\overline{\varphi}(q,T) \equiv \varphi(Q(q,T),\tau(T)) = -\frac{\dot{\rho}(T)}{2\rho(T)}q^2 - \left(\dot{\alpha}(T) - \frac{\dot{\rho}(T)}{\rho(T)}\alpha(T)\right)q - \frac{1}{2}\int_{T_0}^{T} dT'\left[\frac{\alpha^2}{\rho^4} - \left(\dot{\alpha} - \frac{\dot{\rho}}{\rho}\alpha\right)^2\right] + \varphi_0, \tag{20}$$

where φ_0 is an arbitary real phase, independent of time and coordinate. Now, the results of the quantum reduction method can be summarized as follows.

Let $\Psi(Q,\tau)$ be *any* solution of the ''reduced'' Schrödinger equation (14) (unit frequency and no driving). Then, a *class* of solutions of Eq. (6) is obtained as

$$\overline{\Psi}^{(\xi)}(q,T) = \frac{1}{\sqrt{\rho(T)}}\Psi\left(\frac{q-\alpha(T)}{\rho(T)}, \int_{T_0}^{T}\frac{dT'}{\rho^2}\right) \times \exp\left\{i\left[\frac{\dot{\rho}(T)}{2\rho(T)}q^2 + \left(\dot{\alpha}(T) - \frac{\dot{\rho}(T)}{\rho(T)}\alpha(T)\right)q + \frac{1}{2}\int_{T_0}^{T} dT'\left[\frac{\alpha^2}{\rho^4} - \left(\dot{\alpha} - \frac{\dot{\rho}}{\rho}\alpha\right)^2\right] - \varphi_0\right]\right\}, \tag{21}$$

where the functions ρ, α satisfy the classical equations (8), with *arbitary* initial conditions

$$\xi \equiv \{\rho(T_0), \dot{\rho}(T_0), \alpha(T_0), \dot{\alpha}(T_0)\} \in \mathsf{R}^4.$$

In view of practical applications of Eq. (21), it may be useful to express α as a functional of ρ and f. One can verify that

$$\alpha(T) = \rho(T)\sin[\tau(T)]\left[\int_{T_0}^{T} dT' f \;\; \rho\cos(\tau) + A_0\right] - \rho(T)\cos[\tau(T)]\left[\int_{T_0}^{T} dT' f \;\; \rho\sin(\tau) + B_0\right] \tag{22}$$

satisfies the second Eq. (8), with arbitary initial conditions determined by A_0, B_0, $\rho(T_0)$, $\dot{\rho}(T_0)$. In this way, the mathematical effort for the reduction method is limited to the solution of the first Eq. (8). By using expression (22) as the solution for α in the second equation of (8), the equation for ρ in (8) becomes the cornerstone for finding a concrete solution of any particular linear problem, whether it be quantum or classical. Indeed, the reduction method leading to Eq. (8) is independent of the classical or quantum framework. Any choice of the real initial values $\xi \in \mathsf{R}^4$ determines an element of the class of solutions. For numerical work, it may be preferable in particular cases to choose the initial conditions for Eqs. (8) in certain ways. For example, it may be preferable to obtain a numerical solution for ρ that is associated with the adiabatic behavior. Hence the reduction method has a large amount of flexibility in approaching the solution of the quantum problem with *given* initial wave function $\Phi(q,T_0)$. First, one may choose a basis $\{\Psi_n(Q,\tau); n = 0,1,2,\ldots\}$, evolving with the reduced Schrödinger equation (14). Then, Eq. (21) yields a class of possible bases $\{\overline{\Psi}_n^{(\xi)}; n = 0,1,2,\ldots; \xi \in \mathsf{R}^4\}$, each element of which satisfies the original Eq. (6). Second, one may further choose the set ξ of initial values according to specific requirements of simplicity. The evolving solution with initial wave function $\Phi(q,T_0)$ is finally given by

$$\Phi(q,T) = \sum_{n=0}^{\infty} \Phi_n(\xi)\overline{\Psi}_n^{(\xi)}(q,T),$$
$$\Phi_n(\xi) = \int_{-\infty}^{\infty} dq[\overline{\Psi}_n^{(\xi)}(q,T_0)]^*\Phi(q,T_0). \tag{23}$$

As far as the choice of $\{\Psi_n(Q,\tau); n = 0,1,2,\ldots\}$ is concerned, the simplest possibility is choosing the basis of eigenstates of $K(P,Q)$:

$$\Psi_n(Q,\tau) = \frac{\exp[-Q^2/2 - i(n+1/2)\tau]}{(\sqrt{\pi}2^n n!)^{1/2}}H_n(Q),$$

where $H_n(Q)$ are the Hermite polynomials. On setting $\Psi = \Psi_n$ in Eq. (21), with $\alpha(T) = 0$, one gets the same expression found in Refs. [7,8] for an evolving basis in the case of

no driving. However, we stress that *any* basis evolving with the standard Hamiltonian $K(P,Q)$ can be used in its own right.

In conclusion, the reduction method starts with an argument connecting the Hamilton and Heisenberg equations for Eq. (1) to those for the harmonic-oscillator Hamiltonian with unit frequency. This is realized by the canonical transformation equation (7) and by the redefinition of the time, Eqs. (2),(10). The method involves two auxiliary functions ρ and α, satisfying second-order differential equations. On taking the square modulus of both sides of Eq. (21), it is seen that the quantum probability density in the coordinate space is shifted by α and rescaled by ρ^{-1}. Classically, this means that the amplitude of the oscillator is determined by the evolution of ρ, while its position, relative to an inertial frame of reference, is determined by α. The time dependence of ρ and α is reflected in the explicit dependence of the *phase* on the coordinate [Eq. (21)]. The resulting *quantum* effect is a probability current that is necessary to satisfy the continuity equation [Eq. (16)].

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