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RAPID COMMUNICATIONS

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Dual symmetric Lagrangians and conservation laws

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By using a complex field with a symmetric combination of electric and magnetic fields, a first-order covariant Lagrangian for Maxwell's equations is obtained. This leads to a dual-symmetric quantum-field theory with an infinite set of local conservation laws. The dual symmetry is shown to correspond to a helical phase, conjugate to a conserved helicity number. [S1050-2947(99)50809-3]

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The complex, dual symmetric form of Maxwell's equations [1] has a number of useful and intriguing properties. Dual symmetry [2] (interchanging B and E) has become a topical subject recently, due to its relevance to problems involving massive quantum-field theories and solitons [3]. In this Rapid Communication, I will introduce a Lagrangian for the complex form of Maxwell's equations. This allows an elegant reformulation of quantum electrodynamics using a Dirac-type covariant Lagrangian, with only first-order time derivatives and a six-dimensional complex field. In this form, the dual symmetry corresponds to a type of phase rotation, which generates the conserved helicity number of the field. In addition to this, I show that these results can be extended to obtain an infinite set of conservation laws, including conservation of local photon number and quantum squeezing density operators.

In this type of electrodynamics, the dual symmetry is generated simply through a phase rotation of the complex electromagnetic field, and is therefore similar to the chiral symmetry of Weyl [4] neutrino theory. A useful property of the dual phase defined in this way is that it is conjugate to a photon-number difference operator, and hence is free from the well-known problems of single-mode phase operators [5,6]. It is also possible to develop a local current and density operator for the helicity number, which does not have the drawbacks usually found with local photon-number density [7,8]. The advantage is that the helicity density is related to physical fields acting at a point, has well-defined transformation properties, and gives information about particle number rather than energy. This operator directly utilizes the explicit dual symmetry in the present formulation of electrodynamics. Another local symmetry, together with a conserved current, is obtained from a rescaling symmetry found by using a covariant Lagrangian with complex fields.

The complex electromagnetic field [1] is defined as $\boldsymbol{\mathcal{F}} = (\boldsymbol{E}_\perp + i\boldsymbol{B})/\sqrt{2}$ in units where $\hbar = c = \varepsilon_0 = \mu_0 = 1$. This technique has been known for some time [9–14] and has been investigated recently as a means of defining a photonic wave function [15,16]. The motivation for the theory is a logical extension of Dirac's technique [17] of finding the square root of the Laplacian operator, except using representations of $O(3)$ rather than the Dirac gamma matrices. I extend this idea to derive a covariant Lagrangian, and obtain the relevant conservation laws.

The procedure I will follow is to consider possible local Lagrangian field theories that have only first-order terms in the time derivatives, so that they directly generate the first-order Maxwell equations, just as the Dirac Lagrangian directly generates the Dirac equation. Since radiation field operators have dimension $[L]^{-2}$, and the Lagrangian has dimension $[L]^{-4}$, it is clear that a Lagrangian that includes products of fields and derivatives must also involve gauge fields with dimension $[L]^{-1}$. Thus, it is necessary to introduce a transverse complex vector potential $\boldsymbol{\mathcal{A}} = (\boldsymbol{\Lambda} + i\boldsymbol{A})/\sqrt{2}$, where $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{A}} = 0$. The field $\boldsymbol{\Lambda}$ is the dual potential

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for the transverse part of the electric field, while $\mathcal{A}$ is just the usual magnetic vector potential. Here the complex potential field $\boldsymbol{\mathcal{A}}$ is related to the free complex Maxwell field $\boldsymbol{\mathcal{F}}$, by $\boldsymbol{\mathcal{F}}=\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}$.

In general, $\boldsymbol{E}$ can have a longitudinal component, which has no representation in this form. This problem only arises in the presence of interactions, and is inherent in the idea of a photonic wave function with just transverse components. Interacting fields and details of applications to dielectric and atomic systems will be treated in more detail in a following paper. The first question is whether there is a Lagrangian that can generate Maxwell's equations, while having a Hamiltonian density:

$$\mathcal{H}=\boldsymbol{\mathcal{F}}^{*}\cdot\boldsymbol{\mathcal{F}}=\frac{1}{2}[|\boldsymbol{E}_{\perp}|^{2}+|\boldsymbol{B}|^{2}]. \tag{1}$$

There is a straightforward noncovariant Lagrangian density with these properties. It can be written as

$$\mathcal{L}_{nc}(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{A}}^{*})=i\dot{\boldsymbol{\mathcal{A}}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}^{*}-|\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}|^{2}. \tag{2}$$

It is simple to verify that this generates the Maxwell field equation, since one immediately obtains

$$i\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t}=\boldsymbol{\nabla}\times\boldsymbol{\mathcal{F}}. \tag{3}$$

This equation is identical to the free-field Maxwell equations. The resulting Hamiltonian density is also identical to the energy density, as required.

There is a covariant form of the Lagrangian due to Schwinger [18], which has only first-order derivatives, like the Dirac equation. It is also possible to extend the present complex electrodynamic theory to a covariant, first-order Lagrangian, on defining a six-dimensional spinor field $\boldsymbol{\psi}$, where

$$\boldsymbol{\psi}=\frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\mathcal{F}}\\ \boldsymbol{\mathcal{A}}\end{pmatrix}. \tag{4}$$

This has a Lagrangian — apparently different from previous first- or second-order forms [19] — given by

$$\mathcal{L}_{c}=\Psi^{\dagger}p_{\mu}\alpha^{\mu}\Psi=[(i\dot{\boldsymbol{\mathcal{A}}}-\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}})\cdot\boldsymbol{\mathcal{F}}^{*}+(\boldsymbol{\mathcal{F}}\leftrightarrow\boldsymbol{\mathcal{A}})]/2. \tag{5}$$

Here $p_{\mu}=i\partial_{\mu}$, and I introduce the six-dimensional complex matrices α^{μ} which are given by

$$\alpha^{0}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \alpha^{i}=\begin{pmatrix}0 & S^{i}\\ S^{i} & 0\end{pmatrix},\quad \alpha^{4}=\begin{pmatrix}iI & 0\\ 0 & -iI\end{pmatrix}. \tag{6}$$

The matrix vector $\boldsymbol{S}$ comprises 3×3 complex matrices such that $[S^{i},S^{j}]=i\varepsilon_{ijk}S^{k}$. These S^{i} matrices can be represented as 3×3 Hermitian rotation matrices:

$$\boldsymbol{S}=\left(\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}\right). \tag{7}$$

This Lagrangian density generates the covariant form of the wave equation, which is $\partial_{\mu}\alpha^{\mu}\boldsymbol{\psi}=0$. In more detail, the relation $[\nabla\cdot\boldsymbol{S}]\boldsymbol{\psi}=i\boldsymbol{\nabla}\times\boldsymbol{\psi}$ means that the two three-vector components of $\boldsymbol{\psi}$ each satisfy an independent wave equation of the form given by Eq. (3).

To ensure that the the canonical Hamiltonian corresponds to the classical energy, the constraint is imposed that $\boldsymbol{\mathcal{F}}=\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}$. The Hamiltonian then equals the energy at all times, and the canonical momentum field is $\boldsymbol{\Pi}=i\Psi^{\dagger}\alpha^{0}$. The fundamental fields are regarded here as being a combination of the physical fields and the potentials, rather than the usual situation of just the potentials being regarded as dynamical variables. This does not change the number of physical degrees of freedom.

Spinor theories of the electric and magnetic fields are well known, and generate a finite-dimensional representation of the Lorenz group. Here, the potential transformation belongs to an infinite-dimensional representation of the Lorenz group [20], since the Lorenz-transformed potentials depend nonlocally on the potentials in the original reference frame. This is due to the known difficulty of defining a unique photon position. An advantage is that there is no time-like photon momentum with a negative-metric Hilbert space in this approach, as there is with the Gupta-Bleuler quantization of the covariant Fermi Lagrangian. Thus, the normal axioms of quantum theory apply.

An important symmetry property of the present equations is the dual symmetry generating by changing the phase of $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{A}}$ simultaneously, thus rotating $\boldsymbol{E}_{\perp}$ into $\boldsymbol{B}$. This symmetry must generate a globally conserved quantity, which I will show to be the helicity number of the photon field, i.e., the difference between the number of right and left circularly polarized photons. The conserved dual current J_{d} is given by

$$J_{d}^{\mu}=\Psi\alpha^{\mu}\Psi. \tag{8}$$

There is a corresponding globally conserved charge, which is the integral of the density, $J_{d}^{0}=(\boldsymbol{\Lambda}\cdot\boldsymbol{E}_{\perp}+\boldsymbol{A}\cdot\boldsymbol{B})/2$. The global helicity ''charge'' is

$$N_{h}=\int\Psi^{\dagger}\alpha^{0}\Psi dV=\frac{1}{2}\int[\boldsymbol{\Lambda}\cdot\boldsymbol{E}_{\perp}+\boldsymbol{A}\cdot\boldsymbol{B}]dV. \tag{9}$$

It is important to notice that J_{d}^{0} is always zero whenever the electric and magnetic fields both vanish. This property of ''quasilocality'' is invariant under Lorenz transformations, even though the potentials are not local.

Dual symmetry is not the only internal symmetry of the covariant Lagrangian formulation, since it is also possible to rescale $\boldsymbol{\mathcal{F}}\rightarrow(1+\epsilon)\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{A}}\rightarrow(1-\epsilon)\boldsymbol{\mathcal{A}}$ simultaneously, without changing the Lagrangian. This generates yet another locally conserved current. I call this the scaling current J_{s}^{i}, with scaling density J_{s}^{0}:

$$J_{s}^{\mu}=\Psi^{\dagger}\alpha^{\mu}\alpha^{4}\Psi. \tag{10}$$

There is a corresponding globally conserved charge, which is the integral of the scaling density, $J_{s}^{0}(\boldsymbol{A}\cdot\boldsymbol{E}_{\perp}-\boldsymbol{\Lambda}\cdot\boldsymbol{B})/2$. Hence, define

$$M_h = \int \Psi \alpha^0 \alpha^4 \Psi \, dV = \frac{1}{2} \int [\boldsymbol{A} \cdot \boldsymbol{E}_\perp - \boldsymbol{\Lambda} \cdot \boldsymbol{B}] dV. \quad (11)$$

More generally, an infinite set of related conserved densities is obtained by first defining a hierarchy of fields $\boldsymbol{\mathcal{A}}^{(n)}$ through the relation $\boldsymbol{\mathcal{A}}^{(n+1)} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}^{(n)}$, where $\boldsymbol{\mathcal{A}}^{(1)} = \boldsymbol{\mathcal{A}}$. Each field obeys an identical Maxwell equation, and it can be verified that there is a conserved density $\rho^{(m,n)} = \boldsymbol{\mathcal{A}}^{(m)*} \cdot \boldsymbol{\mathcal{A}}^{(n)}$, with conserved current $\boldsymbol{J}^{(m,n)} = \boldsymbol{\mathcal{A}}^{(m)*} \times \boldsymbol{\mathcal{A}}^{(n)}/i$, for every integer pair (m,n). The fields have dimension $[L]^{-n}$, and the currents have dimension $[L]^{-(n+m)}$. Thus, for example, $\rho^{(2,2)} = \mathcal{H}$ is the energy density, $\boldsymbol{J}^{(2,2)} = \boldsymbol{E}_\perp \times \boldsymbol{B}$ is the Poynting vector, and $\rho^{(2,1)} = J_d^0 + i J_s^0$ is a complex density that includes both the dual and scaling densities, giving a single conservation law.

Either form of the Lagrangian can be quantized by creating commutators from the fundamental constrained Dirac brackets [21]. The canonical momentum conjugate to $\boldsymbol{\mathcal{A}}$ exactly interchanges the magnetic and electric fields, since:

$$\hat{\boldsymbol{\Pi}} = i \hat{\mathcal{F}}^* = i \boldsymbol{\nabla} \times \hat{\mathcal{A}}^* = [\hat{\boldsymbol{B}} + i \hat{\boldsymbol{E}}_\perp]/\sqrt{2}. \quad (12)$$

Imposing canonical commutators at equal times, and assuming transverse field constraints, gives

$$[\hat{\mathcal{A}}_i(t,\boldsymbol{x}), \hat{\Pi}_j(t,\boldsymbol{x}')] = i \delta_{ij}^\perp(\boldsymbol{x} - \boldsymbol{x}'). \quad (13)$$

Here $\delta_{ij}^\perp(\boldsymbol{x} - \boldsymbol{x}')$ is the transverse delta function. The resulting field commutators agree with those derived in more standard ways, and hence agree with known quantum-field theoretic results. Related dual symmetric Lagrangians of a different type were introduced by Zwanziger [22], and by Schwarz and Sen [23], using pairs of real fields, rather than the present approach.

On expanding in terms of annihilation and creation operators for the different helicities, the complete mode expansion becomes: $\hat{\boldsymbol{\mathcal{A}}}(\boldsymbol{x}) = \hat{\boldsymbol{\mathcal{A}}}_+(\boldsymbol{x}) + \hat{\boldsymbol{\mathcal{A}}}_-^\dagger(\boldsymbol{x})$, where

$$\hat{\boldsymbol{\mathcal{A}}}_\sigma(\boldsymbol{x}) = \int d^3\boldsymbol{k} \left[\frac{1}{(2\pi)^3 k} \right]^{1/2} \boldsymbol{e}_{\boldsymbol{k}\sigma} \hat{a}_{\boldsymbol{k}\sigma} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \quad (14)$$

Here $k = |\boldsymbol{k}|$, while $\mathrm{e}_{\boldsymbol{k}\pm}$ represent the right and left circularly polarized photons, respectively [24], with $\boldsymbol{e}_{\boldsymbol{k}\sigma} \cdot \boldsymbol{e}_{\boldsymbol{k}\sigma'}^* = \delta_{\sigma\sigma'}$, $i\boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{k}\sigma} = \sigma \omega \boldsymbol{e}_{\boldsymbol{k}\sigma}$, and $|\boldsymbol{e}_{\boldsymbol{k}\sigma}| = 1$, where $\sigma = \pm 1$ is the helicity and $k = |\boldsymbol{k}|$.

Clearly, the positive frequency parts of $\hat{\boldsymbol{\mathcal{A}}}(\boldsymbol{x})$ and $\hat{\boldsymbol{\mathcal{A}}}^\dagger(\boldsymbol{x})$ correspond to different helicities. Thus, number operators can be formed in each helicity case, with

$$\hat{N}_\sigma = \int d^3\boldsymbol{k} \hat{N}_{\boldsymbol{k},\sigma}, \quad (15)$$

where $\hat{N}_{\boldsymbol{k},\pm} = \hat{a}_{\boldsymbol{k}\pm}^\dagger \hat{a}_{\boldsymbol{k}\pm}$. The above result can be used to calculate the conserved quantity $\hat{N}_h$, which shows that the normal-ordered operator equals the difference between left and right circularly polarized photon numbers [25]:

$$\hat{N}_h = \hat{N}_+ - \hat{N}_-. \quad (16)$$

Since the dual charge equals the helicity number, which is a difference operator, it appears possible to define a Hermitian dual phase operator conjugate to n_h, which could have a physical realization. By comparison, a Hermitian quantum phase operator conjugate to a positive-definite number operator is *not* well defined, except as a singular limit [5,6]. This is in agreement with the experimental situation, which is that optical phase measurements typically involve two modes, whose relative phase is then conjugate to a photon-number difference. Relative phases are also involved in recent work on optimal quantum measurement [26].

Operational measurements of photon flux usually involve time-averaging detectors that distinguish positive and negative frequency components. With this in mind, it is useful to note that the helicity potentials $\hat{\boldsymbol{\mathcal{A}}}_\sigma$ contain only positive frequency components. Defining a hierarchy of fields like this, according to $\hat{\boldsymbol{\mathcal{A}}}_\sigma^{(n+1)} = \sigma \boldsymbol{\nabla} \times \hat{\boldsymbol{\mathcal{A}}}_\sigma^{(n)}$ (where $\hat{\boldsymbol{\mathcal{A}}}_\sigma^{(1)} = \hat{\boldsymbol{\mathcal{A}}}_\sigma$), I find that these helicity fields satisfy modified complex Maxwell equations of the form

$$i \frac{\partial \boldsymbol{\mathcal{A}}_\sigma^{(n)}}{\partial t} = \sigma \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_\sigma^{(n)}. \quad (17)$$

This implies the existence of four conservation laws for each (m,n) pair, which divide the earlier conservation laws into their helicity components. The locally conserved helicity densities are

$$\hat{\rho}_{++}^{(m,n)} = \hat{\boldsymbol{\mathcal{A}}}_+^{(m)\dagger} \cdot \hat{\boldsymbol{\mathcal{A}}}_+^{(n)}/2, \quad (18)$$

$$\hat{\rho}_{+-}^{(m,n)} = \hat{\boldsymbol{\mathcal{A}}}_+^{(m)} \cdot \hat{\boldsymbol{\mathcal{A}}}_-^{(n)}/2,$$

together with corresponding equations found by exchanging the helicity signs, and the corresponding conserved currents. It is remarkable that these can all be directly obtained simply by using the complex form of Maxwell's equations. This gives rise to a definition of the total photon-number operator as

$$\hat{N} = \int \hat{n}(\boldsymbol{x}) dV = \sum_\sigma \int d^3\boldsymbol{k} \hat{N}_{\boldsymbol{k},\sigma}, \quad (19)$$

where I have introduced a conserved photon density operator $\hat{n}(\boldsymbol{x}) = \hat{n}_+(\boldsymbol{x}) + \hat{n}_-(\boldsymbol{x})$, which is defined in terms of its helicity components: $\hat{n}_\sigma(\boldsymbol{x}) = (\hat{\rho}_{\sigma\sigma}^{(1,2)} + \hat{\rho}_{\sigma\sigma}^{(2,1)})/2$. The definition in terms of covariant positive and negative helicity quantum fields $\hat{\Psi}_\sigma = (\hat{\boldsymbol{\mathcal{F}}}_\sigma, \hat{\boldsymbol{\mathcal{A}}}_\sigma)^T/\sqrt{2}$ (where $\hat{\boldsymbol{\mathcal{F}}}_\sigma = \hat{\boldsymbol{\mathcal{A}}}_\sigma^{(2)}$) is

$$\hat{n}_\sigma(\boldsymbol{x}) = \sigma : \hat{\Psi}_\sigma^\dagger(\boldsymbol{x}) \alpha^0 \Psi_\sigma(\boldsymbol{x}) :. \quad (20)$$

After time-averaging, it is possible to show that the normally ordered dual density $\hat{J}_h^0(\boldsymbol{x})$ is equal to the helicity density $\hat{n}_h(\boldsymbol{x}) = \hat{n}_+(\boldsymbol{x}) - \hat{n}_-(\boldsymbol{x})$.

Similarly, the scaling density $\hat{J}_s$ can be reexpressed in terms of helicity components. There are two main contributions, one from photon-density-type terms (which vanishes on time averaging), and another from terms proportional to $a_{k+} a_{k'-}$ and their conjugates, which correspond to a quadrature-squeezing [27] measurement, in which opposite helicities are correlated. These squeezing-related terms form a conserved current. I therefore define the conserved helicity squeezing density $\hat{m}_h(\boldsymbol{x})$, which is measurable using the technique of homodyne detection with a local oscillator field:

$$\hat{m}_h(\boldsymbol{x}) = (\hat{\boldsymbol{\mathcal{F}}}^\dagger_+ \cdot \hat{\boldsymbol{\mathcal{A}}}^\dagger_- + \hat{\boldsymbol{\mathcal{F}}}^\dagger_- \cdot \hat{\boldsymbol{\mathcal{A}}}^\dagger_+ - \text{H.c.})/(2i). \qquad (21)$$

These definitions of the helicity density, and squeezing density operators are obtained in terms of quasilocal fields with well-defined geometrical properties. The corresponding densities and currents have a zero expectation value at a given spatial location $\boldsymbol{x}$, provided $\hat{\boldsymbol{\mathcal{F}}}_\sigma(\boldsymbol{x})|\Psi\rangle = 0$. This is in accordance with the requirement of quasilocality, stated earlier. The photon-number density reduces to the earlier known forms [7,8]. The field $\hat{\boldsymbol{\psi}}$ therefore has the required attributes of a photon field operator.

The construction of the photon density operator from positive and negative frequency parts of the total operator means that $\hat{n}(\boldsymbol{x})$ corresponds physically to a detector that cycle averages the input field to obtain the photon number. A detector of this type implicitly involves time averaging. Because of this, the *instantaneous* value of $\hat{n}(\boldsymbol{x})$ is not guaranteed positive-definite. However, it is easily verified that the time average of $\hat{n}(\boldsymbol{x})$ on time scales longer than any relevant period *is* positive-definite. This seems to imply that any attempt to measure the conserved photon flux on time scales less than a cycle period can cause dark counts, photoemission, and similar effects. These are not always observed, due to the very slow response times of most current photo detectors (especially efficient ones), relative to an optical cycle. The related question of photon position in a state containing exactly one photon has also been analyzed recently [28].

In summary, the use of a complex field combining electric and magnetic components has a number of interesting features in developing a canonical theory of the radiation field. The Lagrangian generates first-order field equations. A covariant form is possible, resulting in a six-component wave equation, analogous to the Dirac equation. The fields themselves have a nonlocal Lorenz transformation property. They are treated here as massless spinors transforming under an infinite-dimensional representation of the Lorenz group. This does not appear to contradict either causality or unitarity.

The complex Lagrangian form allows a direct implementation of symmetry transformations that generate the conserved helicity density, as well a conserved scaling density. The dual phase can be regarded as the prototype of a physical phase measurement, since it comes from the electric and magnetic fields themselves. This phase is also a relative phase of two propagating electromagnetic modes, which is how it is operationally measured. It is significant that the corresponding conservation law is for a number-difference operator, which is known to be an essential requirement for obtaining a Hermitian phase operator. This gives an alternative point of view on the question of a phase operator in quantum optics, since the definition of a phase operator for a single mode of the radiation field is known to result in difficulties. I have also shown how an infinite set of local, normally ordered conserved densities can be generated, including a type of photon-number density and a squeezing density.

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