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Wave-function monopoles in Bose-Einstein condensates

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Experimental preparation of multispecies Bose-Einstein condensates should permit the creation of topologically stable defects beyond the superfluid vortex. But the coldness and isolation of condensates should also permit the survival for observable durations of ''pseudodefects,'' such as the one-dimensional dark soliton: localized structures related to a defect but not topologically stable. In this paper we investigate the viability of pseudodefects beyond one dimension, by examining ''wave-function monopoles'' in two-species condensates in two dimensions. We identify interesting instabilities, including a ''dancing mode'' for monopoles of higher winding number, and (in a one-dimensional limit) ''superfluid roulette.'' [S1050-2947(99)51510-2]

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A new natural goal for state engineering has appeared with the advent of multicomponent Bose-Einstein condensates [1,2]: topological defects, in which the order parameter field forms a localized ''knot.'' The superfluid vortex is an example that is well known in liquid helium. Other defects also exist, however, such as the ''textures'' found in superfluid helium-3 [3]; and skyrmions [4] and spin monopoles [5] have been suggested for dilute gas condensates. Since these condensates are weakly interacting and very well isolated, however, it is not obviously necessary to restrict one's attention to defects that are truly topologically stable; one can also consider particlelike ''pseudo defects,'' in which topologically stable configurations are embedded in the larger order-parameter spaces afforded by the $2n$ independent real components of the macroscopic wave function. The larger space allows the ''knot'' to be untied, and so such objects are topologically trivial; but their physical time evolution need not be trivial at all. The example of the one-dimensional dark soliton, a ''kink'' defect in a real Ψ that can be untwisted by complex deformations [6], indicates that pseudodefects can inherit particlelike robustness from their topological ''parents,'' but exhibit greater motional freedom than true defects (a larger effective phase space). To test the viability of the pseudodefect concept beyond one dimension (1D), we examine some more complex pseudodefects: 2D monopoles embedded in the 4D wave-function space of a two-component condensate. Our results indicate interesting phenomena that may be observable in future experiments, and also illustrate some useful general principles concerning multicomponent condensates.

Our paper is organized as follows. After presenting our concept of a ''wave-function monopole,'' we examine an analytically tractable one-dimensional limit of large radius. In this context we briefly discuss the consequences of having different scattering lengths among the two atomic species, indicating why neglecting these differences is a good first approximation. We then return to two dimensions and present numerical solutions of the two-component Gross-Pitaevskii equation (GPE), which describes the very-low-temperature behavior of condensates. We identify some intriguing effects, including vortex production and ''dancing modes'' of multiple monopoles. We remark on experimental approaches to creating wave-function monopoles, and on the limitations of our zero-temperature mean-field calculations.

A monopole is a configuration of an order-parameter field having as many components as there are spatial dimensions, in which the order parameter considered as a vector field points outwards all around a defect core. In the two-dimensional case we will consider, the field will be the two-component wave function of a two-species condensate, in a configuration in which both components happen to be real: $(\psi_1,\psi_2)=f(r)(\cos\theta,\sin\theta)$, where r,θ are the usual polar coordinates. Since the modulus of the order parameter is the total density of both components, the ''particlelike'' core of the monopole is in fact a local minimum of density: a void or bubble, maintained by destructive interference of matter waves. Our ''wave-function monopoles'' resemble both vortices and dark solitons in this respect; but since continuously deforming ψ_j into the complex plane can eliminate these monopoles, they are not true defects like the vortex but pseudodefects like the dark soliton.

It is instructive first to avoid the core $r\to 0$ of the monopole, and consider only its behavior at large r, where the radial dimension becomes unimportant, leaving the effectively one-dimensional problem of a two-species condensate on a circle of fixed radius R. So we replace $f(r)\to 1$, and examine the ''ring monopole'' (ψ_1,ψ_2) as a stationary solution to the one-dimensional GPE

$$i\dot{\psi}_j=-\frac{1}{2R^2}\frac{\partial^2}{\partial\theta^2}\psi_j+\left(\sum_j g_{ij}|\psi_i|^2-\mu_j\right)\psi_j, \qquad (1)$$

where at first we assume $g_{ij}=g$ (all scattering lengths equal), so that $\mu_j=1-(2gR^2)^{-1}$ provides $\dot{\psi}_j=0$.

For the ring monopole we can obtain analytically the Bogoliubov spectrum of perturbations, $\psi_j\to\psi_j+\epsilon\phi(\theta,t)$. Working to linear order in ϵ yields the modes

$$\begin{bmatrix}\phi^{\pm}_{1k}\\ \phi^{\pm}_{2k}\end{bmatrix}=\begin{bmatrix}X^{\pm}_{1k}\\ X^{\pm}_{2k}\end{bmatrix}\cos\Omega^{\pm}_k t-\frac{2iR^2\Omega^{\pm}_k}{k(k+2)}\begin{bmatrix}Y^{\pm}_{1k}\\ Y^{\pm}_{2k}\end{bmatrix}\sin\Omega^{\pm}_k t,$$

$$\begin{bmatrix}X^{\pm}_{1k}\\ X^{\pm}_{2k}\end{bmatrix}=\begin{bmatrix}(k-2)\cos(k-1)\theta+C^{\pm}_k\cos(k+1)\theta\\ -(k-2)\sin(k-1)\theta+C^{\pm}_k\sin(k+1)\theta\end{bmatrix}, \qquad (2)$$

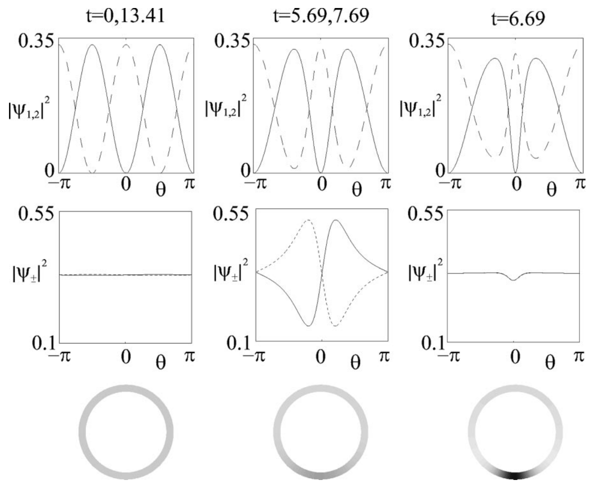

FIG. 1. Dynamical instability of the ring monopole, for $gR^2 = 300/\pi$. Five times are shown, the configurations at the last two times being precise revivals of those at the first two. The top three plots show $|\psi_1|^2$ (broken) and $|\psi_2|^2$ (solid); the middle three show $|\psi_+|^2$ and $|\psi_-|^2$ for the same configurations. (In the left middle plot the small initial perturbation can just be seen.) The rings at the bottom are overhead views of the $|\psi_1|^2+|\psi_2|^2$, with darker shading for lower total density.

$$\begin{bmatrix} Y^{\pm}_{1k} \\ Y^{\pm}_{2k} \end{bmatrix} = \begin{bmatrix} (k+2)\cos(k-1)\theta + C^{\pm}_k \cos(k+1)\theta \\ -(k+2)\sin(k-1)\theta + C^{\pm}_k \sin(k+1)\theta \end{bmatrix}.$$

The $\pm$ index distinguishes acoustic and optical branches, in which the two species' density perturbations are respectively in and out of phase:

$$C^{\pm}_k = 2 + 2k^2(gR^2)^{-1} \pm k\chi_k, \tag{3}$$

$$(2R^2\Omega^{\pm}_k)^2 = k^2[k^2+4+2gR^2(1\pm\chi_k)],$$

$$\chi_k \equiv \sqrt{1+8(gR^2)^{-1}+4k^2(gR^2)^{-2}}.$$

Rotating these modes by $\theta \to (\theta+\pi/2)$ produces a second set of independent modes, with the same frequencies.

The modes $k=0$ (for which the $\pm$ branches coincide) and $k\pm=2$ provide the four zero modes due to the model's U(2) symmetry (which already includes spatial rotation). All the other modes are positive frequency excitations, except for the two $k\pm=1$ modes; and for $gR^2>3/4$, these become *dynamical instabilities*. (See [5] for a detailed example of this type of stability analysis.) Infinitesimal excitation of these modes then grows exponentially until it becomes finite, and the Bogoliubov theory is inadequate. To follow the evolution into this regime, we solve the GPE numerically, using the split operator method as described in [7]. As shown in Fig. 1, the finite perturbation does not mix with other modes to produce irreversible (long revival time) decay, but grows to a maximum size, then shrinks back; and the cycle repeats. Further investigation confirms that the two unstable modes form an isolated subsystem of two degrees of freedom, resembling a two-dimensional particle in a ''Mexican hat'' potential.

This 1D model could in fact be realized in a tight toroidal trap. One could then visualize the unstable subsystem as a ''virtual monopole'' moving about inside the 1D ring to which the actual condensate is confined. At maximum amplitude the instabilities produce a density depression at some point around the trap: the virtual monopole ''touches the ring.'' Small perturbations of the unstable uniform density state, due to noise or even to quantum corrections to mean-field theory, make successive density minima appear at random points, and might thus be studied by observing this ''superfluid roulette.''

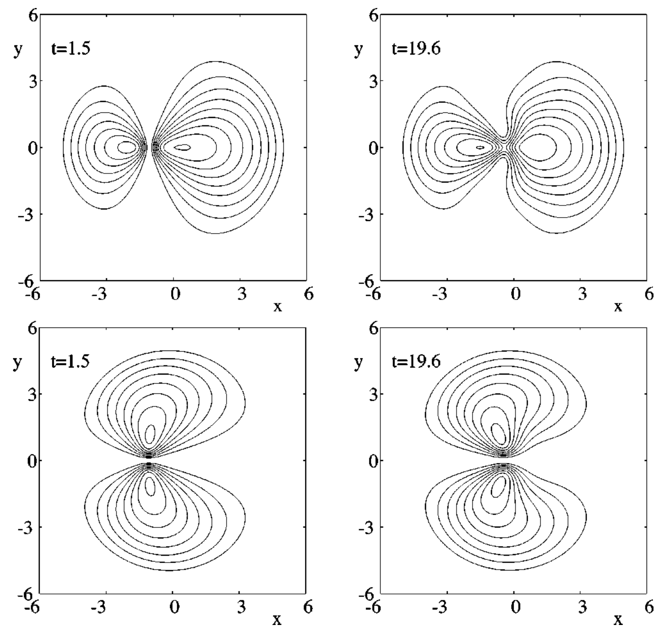

FIG. 2. $|\psi_1|^2$ (top) and $|\psi_2|^2$ for a monopole initially displaced in the direction $\theta=\pi$, at early and late times ($g\int d^2x|\psi_j|^2=2000$). Note the ''throttle'' formed by $|\psi_2|^2$.

Before going on to examine nonvirtual monopoles in two dimensions, we remark here that the simple solution we have examined can be generalized to encompass different (positive) scattering lengths g_{ij}. When phase separation is favored ($\det g_{ij}<1$) [8], one can obtain solutions in which our cosine and sine are replaced by Jacobi elliptic functions. When mixing of the two components is favored ($\det g_{ij}>1$), one can also generalize monopoles, but in a somewhat surprising way. As in changing from rectangular to circular polarization bases in fiber optics [9], one can change basis for the two condensate components, by defining $\psi_\pm = (1/\sqrt{2})(\psi_1 \pm i\psi_2) = e^{\pm i\theta}/\sqrt{2}$. Thus our monopole can also be described as two interpenetrating, counterpropagating currents. Changing scattering lengths so as to favor mixing of the two species $\psi_\pm$ will then stabilize this configuration. (Of

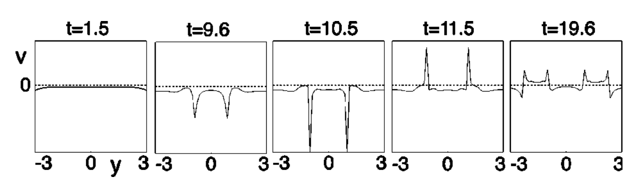

FIG. 3. Superfluid velocity component $v_{x1}=\mathrm{Im}(\partial_x \ln\psi_1)$, at various times in the same simulation illustrated in Fig. 4. Plots are ''edge-on'' views of the velocity field $v_{x1}(x,y)$, looking in the positive x direction: the velocity is very small everywhere except near the arc on which $|\psi_1|^2$ vanishes, so the edge-on view is convenient. Initial fluid velocity is negative as the monopole moves in the positive ($\theta=0$) direction.

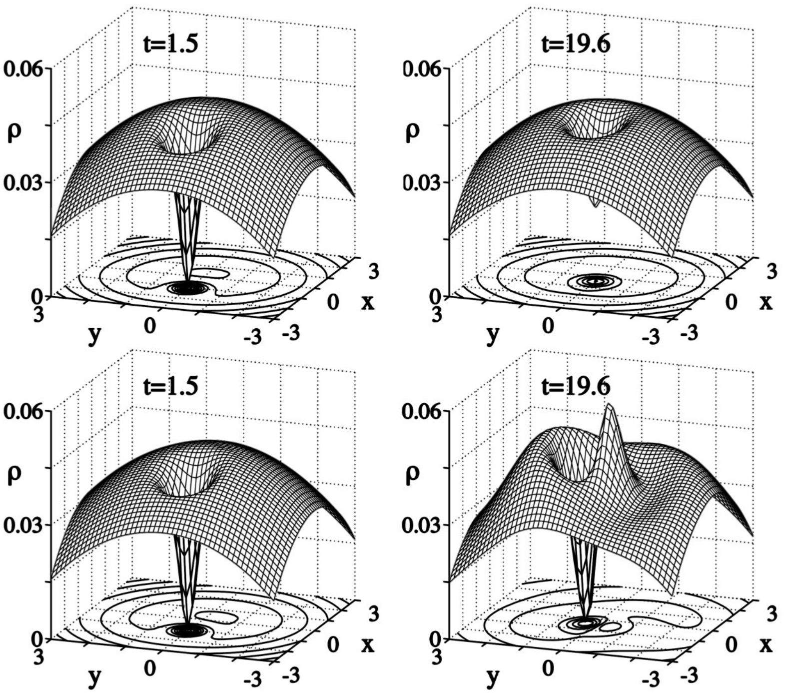

FIG. 4. Monopole instability in the SCV basis. Top figures show total density $\rho=|\psi_+|^2+|\psi_-|^2$ at early and late times; bottom figures show $|\psi_+|^2$ ($|\psi_-|^2$ being the mirror image in the x axis.) No new vortices form; instead, the initially superimposed vortices slip apart, and each is pinned.

course, if the scattering lengths are not exactly equal the transformation between monopole and countercurrents is only an approximate symmetry of the system.) From these exact generalizations one sees that the behavior for slightly unequal g_{ij}, as in the experimental case, is indeed only slightly different from what is seen with $g_{ij}=g$. We have also investigated such perturbations in two dimensions, and confirmed that equal scattering lengths are a very close approximation to slightly unequal, and even qualitatively similar to greatly unequal, g_{ij}. This is not surprising, since one expects qualitative sensitivity to small $\det g_{ij}$ to appear only over large space or time scales. Turning to the monopole in two dimensions, we therefore let $g_{ij}=g$ and note that $(\psi_1,\psi_2)=f(r)(\cos\theta,\sin\theta)$ is a stationary solution to the GPE,

$$i\dot{\psi}_j=-\frac{1}{2}\nabla_2^2\psi_j+\left(\sum_i |\psi_i|^2-\mu_j\right)\psi_j+\frac{1}{2}r^2\psi_j, \qquad (4)$$

as long as $f(r)$ satisfies the same nonlinear equation as the modulus of the two-dimensional vortex in the trap:

$$f''+\frac{1}{r}f'+\left(\frac{r^2}{2}-\frac{1}{r^2}\right)f=2g(f^2-\mu)f. \qquad (5)$$

We have studied these two-dimensional monopoles by numerically solving Eq. (4); we report representative results from investigations over a wide range of parameters. If a monopole is displaced from the center of the trap, it begins to fall back towards the center. (A displaced vortex, in contrast, precesses around the trap center [10].) But the monopole motion is unstable, as can be deduced by diagonalizing the linearized energy functional. More importantly, the instability is dynamical (self-driving without external dissipation): in numerical solutions the amplitude of small oscillations grows exponentially.

The instability, when infinitesimal, is a self-amplifying oscillation of the monopole back and forth across the trap. As can be seen from Fig. 2, the motion of the monopole involves the pouring of one of the two species through a narrow channel formed by the other species (which is comparatively passive throughout). As shown in Fig. 3, very rapid flow develops along the sides of this channel, until vortex pairs suddenly nucleate at the maxima of the passive species' density. Each pair then separates into an inner vortex, pinned by the passive species, and an outer vortex, which drifts away. This vortex formation spoils the nonlinear revivals seen on the 1D ring, and indicates that instabilities of pseudodefects in higher dimensions can be qualitatively more severe.

Although the 2D monopole never fully revives from its instability, neither does it rapidly die. Despite spawning vortices, an initially displaced monopole continues its fall towards the center of the trap with little apparent impediment. As (or just before) it reaches the center, however, its motion changes. In most cases we have investigated, it abruptly slows (though in some cases, and for reasons that are not clear to us, it seems instead to accelerate, and then does decay rather violently). Thereafter, the minimum in total density at the core gradually fills in, until only a slight depression remains. But even at this late stage, a double-lobed pattern of the two densities (as in Fig. 2) remains; and in each species a phase difference of π persists across the monopole, although the sharp jump in phase has been smoothed out. The instability appears to have saturated, leaving a ''smeared'' monopole that appears to be dynamically (although not energetically) stable. This eventual stability is much easier to understand in the $\psi_\pm$ basis, in which the initial monopole appears as a pair of superimposed, counter-rotating vortices (SCV's). In this basis no new vortices form. The instability is simply that the two initial SCV's slip apart, becoming a pair of reciprocally pinned vortices: each condensate species has one vortex, and each also fills a strong

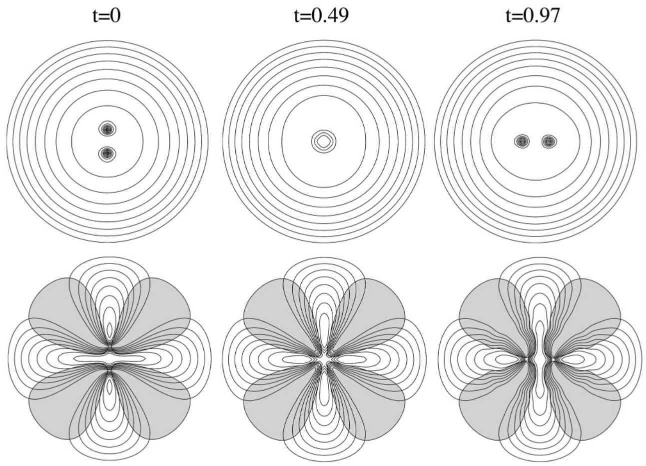

FIG. 5. Two aligned monopoles scattering in a harmonic trap. Top figures show total density; bottom figures show $|\psi_1|^2$ and $|\psi_2|^2$ separately. Since $|\psi_2|^2$ essentially does not change, it is merely indicated as uniformly shaded lobes.

potential well formed by the other species' vortex. (Compare the countercurrent picture of the ring instability, in Fig. 1.) Once this separation has occurred, the stability of two pinned vortices is unsurprising. See Fig. 4.

Using the monopole basis can be more illuminating of other phenomena, especially a particularly intriguing one: a pair of directly aligned monopoles perform a curious dance in a trap. They attract each other, merging into a single monopole of winding number $n=2$ [i.e., $(\psi_1,\psi_2) \propto (\cos n\theta, \sin n\theta)$]; this then separates again, but in the perpendicular direction; and the cycle repeats. In the SCV basis, this is a rather confusing system of orbiting vortices, and pinning peaks that form and dissolve, but in the monopole basis, it is an obvious mode of excitation of the $n=2$ monopole, see Fig. 5. And the peculiar scattering of monopoles at right angles in a head-on collision is actually what is expected for monopoles in gauge theories as well [11]. We find analogous dances, with π/n scattering, for monopoles of higher winding number n. In fact, monopoles of $|n|>1$ decay by these dances, since their amplitude *increases* over several periods.

We conclude by assessing the realism of the prospects we have raised. First, the Gross-Pitaevskii mean-field theory on which we rely should be an excellent first approximation for the extremely cold condensates that are already feasible (though in the case of a dynamical instability quantum corrections will be required after a logarithmically short time [12]). Second, very tightly confining traps in which effective two or one dimensionality is realized are already under development and are widely believed to be realistic prospects for the near future. Third, various methods of condensate state engineering should indeed be capable of creating the states we have proposed, or at least reasonable facsimiles of them. For instance, we have simulated the adiabatic transfer technique of [13], and achieved over 95% population transfer into the 2D monopole state, using realistic but unoptimized parameters. (The remaining population resides as a third condensate component in the monopole core.) Fourth, detection of monopoles is straightforward if the two species can be imaged separately [1], since the pattern of density lobes is large and obvious, and (in 2D) clearly indicates the presence of a monopole core.

Obviously the structures we have presented are examples of a wide range of conceivable pseudo- or true defects realizable in multicomponent condensates. Elementary but somewhat surprising changes of basis like the one we have shown should be useful in general, both for interpreting or predicting behavior and for relating apparently distinct configurations to each other. One should also expect vortex pair nucleation to be a common instability. Although such vortices may be hard to detect directly, their appearance in flow through a narrow channel vividly illustrates a potentially useful fact on which advanced multicondensate engineering might be based: finely detailed potentials are much more easily formed with matter waves than with light waves. If one wishes to manipulate a condensate with high resolution, therefore, the best tool may well be another condensate.

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