

## Dissipative dynamics of a vortex state in a trapped Bose-condensed gas

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We discuss dissipative dynamics of a vortex state in a trapped Bose-condensed gas at finite temperature and draw a scenario of decay of this state in a static trap. The interaction of the vortex with the thermal cloud transfers energy from the vortex to the cloud and induces the motion of the vortex core to the border of the condensate. Once the vortex reaches the border, it immediately decays through the creation of excitations. We calculate the characteristic lifetime of a vortex state and address the question of how the dissipative dynamics of vortices can be studied experimentally. [S1050-2947(99)50609-4]

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The recent successful experiments on Bose-Einstein condensation (BEC) in trapped clouds of alkali-metal atoms [1–3] have stimulated a great interest in the field of ultracold gases [4]. One of the goals of ongoing studies is to investigate the nature of a superfluid phase transition in ultracold gases and to make a link to more complicated quantum systems, such as superfluid helium. Of particular interest is the relation between Bose-Einstein condensation and superfluidity. However, being the most spectacular manifestation of the phase transition in <sup>4</sup>He, superfluidity has not yet been observed in trapped gases. A promising way of studying superfluidity in trapped gases is the creation of quantum vortices, as quantization of circulation and the related phenomenon of persistent currents are the most striking properties of superfluids.

A widely discussed option of creating vortices in trapped gases assumes the rotation of a slight asymmetry of a cylindrical trap after achieving BEC, or cooling down the gas sample below the Bose-condensation temperature in an already rotating trap [5,6]. Another possibility is a rapid quench of a gas sample near the critical temperature, which should lead to creation of vortices, even in a nonrotating trap [7]. It is worth mentioning the ideas of creating the vortex state in a Bose-condensed gas by optical means [8,9], and the idea of forming vortex rings in the regime of developed turbulence [10]. The spatial size of the vortex core in the Thomas-Fermi regime is too small to be observed, and for visualizing the vortex state it is suggested to switch off the trap and let the cloud ballistically expand. Then the size of the vortex core will be magnified approximately by the same factor as the size of the expanding condensate [11].

Similar to the recently studied kinkwise condensates [8,12,13], vortices are examples of macroscopically excited Bose-condensed states. In a nonrotating trap the vortex state has a higher energy than the ground-state Bose condensate; i.e., the vortex is thermodynamically unstable [14–16]. On the other hand, a quantum vortex with the lowest possible circulation (the vortex “charge” equal to 1) is dynamically

stable (small perturbations do not develop exponentially with time; see [17,11] and references therein). Therefore, the vortex state can only decay in the presence of dissipative processes.

In this Rapid Communication we discuss dissipative dynamics of a vortex state in a trapped Bose-condensed gas at finite temperatures and show how the interaction of the vortex with the thermal cloud leads to decay of the vortex state in a static trap. According to our scenario, the scattering of thermal excitations by a vortex provides the energy transfer from the vortex to the thermal cloud and induces the motion of the vortex core to the border of the condensate, where the vortex decays by creating elementary excitations. We calculate the characteristic lifetime of the vortex state and discuss how the dissipative dynamics of vortices can be studied experimentally.

We first briefly outline the main features of the vortex behavior in a superfluid, known from the studies of liquid helium. The motion of a vortex in a superfluid of density  $\rho_s$  satisfies the Magnus law (see [18–20] and references therein)

$$\rho_s(\mathbf{v}_L - \mathbf{v}_S) \times \boldsymbol{\kappa} = \mathbf{F}. \quad (1)$$

Here  $\mathbf{v}_L$  is the velocity of the vortex line and  $\mathbf{v}_S$  the velocity of superfluid at the vortex line. The vector  $\boldsymbol{\kappa}$  is parallel to the vortex line and is equal to the circulation carried by the vortex. The force  $\mathbf{F}$  acting on the vortex originates from the mutual friction between the normal component and the moving vortex line, and is usually small. Assuming the absence of friction ( $\mathbf{F} = \mathbf{0}$ ), the vortex moves together with the superfluid component ( $\mathbf{v}_L = \mathbf{v}_S$ ). The superfluid velocity  $\mathbf{v}_S(\mathbf{r})$  in the presence of a vortex at the point  $\mathbf{r}_0$  satisfies the equations

$$\text{rot} \mathbf{v}_S = 2\pi \boldsymbol{\kappa} \delta(\mathbf{r} - \mathbf{r}_0), \quad \text{div} \mathbf{v}_S = 0 \quad (2)$$

and is related to the phase  $\phi$  of the condensate wave function as  $\mathbf{v}_S = \nabla \phi$ . This leads to quantization of the circulation,  $\kappa = Z\hbar/m$  [21], where  $Z$  is an integer (the charge of the vortex) and  $m$  is the mass of the condensate particle. Below we will consider vortex states with  $Z=1$ , which are dynamically stable ([17,11] and references therein).

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Equations (2) are similar to the equations of the magnetostatic problem, with the magnetic field replaced by the velocity  $\mathbf{v}_S$  and the electric current replaced by  $\boldsymbol{\kappa}$ . The velocity field around an infinitely long straight vortex line is analogous to the magnetic field of a straight current:

$$\mathbf{v}_S(\mathbf{r}) = [\boldsymbol{\kappa} \times \mathbf{r}] / r^2. \quad (3)$$

The vortex itself can experience small oscillations of its filament, characterized by the dispersion law  $|\omega(k)| = \kappa k^2 \ln(1/ak)/2$  [22], where  $k$  is the wave vector of the oscillations, and  $a$  the radius of the vortex core. In a weakly interacting Bose-condensed gas the core radius is of the order of the healing length  $a = (\hbar^2/m\mu)^{1/2}$ , where  $\mu$  is the chemical potential.

We will see that the dissipative dynamics of a vortex state is insensitive to the details of the density distribution in a gas. The spatial size of the Thomas-Fermi condensate trapped in a harmonic potential of frequency  $\omega$  is  $R = (2\mu/m\omega^2)^{1/2}$ . Therefore, for finding the superfluid velocity  $\mathbf{v}_S$  in this case, we may consider a vortex in a spatially homogeneous condensate in a cylindrical vessel of radius  $R$ , with the vortex line parallel to the axis of the cylinder. For the vortex line at distance  $x_0$  from the axis, the velocity field can be found by using the ‘‘reflection’’ method [18]. In a nonrotating trap, in order to compensate the normal component of the velocity field (3) everywhere on the surface of the cylinder, we introduce a fictitious vortex with opposite circulation on the other side of the vessel wall, i.e., at distance  $R^2/x_0$  from the cylinder axis. At the position of the vortex the ‘‘reflection’’ induces the velocity

$$\mathbf{v}_S = \frac{[\boldsymbol{\kappa} \times \mathbf{x}_0]}{R^2 - x_0^2}. \quad (4)$$

As  $v_S \approx v_L$ , the vortex line will slowly drift around the axis of the trap. A characteristic time responsible for the formation of the velocity field (4) is  $\tau_R \sim R/c_s$ , where  $c_s = \sqrt{\mu/m}$  is the velocity of sound. Sufficiently far from the border of the Thomas-Fermi condensate, i.e., outside the spatial region where  $R - x_0 \ll R$ , the drift period is  $\tau_{dr} \sim x_0/v_S \sim R^2/\kappa$  and greatly exceeds the time  $\tau_R$ :

$$\frac{\tau_R}{\tau_{dr}} \sim \frac{R}{c_s \tau_{dr}} \sim \frac{a}{R} \ll 1.$$

This means that we can neglect the retardation effects and, in particular, the emission of phonons by the moving vortex. In other words, the ‘‘cyclotron’’ radiation is prohibited, since the wavelength  $c_s \tau_{dr}$  of sound that would be emitted exceeds the size  $R$  of the condensate.

According to the above-mentioned magnetostatic analogy, in a nonrotating trap the potential energy of the system (vortex plus its reflection) can be thought of as the energy of two counterflowing currents. Since the currents attract each other, the energy is negative and decreases with displacement of the vortex core towards the wall. In other words, it is energetically favorable for the vortex to move to the border of the vessel. Near the border the velocity of the vortex exceeds the Landau critical velocity, and in a homogeneous superfluid the vortex decays through the creation of phonons [18]. In a

trapped gas the condensate density strongly decreases near the border, and the vortex can decay by emitting both collective and single-particle excitations. The motion of the vortex towards the wall requires the presence of dissipation; as in the frictionless approach the velocity of the vortex core coincides with the velocity (4), which does not contain a radial component. Thus, just the presence of dissipative processes provides a decay of the vortex state (see [15] and related discussion [13] of the stability of a kink state).

The dissipation originates from the scattering of elementary excitations by the vortex and is related to the friction force  $\mathbf{F}$  in Eq. (1), which is nothing other than the momentum transferred from the excitations to the vortex per unit time. This force can be decomposed into longitudinal and transverse components:

$$\mathbf{F} = -D\mathbf{u} - D'\mathbf{u} \times \boldsymbol{\kappa} / \kappa, \quad (5)$$

where  $\mathbf{u} = \mathbf{v}_L - \mathbf{v}_n$ ,  $\mathbf{v}_n$  is the velocity of the normal component, and  $D, D'$  are longitudinal and transverse friction coefficients, respectively. In a static trap,  $\mathbf{u} = \mathbf{v}_L$ , as the normal component is at rest ( $\mathbf{v}_n = 0$ ). The friction force has been investigated in relation to the attenuation of the second sound in superfluid  $^4\text{He}$ , where the most important feature is the transverse component [23–25] (see also [19] for a review). For a straight infinite vortex line (parallel to the  $z$  axis), a general expression for the friction force in a homogeneous superfluid is obtained in terms of the scattering amplitude  $f(\mathbf{k}, \mathbf{k}')$  [25]:

$$\mathbf{F} = \left[ \int \frac{\partial n}{\partial E_k} \hbar(\mathbf{k}\mathbf{u}) \int (\mathbf{k} - \mathbf{k}') \frac{\delta(E_k - E_{k'})}{\delta(k_z - k'_z)} \times \left| f(\mathbf{k}, \mathbf{k}') \right|^2 \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \right] - [\mathbf{u} \times \boldsymbol{\kappa}] \rho_n. \quad (6)$$

Here  $\rho_n$  is the local mass density of the normal component,  $\mathbf{k}, \mathbf{k}'$  are the wave vectors of the incident and scattered excitations,  $n(E_k) = [\exp(E_k/T) - 1]^{-1}$  are the Bose occupation numbers for the excitations,  $E_k$  is the excitation energy, and  $T$  the gas temperature. Comparing the second terms of Eqs. (5) and (6), one immediately arrives at the universal expression for the transverse friction coefficient,  $D' = \kappa \rho_n$ , assuming that the first term of Eq. (6) does not contribute to  $D'$  [19,25].

We now turn to our analysis of the dissipative dynamics of the vortex state in a nonrotating trap, which is related to the motion of the vortex core (line) to the border of the condensate. This motion occurs on top of small oscillations of the vortex filament and a slow drift (4) of the vortex core. The radial component of the velocity of the vortex core is determined by the longitudinal friction coefficient  $D$ . For finding these quantities in dilute Bose-condensed gases, the analysis of [23–25,19] can only be used at very low temperatures ( $T \ll \mu$ ), where the number of thermal excitations is very small and, hence, the longitudinal friction force is extremely weak.

The situation is drastically different in the temperature range  $T \geq \mu$ , which is the most interesting for trapped Bose-condensed gases. We will consider the limit  $T \gg \mu$  and first analyze how the vortex scatters excitations with energies

$E_k \gg \mu$ . These excitations are single particles, and their de Broglie wave length is much smaller than the spatial size  $R$  of the condensate. Most important is the interaction of the excitations with the vortex at distances from the vortex line  $r \sim a \ll R$ . Therefore, the corresponding friction force in a trapped condensate can be found in the local-density approximation: We may use Eq. (6), derived for a homogeneous superfluid, and then replace the condensate density  $n_0$  by the Thomas-Fermi density profile of the trapped condensate.

The Hamiltonian of the single-particle excitations is  $\hbar^2 \mathbf{k}^2/2m + 2n_0(\mathbf{r})g - \mu$ , where the second term originates from the mean-field interparticle interaction, where  $n_0(\mathbf{r})$  is the density of the vortex state,  $g = 4\pi\hbar^2 a_{sc}/m$ ,  $a_{sc}$  is the scattering length, and  $\mu = n_0(\infty)g[n_0(\infty) \equiv n_0]$ . For  $r \rightarrow \infty$  we have  $\hat{H}(\mathbf{k}, \mathbf{r}) = \hbar^2 \mathbf{k}^2/2m + \mu$ . Hence, the interaction Hamiltonian responsible for the scattering of excitations from the vortex can be written as

$$\hat{H}_{int} = 2[n_0(\mathbf{r})g - \mu].$$

For the vortex charge  $Z=1$ , at distances  $r \ll a$ , the interaction Hamiltonian  $\hat{H}_{int} \approx -2\mu$ . For  $r \gg a$  we have  $n_0(\mathbf{r}) \approx (\mu - \hbar^2/2mr^2)/g$  and  $\hat{H}_{int} \approx -\hbar^2/mr^2$ . The scattering amplitude in Eq. (6) can be written as  $f(\mathbf{k}, \mathbf{k}') = 2\pi\delta(k_z - k'_z)\tilde{f}(\mathbf{k}, \mathbf{k}')$ , where the two-dimensional scattering amplitude in the Born approximation is given by

$$\tilde{f}(\mathbf{k}, \mathbf{k}') = \int d^2r H_{int}(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}. \quad (7)$$

Here  $\mathbf{q} = \mathbf{k} - \mathbf{k}'$  is the momentum transferred from the excitation to the vortex. As the amplitude  $\tilde{f}$  only depends on  $|\mathbf{q}|$ , the first term in Eq. (6) is purely longitudinal.

For  $qa \ll 1$ , which corresponds to small-angle scattering, from Eq. (7) we obtain  $\tilde{f} \sim (\hbar^2/m)\ln(1/qa)$ . For  $qa \gg 1$  we find  $|\tilde{f}(q)|^2 \sim (\hbar^2/m)^2 \sin^2(qa - \pi/4)/(aq)^3$ . Using these results in Eq. (6), we see that the main contribution to the integral over momenta comes from energies  $E_k$  satisfying the inequality  $\mu \lesssim E_k \lesssim T$ . A direct calculation of the longitudinal friction coefficient gives

$$D \approx \kappa \rho_n(T) (n_0 g/T)^{1/2}, \quad (8)$$

where the density of the normal component

$$\rho_n = -\frac{1}{3} \int \frac{\partial n}{\partial E_p} p^2 \frac{d^3 p}{(2\pi\hbar)^3} \approx 0.1 \frac{m^{5/2} T^{3/2}}{\hbar^3}.$$

A collective character of excitations with energies  $E_k \sim \mu$  can influence the numerical coefficient in Eq. (8), and for this reason we did not present the exact value of this coefficient in the single-particle approximation. In cylindrical traps the behavior of excitations with energies  $E \sim \mu$  (and somewhat larger) is stochastic [26], and hence the discrete structure of the spectrum is not important [27].

The coefficient  $D \propto T$ , and Eq. (8) can be rewritten as  $D \propto \hbar n_0 \xi$ , where the quantity

$$\xi = (n_0 a_{sc}^3)^{1/2} (T/\mu) \ll 1 \quad (9)$$

is a small parameter of the finite-temperature perturbation theory at  $T \gg \mu$ . The inequality  $\xi \ll 1$  remains valid even near the BEC transition temperature, except in the region of critical fluctuations [27].

Relying on Eq. (8) for the longitudinal friction force, we consider the motion of the vortex line to the border of the condensate in a static trap, where the normal component is at rest. Assuming a small friction in Eqs. (1) and (5), for finding a friction-induced small quantity  $\mathbf{v}_L - \mathbf{v}_S$ , we only retain the terms linear in the dissipation coefficients  $D$  and  $D'$ . Then we obtain the equation

$$\rho_s [(\mathbf{v}_L - \mathbf{v}_S) \times \boldsymbol{\kappa}] = -D \mathbf{v}_S - D' [\mathbf{v}_S \times \boldsymbol{\kappa}] / \kappa,$$

which has a solution of the form  $\mathbf{v}_L = \mathbf{v}_L^{(r)} \hat{r} + \mathbf{v}_L^{(\phi)} [\boldsymbol{\kappa} \times \mathbf{r}] / \kappa r$ . For the radial ( $\mathbf{v}_L^{(r)}$ ) and tangential ( $\mathbf{v}_L^{(\phi)}$ ) components of the velocity of the vortex line we find

$$\mathbf{v}_L^{(r)} = D \mathbf{v}_S / \rho_s \kappa, \quad \mathbf{v}_L^{(\phi)} = \mathbf{v}_S (1 - D' / \rho_s \kappa). \quad (10)$$

From Eqs. (10) it is clear that the radial motion of the vortex is governed by the value of the longitudinal friction coefficient, whereas the transverse friction (Iordanskii force) simply slows down the drift velocity (4) of the vortex. The radial velocity  $\mathbf{v}_L^{(r)} \ll \mathbf{v}_S$ , which is guaranteed by the inequality (9).

The time dependence of the distance  $x_0$  of the vortex line from the axis of a cylindrical trap follows from the equation of radial motion for the vortex,  $dx_0/dt = \mathbf{v}_L^{(r)}$ . With Eq. (10) for  $\mathbf{v}_L^{(r)}$  and Eq. (4) for  $\mathbf{v}_S$ , for the characteristic time of motion of the vortex from the center of the trap to the border, we obtain

$$\tau \approx \int_{x_{\min}}^R \frac{dx_0 m (R^2 - x_0^2) \rho_s \left( \frac{n_0 g}{T} \right)^{1/2}}{\hbar x_0 \rho_n}, \quad (11)$$

where  $x_{\min}$  is the initial displacement of the vortex line from the axis of the trap. The vortex velocity is the smallest near the axis, and the main contribution to the integral in Eq. (11) comes from distances  $x_0 \ll R$ . Therefore, we can neglect  $x_0$  in the numerator of the integrand and put  $\rho_s = \rho_s(0)$ ,  $n_0 = n_0(0)$ . Then, Eq. (11) yields

$$\tau \approx \frac{m R^2 \rho_s \left( \frac{n_0 g}{T} \right)^{1/2}}{\hbar \rho_n} \ln(R/x_{\min}). \quad (12)$$

This result is only logarithmically sensitive to the exact value of  $x_{\min}$ , and we can put  $x_{\min} \sim a$ .

Once the vortex reaches the border of the condensate, it immediately decays. Hence, the time  $\tau$  can be regarded as a characteristic lifetime of the vortex state in a static trap. Interestingly, the decay rate can be written as

$$\tau^{-1} \sim \frac{E_0}{\hbar} (n_{0m} a_{sc}^3)^{1/2} \left( \frac{T}{\mu} \right), \quad (13)$$

where  $n_{0m}$  is the maximum condensate density, and  $E_0 \sim \hbar^2/mR^2$  is the energy of excitation corresponding to the motion of the vortex core with respect to the rest of the condensate (excitation with negative energy, found in the recent calculations [28,29,17,14,16]). Equation (13) is similar to the damping rate of low-energy excitations of a trapped

condensate, found beyond the mean-field approach [27,4]. Both rates are proportional to the small parameter  $\xi$  [Eq. (9)].

For Rb and Na condensates at densities  $n_0 \sim 10^{14} \text{ cm}^{-3}$  and temperatures  $100 \leq T \leq 500 \text{ nK}$ , in the static traps with frequencies  $10 \leq \omega \leq 100 \text{ Hz}$ , the lifetime  $\tau$  of the vortex state ranges from 0.1 to 10 s. This range of times is relevant for experimental studies of the dissipative vortex dynamics.

A proposed way of identifying the presence of a vortex state in a trapped Bose-condensed gas assumes switching off the trap and observing a ballistically expanding gas sample [11]. As follows from the numerical simulations [11], at zero temperature the expansion of a condensate with a vortex occurs along the lines of the scaling theory [30,31]. The shape of the Bose-condensed state is nearly preserved and its spatial size is increasing. Due to expansion, the density of the condensate decreases, and the size of the vortex core increases to match the instantaneous value of the healing length. This should allow one to detect the vortex through the observation of a hole in the density profile of the condensate.

It is important to emphasize that at temperatures  $T \gg \mu$  the thermal cloud will expand with the thermal velocity  $v_T \sim \sqrt{T/m}$ , which is much larger than the expansion velocity of the condensate (the latter is of order the sound velocity  $c_s$ ).

Therefore, after a short time  $R/v_T$  the thermal component flies away, and the dissipation-induced motion of the vortex core ceases. Accordingly, the expansion of the Bose-condensed state will be essentially the same as that at zero temperature. This means that the relative displacement of the vortex core from the trap center practically remains the same as before switching off the trap. Therefore, the dissipative motion of the vortex towards the border in the initial static trap can be studied by switching off the trap at different times and visualizing the position of the vortex core in a ballistically expanding condensate.

In conclusion, we have developed a theory of dissipative dynamics of a vortex state in a trapped Bose-condensed gas at finite temperatures and calculated the decay time of the vortex with charge equal to 1 in a static trap. Our theory can be further developed to analyze the motion of vortices in rotating traps and, in particular, to calculate a characteristic time of the formation of the vortex state in a trap rotating with supercritical frequency.

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- [1] M.H. Anderson *et al.*, *Science* **269**, 198 (1995).  
 [2] K.B. Davis *et al.*, *Phys. Rev. Lett.* **75**, 3969 (1995).  
 [3] C.C. Bradley, C.A. Sackett, J.J. Tolett, and R.G. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995).  
 [4] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).  
 [5] A. J. Leggett, in *Topics in Superfluidity and Superconductivity, Low Temperature Physics*, edited by M. Hoch and R. Lemmer (Springer-Verlag, New York, 1992).  
 [6] S. Stringari, *Phys. Rev. Lett.* **82**, 4371 (1999).  
 [7] J.R. Anglin and W.H. Zurek, e-print quant-ph/9804035.  
 [8] R. Dum, I. Cirac, M. Lewenstein, and P. Zoller, *Phys. Rev. Lett.* **80**, 2972 (1998).  
 [9] M. Olshanii and M. Naraschewski, e-print cond-mat/9811314.  
 [10] B. Jackson, J.F. McCann, and C.S. Adams, *Phys. Rev. Lett.* **80**, 3903 (1998).  
 [11] Y. Castin and R. Dum, *Eur. Phys. J. D* (to be published).  
 [12] T. Busch and J.R. Anglin, e-print cond-mat/9809408.  
 [13] A.E. Muryshev, H.B. van Linden van den Heuvell, and G.V. Shlyapnikov, e-print cond-mat/9811408.  
 [14] D.S. Rokhsar, *Phys. Rev. Lett.* **79**, 2164 (1997).  
 [15] A.L. Fetter, *J. Low Temp. Phys.* **113**, 189 (1998).  
 [16] A. Svidzinsky and A.L. Fetter, e-print cond-mat/9811348.  
 [17] H. Pu, C.K. Law, J.H. Eberly, and N. Bigelow, *Phys. Rev. A* **59**, 1533 (1999).  
 [18] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).  
 [19] E.B. Sonin, *Phys. Rev. B* **55**, 485 (1997).  
 [20] D.J. Thouless, P. Ao, and Q. Niu, *Phys. Rev. Lett.* **76**, 3758 (1996).  
 [21] L. Onsager, *Nuovo Cimento* **6**, 249 (1949).  
 [22] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1980).  
 [23] L.P. Pitaevskii, *Zh. Éksp. Teor. Fiz.* **35**, 1271 (1959) [*Sov. Phys. JETP* **8**, 888 (1959)].  
 [24] A.L. Fetter, *Phys. Rev.* **136**, A1488 (1964).  
 [25] S.V. Iordanskii, *Zh. Éksp. Teor. Fiz.* **49**, 225 (1965) [*Sov. Phys. JETP* **22**, 160 (1966)].  
 [26] M. Fliesser, A. Cordas, R. Graham, and P. Szepfalusy, *Phys. Rev. A* **56**, R2533 (1997); **56**, 4879 (1997).  
 [27] P.O. Fedichev, G.V. Shlyapnikov, and J.T.M. Walraven, *Phys. Rev. Lett.* **80**, 2269 (1998); P.O. Fedichev and G.V. Shlyapnikov, *Phys. Rev. A* **58**, 3146 (1998).  
 [28] R.J. Dodd, K. Burnett, M. Edwards, and C. Clark, *Phys. Rev. A* **56**, 587 (1997).  
 [29] T. Isoshima and K. Machida, *J. Phys. Soc. Jpn.* **66**, 3602 (1997).  
 [30] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, *Phys. Rev. A* **54**, R1753 (1996); **55**, R18 (1997).  
 [31] Y. Castin and R. Dum, *Phys. Rev. Lett.* **77**, 5315 (1996).