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Time-asymmetric quantum physics

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A quantum theory that applies to the (closed) universe needs to be time asymmetric because of the cosmological arrow of time. The preparation ⇒ registration arrow of time (a state must be prepared before an observable can be detected in it) of the quantum mechanics of measured systems provides a phenomenological reason for an asymmetric semigroup time evolution. The standard theory in the Hilbert space (HS) is inadequate for either since the mathematics of the HS allows only reversible unitary group evolution and time symmetric boundary conditions. The mathematical theory that describes time-asymmetric quantum physics in addition to providing the mathematics for the Dirac kets is the rigged Hilbert space (RHS) theory. It uses a pair of RHS's of Hardy class with complementary analyticity property, one for the prepared states (''in states'') and the other for the registered observables (''out states''). The RHS's contain Gamow kets which have all the properties needed to represent decaying states and resonances. Gamow kets have only asymmetric time evolution. The neutral kaon system is used to show that quasistationary microphysical systems can be experimentally isolated if their time of preparation can be accurately identified. The theoretical predictions for a Gamow ket have the same features as the observed decay probabilities, including the time ordering. This time ordering is the same as the time ordering in the probabilities of histories for the quantum universe. The fundamental quantum mechanical arrow of time represented by the semigroup in the RHS is therefore the same as the cosmological arrow of time, assuming that the universe can be considered a closed quantum system.
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I. INTRODUCTION

Standard quantum mechanics in Hilbert space $\mathcal{H}$ is a time-symmetric theory with a time-symmetric dynamical (differential) equation and time symmetric boundary conditions. This is in contrast to many time-asymmetric phenomena observed in classical and also in quantum physics. Of the latter we want to discuss in this article two examples, the decay of a quasistable particle [1] and the expanding universe as a whole when considered as a closed quantum system [2]. The time asymmetry that is discussed here is not the irreversible time evolution of open systems under external influences [3] (cf. Appendix C). The time asymmetry is also *not* due to a time reversal noninvariant Hamiltonian; the (self-adjoint and semibounded) Hamiltonians we discuss here are time reversal and CP invariant. Our time asymmetry expresses causality in quantum theory. It is mathematically described by the appropriate choice of spaces of allowed solutions (asymmetric boundary conditions) for the usual time-symmetric Schrödinger equations.

In classical physics solutions of time-symmetric dynamical equations with time asymmetric boundary conditions come in pairs, e.g., big bang–big crunch in general relativity or retarded–advanced in electromagnetism. With the choice of the boundary condition, one of the two time-asymmetric solutions is selected. The Hilbert space theory of quantum mechanics does not allow such time-asymmetric formulations. In the Hilbert space formulation of quantum mechanics the space-time transformations (e.g., Galilean transformations, Poincaré transformations) are described by a unitary group representation in the Hilbert space $\mathcal{H}$. Thus the time evolution is unitary and reversible, and it is given by $U^{\dagger}(t) = \exp(-iHt), -\infty < t < \infty$. This is the consequence of a series of mathematical theorems which follow from the mathematical properties—specifically the topological completeness—of the Hilbert space; they are listed Appendix A. These theorems in particular exclude the existence of nonzero probabilities which are zero before a given finite time t_0 ($t_0 \neq -\infty$), which is the time at which the quasistable particle had been produced or the time of the big bang in the two examples of this article. The decay of resonances and the quantum theory of our universe can therefore not be described consistently in the mathematical theory using the Hilbert space.

Disregarding Hilbert space mathematics, in scattering theory one arrived in a heuristic way at a pair of time-asymmetric boundary conditions by choosing in- and out-plane wave ''states'' $|E^+\rangle$ and $|E^-\rangle$ which have their origin in the $\epsilon = +0$ and $\epsilon = -0$ of the Lippmann-Schwinger equation [4], cf. Appendix B. Still, the widespread opinion remained that asymmetric or irreversible time evolution of closed quantum mechanical systems is impossible.

It could have been that historically the analogy to classical mechanics was the origin of this belief, though the time evolution for the Schrödinger equation could have as well been discussed in analogy to the electromagnetic waves, and for those the radiation arrow of time was well accepted (and by some even considered as fundamental [5]). However, the reversibility of the Hamiltonian generated time evolution in von Neumann's [6] Hilbert space quantum theory must have been a decisive factor for the longevity of this belief.

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Already the Dirac [7] kets $|E\rangle$, $0\leqslant E<\infty$, are not elements of the Hilbert space but generalized eigenvectors and required the extension of the Hilbert space $\mathcal{H}$ to the rigged Hilbert space $\Phi\subset\mathcal{H}\subset\Phi^{\times}$ [8], where Φ is a linear scalar product space of well-behaved vectors $\phi\in\Phi$ (represented by smooth etc. functions $\langle E|\phi\rangle$) and $\Phi^{\times}\ni|E\rangle$ is the space of its antilinear functionals. In the S-matrix element (cf. Appendix B)

$$(\psi^{\text{out}},S\phi^{\text{in}})=(\psi^{-},\phi^{+})=\sum_{bb'}\int_{0}^{\infty}dE\langle\psi^{-}|b,E^{-}\rangle\langle b|S(E)|b'\rangle \times\langle^{+}b',E|\phi^{+}\rangle \tag{1.1}$$

the Dirac ''scattering states'' $|E^{\pm}\rangle$ which are obtained from $|E\rangle$ by the Lippmann-Schwinger equation appear. In order to analytically continue to the resonance pole $z_R=E_R-i\Gamma/2$ of the S matrix $\langle b|S(z)|b'\rangle$ the set of in states $\{\phi^{+}\}\equiv\Phi_{-}$ and out states $\{\psi^{-}\}\equiv\Phi_{+}$ must additionally have some analyticity property. In order to get a Breit-Wigner energy distribution for the pole term we postulate that the energy wave functions $\langle^{-}E|\psi^{-}\rangle$ and $\langle^{+}E|\phi^{+}\rangle$ are well-behaved Hardy class functions of the upper and lower half plane in the second sheet of the energy surface of the S matrix.

The analytically continued Dirac kets $|E^{-}\rangle\in\Phi_{+}^{\times}$ of the Lippmann-Schwinger equations become—using the Cauchy formula—at the resonance pole $z_R=E_R-i\Gamma/2$ the Gamow kets $|z_R^{-}\rangle\in\Phi_{+}^{\times}$. The time asymmetric semigroup evolution of these Gamow kets,

$$e^{-iH^{\times}t}|_{t\geqslant0}|z_R^{-}\rangle=e^{-iE_Rt}e^{-\Gamma/2t}|z_R^{-}\rangle \quad \text{for} \quad t\geqslant0 \quad \text{only}, \tag{1.2}$$

is then derived as a mathematical consequence of the structure of the rigged Hilbert space $\Phi_{+}^{\times}\supset\mathcal{H}\supset\Phi_{+}$ of the Hardy class [9] (in the same way as the time symmetric unitary group evolution given by $e^{-iH^{\dagger}t}$, $-\infty<t<+\infty$, is a mathematical consequence of the Hilbert space structure).

Thus asymmetric time evolution would be a natural property of quantum mechanical states represented by the vector $|z_R^{-}\rangle$ and other elements of the space $\Phi_{+}^{\times}$. In this article we want to discuss the phenomenological evidence for such states and the experimental conditions and phenomenological reason for the asymmetric time evolution.

In Sec. II we review the basic concepts of quantum physics in a way that shows which mathematical properties are important for quantum mechanical calculations and which are idealizations and not directly obtainable from experimental data. We also argue that experimental observations involve a time asymmetry, the preparation $\Rightarrow$ registration arrow of time. We then give two examples of quantum mechanical states with asymmetric time evolution, the quasistable particle and the universe considered as a closed quantum system, and discuss their common features. In Sec. III we provide the mathematical theory for time-asymmetric quantum mechanics and give some of its result. In Sec. IV we discuss an example of a state with an arrow of time prepared in a laboratory experiment; we compare the concept of its preparation time with the initial time for the state of the quantum universe.

II. CALCULATIONAL METHODS, MATHEMATICAL IDEALIZATIONS, AND EXPERIMENTAL OBSERVATIONS

In quantum theory one has states and observables. States are described by density or statistical operators and conventionally denoted by ρ or W; for pure states vectors ϕ are used. Observables are described by operators A $(=A^{\dagger}),\Lambda,\ P(=P^2)$, but we will also use vectors ψ to describe a state P if $P=|\psi\rangle\langle\psi|$.

The vectors ϕ,ψ are elements of a vector space Φ with a scalar product, denoted $(\cdot,\cdot)$ or $\langle\cdot|\cdot\rangle$. The operators A, Λ, are elements of the algebra of linear operators $\mathcal{A}$ in Φ. The linear space Φ, though often called a Hilbert space, is mostly treated like a pre-Hilbert space, i.e., without a topology (or without a definition of convergence) and it is not topologically complete. If we want to emphasize that Φ has no topology we denote it by Φ_{alg}.

Each ''kind'' of quantum physical system is associated to a space Φ.

In experiments, the state W [or the pure (idealized) state ϕ] is prepared by a preparation apparatus and the observable A (or the idealized observable ψ) is registered by a registration apparatus (e.g., a detector). The fundamental aspect of the theory presented here is to clearly distinguish between states (e.g., in states ϕ^{+} of a scattering experiment) and observables (e.g., detected out states ψ^{-} of a scattering experiment), cf. Appendix B.

The measured (or registered) quantities are ratios of (usually) large numbers, the detector counts. They are interpreted as probabilities, e.g., as the probability to measure the observable Λ in the state W at the time t, which is denoted by $\mathcal{P}_W(\Lambda(t))$.

The probabilities are calculated in theory as the scalar product or, in the general case, as the trace. This is shown in relations (2.1a) and (2.1b), below where $\approx$ indicates the equality between the experimental and the theoretical quantities and $\equiv$ is the mathematical definition of the theoretical probabilities in terms of the quantities of the space Φ (which is not yet completely defined):

$$N_i/N\approx\mathcal{P}_{\phi}(P)\equiv|\langle\psi|\phi\rangle|^2, \tag{2.1a}$$

$$N(t)/N\approx\mathcal{P}_W(\Lambda(t))\equiv\text{Tr}[\Lambda(t)W_0]=\text{Tr}[\Lambda_0W(t)]. \tag{2.1b}$$

The parameter t in Eq. (2.1b) is the continuous time parameter and the observable Λ, or the state W, are ''continuous'' functions of time [with $W_0=W(t=0)$]. Thus, $\mathcal{P}_W(\Lambda(t))$ is thought of as a continuous function of t. But $N(t)$ is the number of counts in the time interval between $t=0$ and t, which is an integer. Thus the right hand side of $\approx$ in Eq. (2.1b) changes continuously in t, but the left hand side can only change in steps of rational numbers. This shows that the continuity of $\mathcal{P}_W(\Lambda(t))$ or of $|\langle\psi(t)|\phi\rangle|^2$

$=|\langle\psi|\phi(t)\rangle|^2$ as a function of t, and similar topological questions, are not directly experimentally testable.

For more general observables A, which are expressed in terms of the orthogonal projection operators P_i ($P_iP_j = \delta_{ij}P_j$) as

$$A=\sum_{i=1}^{\infty} a_iP_i\,, \tag{2.2}$$

where a_i are the eigenvalues of A, the probabilities are measured as the average value $\Sigma_{i=1}^{\text{finite}}a_iN_i/N$. Here the sum is finite since an experiment can give only a finite number of data. In the comparison between theory and experiment this finite sum is represented by the infinite sum obtained from Eq. (2.2), thus

$$\sum_{i=1}^{\text{finite}} a_i\frac{N_i}{N}\approx\mathcal{P}(A)=\sum_{i=1}^{\infty} a_i\mathcal{P}(P_i). \tag{2.3}$$

This also shows that the meaning of such topological notions as the convergence of infinite sequences [of, e.g., partial sums of the right hand side of Eqs. (2.2) and (2.3)] cannot be established directly from the experimental data on the left hand side of Eq. (2.3), which provides only a finite sequence. Thus the definition of convergence of infinite sequences in Φ, i.e., the topology of the space Φ, is a mathematical idealization. If one wants a complete mathematical theory one needs to make this mathematical idealization and choose a topology for the space Φ. Usually, for many practical calculations in physics, one does not worry about the completeness and uses instead some calculational rules.

To obtain the rules for calculating the trace and the scalar product on the right hand side of Eq. (2.1) one starts with a basis vector decomposition for the state vector $\phi\in\Phi$ using a discrete set of eigenvectors $|i\rangle=|\lambda_i\rangle$ of an observable (often the Hamiltonian) with eigenvalues λ_i.

$$\phi=\sum\ |i\rangle\langle i|\phi\rangle. \tag{2.4}$$

Often, following Dirac [7], one uses a continuous set of eigenvectors $|\lambda\rangle$ (Dirac kets) and writes

$$\phi=\int d\lambda|\lambda\rangle\langle\lambda|\phi\rangle. \tag{2.5}$$

The trace, scalar product, etc., are then calculated as

$$\mathrm{Tr}(\Lambda W)=\sum_{i}^{\infty}\ \langle i|\Lambda W|i\rangle \tag{2.6a}$$

or

$$\mathrm{Tr}(\Lambda W)=\int d\lambda\langle\lambda|\Lambda W|\lambda\rangle, \tag{2.6b}$$

$$|\langle\psi|\phi\rangle|^2=\left|\sum_{i=1}^{\infty}\ \langle\psi|i\rangle\langle i|\phi\rangle\right|^2 \tag{2.7a}$$

or

$$|\langle\psi|\phi\rangle|^2=\left|\int d\lambda\langle\psi|\lambda\rangle\langle\lambda|\phi\rangle\right|^2. \tag{2.7b}$$

In practical calculations the convergence of infinite sums and the meaning of integration (Lebesgue versus Riemann) are usually not considered. Often one truncates to finite (e.g., two) dimensions such that of the sums in Eqs. (2.6a) and (2.7a) one retains only a finite number of terms. If one has a complete mathematical theory one can define the meaning of the infinite sums in Eqs. (2.6a), (2.7a) and the meaning of the integrals in Eqs. (2.6b), (2.7b) and prove Eqs. (2.4) and (2.5). For instance, one can choose for Φ the Hilbert space $\mathcal{H}$, in which case Eq. (2.4) but not Eq. (2.5) can be proven. Or one can choose for Φ a complete space with some locally convex, nuclear topology and its space of continuous functionals $\Phi^\times$ to obtain a Gelfand triplet $\Phi\subset\mathcal{H}\subset\Phi^\times$. Then the kets are $|\lambda\rangle\in\Phi^\times$ and one can prove the Dirac basis vector expansion (2.5) as the nuclear spectral theorem. Time evolution, i.e., the dynamics of a quantum physical system, is given by the Hamilton operator H of the system. [H is always assumed to be (essentially) self-adjoint, $\overline{H}=H^\dagger$, and semibounded.] The dynamical equation is the von Neumann or Schrödinger equation:

$$\frac{\partial W(t)}{\partial t}=\frac{i}{\hbar}[H,W(t)],\quad i\hbar\frac{\partial\phi(t)}{\partial t}=H^\dagger\phi(t), \tag{2.8}$$

$$\phi(t=0)=\phi_0\,.$$

Equivalently, one gives the time evolution in the Heisenberg picture by

$$\frac{\partial\Lambda(t)}{\partial t}=-\frac{i}{\hbar}[H,\Lambda(t)],\quad i\hbar\frac{\partial\psi(t)}{\partial t}=-H\psi(t),$$

$$\psi(t=0)=\psi_0\,.$$

In a time-symmetric theory, that means if one uses for the time-symmetric differential equation (2.8) *also* time-symmetric boundary conditions, then, one obtains the following solutions of Eq. (2.8):

$$W(t)=e^{-iHt}W_0e^{iHt},\quad -\infty<t<\infty \tag{2.9}$$

$$\phi(t)=U^\dagger(t)\phi_0=e^{-iHt}\phi_0\,,\quad -\infty<t<\infty \tag{2.10}$$

or, in the Heisenberg picture,

$$\Lambda(t)=e^{iHt}\Lambda_0e^{-iHt},\quad -\infty<t<\infty. \tag{2.11}$$

Here $\Lambda_0\equiv\Lambda(t=0)$, $W_0\equiv W(t=0)$.

On the other hand, if one just starts with the differential equations (2.8) and postulates the Hilbert space topology, $\phi(t)\in\mathcal{H}$, then the above unitary group evolution is the only possible solution of the dynamical equations [this follows from some theorems of Gleason and Stone (Appendix A)]. This means time-asymmetric boundary conditions which could result in an irreversible time evolution are not mathematically allowed in a quantum theory in the Hilbert space $\mathcal{H}$. The assumption $\phi(t) \in \mathcal{H}$ always leads to the time evolution (2.10) given by the unitary group $U(t)$ which has always an inverse $U(-t)$. Inserting Eq. (2.9), (2.10), or (2.11) into the right hand side of Eq. (2.1), the probability $\mathcal{P}(t) = \mathrm{Tr}[\Lambda W(t)]$ can be calculated at any time $t_0 + t$ or $t_0 - t$.

In contrast to the results calculated with Eq. (2.9), the probabilities $\mathcal{P}(t)$ cannot be observed at any arbitrary positive or negative time t. The reason is the following: *A state needs to be prepared before an observable can be measured, or registered in it.* We call this truism the preparation ⇒ registration arrow of time [10]; it is an expression of causality. Let $t_0 (=0)$ be the time at which the state has been prepared. Then, $\mathcal{P}(\Lambda(t))$ is measured as the ratio of detector counts

$$\mathcal{P}_W^{\text{expt}}(\Lambda(t)) \approx \frac{N(t)}{N} \tag{2.12a}$$

$$\text{for} \quad t > t_0 = 0. \tag{2.12b}$$

If there are some detector counts before $t = t_0$, they are discounted as noise because the experimental probabilities *cannot* fulfill

$$\mathcal{P}_W^{\text{expt}}(\Lambda(t)) \not\approx 0 \quad \text{for} \quad t < t_0 = 0. \tag{2.13}$$

Though in the Hilbert space theory $\mathcal{P}_W(\Lambda(t)) = \mathcal{P}_{W(t)}(\Lambda)$ can be calculated at positive or negative values of $t - t_0$ using unitary group evolution (2.9), an experimental meaning can be given to $\mathcal{P}_W(\Lambda(t))$ only for $t > t_0$.

In some cases (e.g., stationary states, cyclic evolutions), it should not matter at what time $\mathcal{P}_{W(t)}(\Lambda)$ is calculated because one can extrapolate to negative values of t.

The physical question is, are there quantum physical states in nature that evolve only into the positive direction of time, $t > t_0$, and for which one therefore cannot extrapolate to negative values of $t - t_0$? If there are such states, pure states or mixtures, they cannot be described by the standard Hilbert space quantum theory, because of the unitary group time evolution (2.9) and (2.10), which is a mathematical consequence of the specific (topological, not algebraic) structure of the Hilbert space.

Two prominent examples of states with an asymmetric time evolution, $t > t_0$, are the decaying states (in all areas of physics, relativistic or nonrelativistic) and our universe as a whole, considered as a quantum physical system.

(1) Decaying states and resonances are often thought of as something complicated, because in the Hilbert space there does not exist a vector that can describe them in the same way as stable states are described by energy eigenvectors. However, empirically, quasistable particles are not qualitatively different from stable particles; they differ only quantitatively by a nonzero value of the width Γ. Stability or the value of lifetime is not taken as a criterion of elementarity, at least not by the practitioners [11]. A particle decays if it can and it remains stable if selection rules for some quantum numbers prevent it from decaying. Therefore, stable and quasistable states should be described on the same footing, e.g., define both by a pole of the S matrix at the position $z_R = E_R - i\Gamma/2$, or/and as a generalized eigenvector with eigenvalue z_R (with $\Gamma = 0$ for stable particles). Since the latter is not possible in the Hilbert space, one devises ''effective theories'' in order to obtain a state vector description of quasistable states.

Phenomenological effective theories have been enormously successful. They describe resonances in a finite-dimensional space as eigenvectors of the ''effective Hamiltonian'' with complex eigenvalue $(E_R - i\Gamma/2)$, where E_R is the resonance energy, $\hbar/\Gamma$ is the lifetime, and their time evolution is given by the exponential law. The common feature of these approximate methods is the omission of a continuous sum; the infinite-dimensional theory is truncated to a finite- (e.g., two-) dimensional effective theory. Examples of this approach are the approximate method of Weisskopf and Wigner and of Heitler for atomic decaying states [12]; the Lee-Oehme-Yang effective two-dimensional theory for the neutral kaon system [13]; and many more finite-dimensional models with non-Hermitian diagonalizable Hamiltonian matrices in nuclear physics [14]. Also nondiagonalizable finite-dimensional Hamiltonians were discussed [15]. In the Hilbert space framework ''there does not exist . . ., a rigorous theory to which these methods can be considered as approximations'' [16].

The decay of a quantum physical system, e.g., the transition of an excited state of a molecule into its ground state or the decay of an elementary particle [17] is a profoundly irreversible process. Therefore we should like to introduce state vector $|F\rangle$, $|\psi^G\rangle = |E_R - i\Gamma/2\rangle$ or state operators $W^G(t) = |F\rangle\langle F|$, for which the time evolution is asymmetric and for which the theoretical probabilities $\mathrm{Tr}[\Lambda W^G(t)]$ can be calculated for $t > t_0 = 0$ only.

This means we have to generalize the unitary group evolution (2.9), (2.10) with $-\infty < t < \infty$ to a semigroup evolution with $0 \leqslant t < \infty$. This is accomplished by seeking solutions of the time-symmetric dynamical equations (2.8) with time-*asymmetric* boundary conditions. Since in Hilbert space quantum mechanics semigroup evolution is not possible, we seek a semigroup solution $F(t)$ to the quantum mechanical Cauchy problem (2.8) with Hamiltonian $H_+^\times$ where $F(t)$ is an element of a larger space in which $\mathcal{H}$ is dense and which we denote by $\Phi_+^\times \supset \mathcal{H}$, i.e., the Hamiltonian $H_+^\times$ is the uniquely defined extension of the Hilbert space Hamiltonian $H^\dagger$ to this space $\Phi_+^\times$. Thus the dynamical equation (2.8) is

$$i\hbar \frac{\partial F(t)}{\partial t} = H_+^\times F(t), \tag{2.14}$$

with the initial data $F(t=0) = F_0^- \in \Phi_+^\times$, and the solution is given by the semigroup[1]

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[1]The semigroup (2.15), (2.16) generated by the Hamiltonian H of a closed quantum system is *not* the semigroup of quantum statistical mechanics of open systems. See Appendix C and [18].

$$F(t)=U_{+}^{\times}(t)F_0^- = e_{+}^{-iH^{\times}t}F_0^- \tag{2.15a}$$

$$\text{for } t\geqslant 0 \text{ only.} \tag{2.15b}$$

If we use the quantum mechanical state operators with semigroup time evolution,

$$W^G(t)\equiv|F(t)\rangle\langle F(t)|=e^{-iH^{\times}t}W^G(t_0)e^{iHt},\quad t\geqslant 0 \tag{2.16}$$

to calculate the quantum mechanical probabilities, then for these calculated probabilities we obtain

$$P_{W^G(t)}(\Lambda_0)=\mathrm{Tr}[\Lambda(t_0)W^G(t)]=\mathrm{Tr}[\Lambda(t)W^G(t_0)], \tag{2.17a}$$

$$t\geqslant t_0=0. \tag{2.17b}$$

This means that they fulfill the same conditions as the experimental probabilities (2.12a), (2.12b), and (2.13).

In particular, the probabilities are not defined unless the preparation ⇒ registration arrow of time (2.17b) is fulfilled, because the time evolution

$$W^G(t)=e^{-iH^{\times}t}W_0^G e^{iHt} \tag{2.18a}$$

or

$$\Lambda(t)=e^{iHt}\Lambda_0 e^{-iH^{\times}t} \tag{2.18b}$$

is a semigroup evolution and only defined for

$$t>t_0=0.$$

The physical meaning of the initial time t_0 for a decaying system in the state W^G will be discussed in Sec. IV below. Mathematically, it is given by the initial time $t=0$ of the Cauchy problem (2.14).

This semigroup arrow of time (2.15b), (2.17b), (2.18b) is the formulation in the mathematical theory of the experimental preparation ⇒ registration arrow of time (2.12).[2]

(2) The universe, when considered as a quantum physical system, must also be in a state ρ (a pure state $\rho=|\phi\rangle\langle\phi|$, or a mixture) with asymmetric time evolution [19]. Its arrow of time must be identical with the traditional cosmological arrow of time and the time $t=t_0=0$, at which the initial state of the universe ρ has been prepared, is the time of the big bang.

[2]Since the semigroup time evolution (2.15) or (2.18) is not possible in the Hilbert space, i.e., $F_0^-\notin\mathcal{H}$, people who wanted to retain the standard Hilbert space theory but were aware of the quantum mechanical preparation ⇒ registration arrow of time had to extrapolate Eq. (2.18) to negative times, therewith eliminating the experimental preparation ⇒ registration arrow of time and causality from the mathematical theory [10].

The general quantum mechanical (*a priori*) probabilities predicted for the observable represented by the projection operator $P_{\alpha_1}^1(t_1)$ (''yes-no observations'') are, according to Eq. (2.17),

$$\begin{aligned}\mathcal{P}(\alpha_1,t_1)\equiv\mathcal{P}_\rho(P_{\alpha_1}^1(t_1))&=\mathrm{Tr}[P_{\alpha_1}^1(t_1)\rho]\\&=\mathrm{Tr}[P_{\alpha_1}^1(t_1)\rho P_{\alpha_1}^1(t_1)]\end{aligned} \tag{2.19a}$$

$$\text{for } t_1>t_0=0 \text{ only.} \tag{2.19b}$$

The time ordering (2.19b) is the same as the semigroup arrow of time (2.17b) in the quantum mechanics of measured systems. Applied to experiments performed on quantum systems in the laboratory it leads to the preparation ⇒ registration arrow of time (2.12b). As in the quantum mechanics of measured systems, (2.19b) is an expression of causality.

The quantum mechanical probabilities (2.19) of projection operators $P_{\alpha_i}^i(t_i)$ can be generalized to probabilities of histories [2,20,21].

A history is a time ordered product of different projection operators (labeled by α_i) for different observables (labeled by i):

$$C_\alpha=P_{\alpha_1}^1(t_1)\cdots P_{\alpha_i}^i(t_i)\cdots P_{\alpha_n}^n(t_n),\quad t_n>t_{n-1}>\cdots>t_2>t_1 \tag{2.20}$$

with

$$P_{\alpha_i}^i(t_i)=e^{iH(t_i-t_{i-1})}P_{\alpha_i}^i(t_{i-1})e^{-iH(t_i-t_{i-1})}, \tag{2.21a}$$

$$t_i-t_{i-1}>0. \tag{2.21b}$$

This definition of histories is suggested by the following considerations: Let $P_{\alpha_i}^i$ be the α_ith projector of (what we denote as) the ith observable $A^i=\Sigma_\alpha a_\alpha^i P_\alpha^i$, $i=1, 2, 3, \ldots$. Then, starting with the operator $\rho=\rho(t_0)$ of Eq. (2.19a), one can define a sequence of effective density operators $\rho^{\mathrm{eff}}(t_1),\ldots,\rho^{\mathrm{eff}}(t_{n-1})$, and one can predict a sequence of probabilities $\mathcal{P}(\alpha_2 t_2;\alpha_1 t_1)$, $\mathcal{P}(\alpha_3 t_3;\alpha_2 t_2;\alpha_1 t_1),\ldots,\mathcal{P}(\alpha_n t_n;\cdots\alpha_1 t_1)$. These density operators and probabilities are listed below:

$$\begin{aligned}\rho^{\mathrm{eff}}(t_1)&=\frac{P_{\alpha_1}^1(t_1)\rho(t_0)P_{\alpha_1}^1(t_1)}{\mathrm{Tr}[P_{\alpha_1}^1(t_1)\rho(t_0)P_{\alpha_1}^1(t_1)]}\\&=N_1P_{\alpha_1}^1(t_1)\rho(t_0)P_{\alpha_1}^1(t_1)\end{aligned} \tag{2.22a}$$

$$\text{for } t_1>t_0 \text{ only} \tag{2.22b}$$

[the second equality in Eq. (2.22a) defines the normalization factor N_1] and

$$\mathcal{P}(\alpha_2 t_2;\alpha_1 t_1)=N_1\mathrm{Tr}[P_{\alpha_2}^2(t_2)\rho^{\mathrm{eff}}(t_1)P_{\alpha_2}^2(t_2)] \tag{2.23a}$$

$$\text{for } t_2>t_1 \text{ only.} \tag{2.23b}$$

Continuing in this way for $n=3, 4, \ldots,$

$$\rho^{\mathrm{eff}}(t_{n-1})=\frac{P^{n-1}_{\alpha_{n-1}}(t_{n-1})\rho^{\mathrm{eff}}(t_{n-2})P^{n-1}_{\alpha_{n-1}}(t_{n-1})}{\mathrm{Tr}[P^{n-1}_{\alpha_{n-1}}(t_{n-1})\rho^{\mathrm{eff}}(t_{n-2})P^{n-1}_{\alpha_{n-1}}(t_{n-1})]}=N_{n-1}P^{n-1}_{\alpha_{n-1}}(t_{n-1})\rho^{\mathrm{eff}}(t_{n-2})P^{n-1}_{\alpha_{n-1}}(t_{n-1});$$

$$=\frac{P^{n-1}_{\alpha_{n-1}}(t_{n-1})\cdots P^{1}_{\alpha_1}(t_1)\rho(t_0)P^{1}_{\alpha_1}(t_1)\cdots P^{n-1}_{\alpha_{n-1}}(t_{n-1})}{\mathrm{Tr}[P^{n-1}_{\alpha_{n-1}}(t_{n-1})\cdots P^{1}_{\alpha_1}(t_1)\rho(t_0)P^{1}_{\alpha_1}(t_1)\cdots P^{n-1}_{\alpha_{n-1}}(t_{n-1})]}, \tag{2.24a}$$

$$t_{n-1}>t_{n-2}>\cdots>t_1>t_0 \tag{2.24b}$$

and

$$\mathcal{P}(\alpha_n t_n;\ldots\alpha_1 t_1)=\frac{\mathrm{Tr}[P^{n}_{\alpha_n}(t_n)\rho^{\mathrm{eff}}(t_{n-1})]}{\mathrm{Tr}[P^{n-1}_{\alpha_{n-1}}(t_{n-1})\rho^{\mathrm{eff}}(t_{n-2})P^{n-1}_{\alpha_{n-1}}(t_{n-1})]}, \tag{2.25a}$$

$$t_n>t_{n-1} \tag{2.25b}$$

or

$$\mathcal{P}(\alpha_n,t_n;\ldots;\alpha_1,t_1)=N_{n-1}\mathrm{Tr}[P^{n}_{\alpha_n}(t_n)\cdots P^{1}_{\alpha_1}(t_1)\rho(t_0)\times P^{1}_{\alpha_1}(t_1)\cdots P^{n}_{\alpha_n}(t_n)] \tag{2.26a}$$

$$\text{for}\ \ t_n>\cdots>t_0\ \ \text{only.} \tag{2.26b}$$

The time ordering or arrow of time (2.23b), (2.24b), (2.25b), and (2.26b) is again the same as the semigroup arrow of time (2.18b), and Eq. (2.21) is the same as the semigroup evolution (2.18a′) (in the Heisenberg picture) for the observable Λ in the quantum theory of measured systems.

The probability (2.25a), (2.26a) is the probability of the history defined in Eq. (2.20),

$$\mathcal{P}(\alpha_n t_n,\ldots,\alpha_1 t_1)=N_n\mathrm{Tr}[C_\alpha\rho(t_0)C_\alpha]. \tag{2.27}$$

One can consider alternative projection operators

$$C'_\alpha=P^{1'}_{\alpha'_1}(t_1)P^{2'}_{\alpha_2'}(t_2)\cdots P^{n'}_{\alpha'_n}(t_n) \tag{2.28a}$$

but a physical meaning can only be given to these products for the time ordering

$$t_n>t_{n-1}\ldots>t_1. \tag{2.28b}$$

This time ordering, identical with the time ordering (2.22b), (2.23b), and (2.24b), is a calculational consequence of the restriction (2.21b) postulated [2,19] for the time evolution of the projectors. The restricted time evolution (2.21) is a semigroup evolution generated by the Hamiltonian of the closed quantum system. Obviously the semigroup (2.21), (2.18), and (2.16) is the same semigroup applied to different observables, $P^i_{\alpha_i}$ and Λ, respectively, of different quantum systems, namely, the quantum universe and the quasistable particle. The semigroup character of the time evolution (2.18a′)—or of (2.18) in the Schrödinger picture‘‘—was inferred from restrictions imposed by observational limitations in a laboratory experiment with quantum systems, namely, from the preparation ⇒ registration arrow of time. The semigroup character of the time evolution (2.21) and the time ordering (2.28b) were postulated for the quantum universe because of the special initial state associated to the big bang [19]. From the way the time ordering appears in the probabilities for the laboratory experiments (2.17) and in the probabilities of the histories (2.19), (2.23), and (2.25), it is clear that both time orderings express the same arrow of time. If our universe is a closed quantum system as suggested by [2], the semigroup arrow for the resonances is subsumed under the cosmological arrow of time, or vice versa. This arrow of time ‘‘may *not* be attributed to the thermodynamic arrow of an external measuring apparatus (for the laboratory experiment) or larger universe’’ (for the quantum universe). It is a ‘‘fundamental quantum mechanical distinction between the past and future’’ [2].

As mentioned above, a semigroup evolution that could give a theoretical description of this arrow of time is impossible in the standard Hilbert space quantum mechanics. Therefore, in order to make the semigroup postulate (2.21) possible and to allow for a semigroup solution (2.15) of the quantum mechanical Cauchy problem, one must develop a new mathematics. We shall present the mathematics that is capable of a time-asymmetric quantum theory in the following section.

III. MATHEMATICAL THEORY FOR TIME-ASYMMETRIC QUANTUM PHYSICS

Our empirical consideration in Sec. II has led us to the postulate of a time evolution semigroup (2.21) or (2.18). Here we want to discuss a mathematical theory of quantum physics for which a semigroup evolution exists.

In a linear space with a scalar product Φ_{alg}, which we need for the calculational rules of quantum mechanics, the simplest modification that allows Hamiltonian generated semigroups is to choose instead of the Hilbert space topology a locally convex topology. If one also wants the Dirac formalism [i.e., kets, the basis vector expansion (2.5), etc.], then one has to choose a rigged Hilbert space (RHS) or Gelfand triplet

$$\Phi\subset\mathcal{H}\subset\Phi^\times. \tag{3.1}$$

The triplet of spaces in a rigged Hilbert space $\Phi\subset\mathcal{H}\subset\Phi^\times$ results from three different topological completions of the same algebraic (pre-Hilbert) space Φ_{alg} of Sec. II [22]. Completion means adjoining to Φ_{alg} the (limit elements of) convergent (Cauchy) sequences with respect to a topology. The completion of $\Phi_{\rm alg}$ with respect to the norm $\|\varphi\| = \sqrt{(\varphi,\varphi)}$, $\varphi\in\Phi_{\rm alg}$ is the Hilbert space $\mathcal{H}$. The topology or meaning of convergence defined by the norms we denote by $\mathcal{T}_{\mathcal{H}}$. The completion of $\Phi_{\rm alg}$ with respect to a finer locally convex, nuclear topology, which we denote by $\mathcal{T}_{\Phi}$ (and which is usually given by a countable number of norms [22]), is denoted by Φ. Then one has $\Phi_{\rm alg}\subset\Phi\subset\mathcal{H}$ (because Φ and $\mathcal{H}$ contain all elements of $\Phi_{\rm alg}$ plus the limit elements of Cauchy sequences in $\Phi_{\rm alg}$), and $\Phi\subset\mathcal{H}$ holds because $\mathcal{T}_{\Phi}$ is chosen to be finer or stronger than $\mathcal{T}_{\mathcal{H}}$ and there are consequently more $\mathcal{T}_{\mathcal{H}}$ Cauchy sequences than $\mathcal{T}_{\Phi}$ Cauchy sequences. We also consider the space of $\mathcal{T}_{\mathcal{H}}$-continuous and of $\mathcal{T}_{\Phi}$-continuous functionals. $\mathcal{H}^{\times}$ is the space of $\mathcal{T}_{\mathcal{H}}$-continuous antilinear functionals ψ on the space $\mathcal{H}$: ψ: $\phi\in\mathcal{H}\to\psi(\phi)\in\mathbb{C}$, and $\mathcal{H}=\mathcal{H}^{\times}$, $\psi(\phi)=(\phi,\psi)$, by a mathematical theorem. $\Phi^{\times}$ is the space of $\mathcal{T}_{\Phi}$-continuous, antilinear functionals F on the space Φ: F: $\phi\in\Phi\to F(\phi)\equiv\langle\phi|F\rangle\in\mathbb{C}$. One has $\mathcal{H}^{\times}\subset\Phi^{\times}$ and the bra-ket $\langle|\rangle$ becomes an extension of the scalar product. Thus one obtains the Gel'fand triplet (3.1).

Dirac kets are elements of $\Phi^{\times}$, but there are also other $|F\rangle\in\Phi^{\times}$ besides the Dirac kets. Dirac's algebra of observables is an algebra of continuous operators in Φ (observables cannot be continuous operators in $\mathcal{H}$).

For a $\mathcal{T}_{\Phi}$-continuous linear operator A, its conjugate operator $A^{\times}$ is defined by

$$\langle A\phi|F\rangle=\langle\phi|A^{\times}|F\rangle,\quad \forall\phi\in\Phi \quad\text{and}\quad \forall F\in\Phi^{\times}. \tag{3.2}$$

$A^{\times}$ is a continuous operator in $\Phi^{\times}$. Then for each observable A, one has a triplet of operators

$$A^{\dagger}|_{\Phi}\subset A^{\dagger}\subset A^{\times}, \tag{3.3}$$

where $A^{\dagger}$ is the Hilbert space adjoint operator of A and $A^{\dagger}|_{\Phi}$ is its restriction to the subspace Φ. Generalized eigenvectors are defined for continuous operators. A vector $|F\rangle\in\Phi^{\times}$ is a generalized eigenvector of the $\mathcal{T}_{\Phi}$-continuous operator A if for some complex number ω and for all $\phi\in\Phi$,

$$\langle A\phi|F\rangle=\langle\phi|A^{\times}|F\rangle=\omega\langle\phi|F\rangle. \tag{3.4}$$

This is also written as

$$A^{\times}|F\rangle=\omega|F\rangle \tag{3.5}$$

(or, even as $A|F\rangle=\omega|F\rangle$ if $A^{\dagger}$ is a self-adjoint operator).

If A is the (self-adjoint) Hamiltonian H of a quantum physical system, then $\Phi^{\times}$ contains the Dirac kets

$$H^{\times}|E^{-}\rangle=E|E^{-}\rangle,\quad E\geqslant 0. \tag{3.6}$$

$\Phi^{\times}$ can also contain generalized eigenvectors with complex eigenvalues, as, e.g.,

$$H^{\times}|E_R-i\Gamma/2^{-}\rangle=(E_R-i\Gamma/2)|E_R-i\Gamma/2^{-}\rangle, \tag{3.7}$$

which we call Gamow vectors or Gamow kets [23].

There is not only one space Φ, but there are many (locally convex, nuclear, countably normed) topologies $\mathcal{T}_{\Phi}$, which lead to different completions Φ of $\Phi_{\rm alg}$ (with the same $\mathcal{H}$). The choice of Φ depends on the particular physical problem at hand, e.g., Φ can be chosen such that the algebra of observables of a particular physical system is an algebra of $\mathcal{T}_{\Phi}$-continuous operators.

Further, in Sec. II we said that we need to distinguish meticulously between states and observables. In order to be able also to distinguish mathematically between states and observables we have to introduce one space for states, which we call Φ_{-}, and another space for observables, which we call Φ_{+}. In general $\Phi_{+}\neq\Phi_{-}$, but $\Phi_{+}\cap\Phi_{-}\neq\{0\}$. The state prepared by the preparation apparatus (e.g., accelerator) we denote by ϕ^{+}, thus $\phi^{+}\in\Phi_{-}$. The observable registered by the registration apparatus (e.g., detector) we denote by $|\psi^{-}\rangle\langle\psi^{-}|$, thus $\psi^{-}\in\Phi_{+}$ (cf. Appendix B for the scattering experiment). Therefore we need two rigged Hilbert spaces, one for prepared in states ϕ^{+}:

$$\phi^{+}\in\Phi_{-}\subset\mathcal{H}\subset\Phi_{-}^{\times}, \tag{3.8}$$

and the other for the registered observables $|\psi^{-}\rangle\langle\psi^{-}|$ or detected out states ψ^{-}:

$$\psi^{-}\in\Phi_{+}\subset\mathcal{H}\subset\Phi_{+}^{\times}. \tag{3.9}$$

Here the space $\mathcal{H}$ is the same Hilbert space (with the same physical interpretation).

Mathematically one can define the spaces of the vectors Φ by the spaces of their energy wave functions $\langle E|\phi\rangle$:[3].

$$\phi^{+}\in\Phi_{-}\Leftrightarrow\ \langle{}^{+}E|\phi^{+}\rangle \in\mathcal{S}\cap\mathcal{H}_{-}^{2}|_{\mathbb{R}^{+}}\ \text{(well-behaved Hardy functions in } \mathbb{C}^{-}). \tag{3.10}$$

$$\psi^{-}\in\Phi_{+}\Leftrightarrow\langle{}^{-}E|\psi^{-}\rangle \in\mathcal{S}\cap\mathcal{H}_{+}^{2}|_{\mathbb{R}}^{+}\ \text{(well-behaved Hardy functions in } \mathbb{C}^{+}). \tag{3.11}$$

The notation here is the following: $\mathbb{C}^{+}(\mathbb{C}^{-})$ denotes the open upper (lower) half of the complex energy plane of the second Riemann sheet for the analytically continued S matrix, and $\mathcal{H}^{2}_{\mp}$ denotes the Hardy class functions [25] and $\mathcal{S}$ the Schwartz space functions. This explains the notation Φ_{-} and Φ_{+} for the spaces. The subscript refers to the subscript in the standard notation of mathematics for Hardy class functions ($\mathcal{H}^{p}_{-}$, $\mathcal{H}^{p}_{+}$, respectively). The superscripts for ϕ^{+} (in states) and ψ^{-} (out states) are the most common convention in scattering theory, cf. Appendix B.

Thus, in the physical interpretation, for each species of quantum physical system one has a pair of RHS's, Eqs. (3.8) and (3.9). Whereas the ''in state'' $\phi^{+}\in\Phi_{-}$ describes the state that is physically defined by the preparation apparatus, the ''out state'' $\psi^{-}\in\Phi_{+}$ describes the observable that is physically defined by the registration apparatus.

[3] In the same way as one can define the Hilbert space $\mathcal{H}$ by the space of Lebesgue square integrable functions $\mathcal{H}\ni h\Leftrightarrow h(E)\in L^{2}[0,\infty)$, where the functions $h(E)$ are uniquely determined only up to a set of Lebesgue measure zero, which is a complicated and unphysical notion, cf. Sec. 3, Ref. [24]

It is by this clear differentiation between the set of vectors $\{\phi^+\}$ which are admitted as in states and the set of vectors $\{\psi^-\}$ which are admitted as out observables that the RHS theory differs from the usual scattering theory, where $\{\phi^+\}=\{\psi^-\}=\Phi\subset\mathcal{H}$ (cf. the asymptotic completeness condition according to which $\{\phi^+\}=\{\phi^-\}=\mathcal{H}$). According to Eqs. (3.10) and (3.11), Φ_- and Φ_+ are different dense subspaces of the same Hilbert space $\mathcal{H}$ (which are both complete with respect to a stronger topology than $\mathcal{T}_{\mathcal{H}}$) with

$$\Phi_+\cap\Phi_-\neq\{0\},\quad\text{and}\quad \Phi=\Phi_++\Phi_-\quad\text{is also dense in }\mathcal{H}. \tag{3.12}$$

After the RHS's (3.8) and (3.9) have been chosen to be the Hardy class spaces (3.10) and (3.11), the semigroup of Sec. II turns up naturally from the mathematics. How one could empirically conjecture the RHS's of the Hardy class will not be discussed here [26].

To obtain the semigroups we start with the unitary group of time evolutions in the Hilbert space $\mathcal{H}$.

$$U(t)=e^{iHt},\quad U^\dagger(t)=e^{-iHt}, \tag{3.13}$$

where $U^\dagger(t)$ denotes the Hilbert space adjoint of $U(t)$.[4]

We first turn to the RHS (3.9) and consider

$$U_+(t)\equiv U(t)|_{\Phi_+}\subset U(t),\quad\text{and}\quad U(t)^\dagger\subset U_+^\times(t). \tag{3.14}$$

It can be shown that, as a consequence of the mathematical properties of Φ_+, the restriction of $U(t)$ to Φ_+, $U_+(t)$, is a $\mathcal{T}_{\Phi_+}$-continuous operator only for $0\leqslant t<\infty$ [8,27]. Therefore its conjugate operator $U_+^\times(t)$, which is an extension of the Hilbert space adjoint operator $U^\dagger(t)$, is well defined [by Eq. (3.2)] and continuous for $0\leqslant t<\infty$ only. Thus in $\Phi_+^\times$ we have only the semigroup

$$U_+^\times(t)=(e^{iHt}|_{\Phi_+})^\times\equiv e_+^{-iH^\times t},\quad 0\leqslant t<\infty. \tag{3.15}$$

The same considerations apply to the other RHS (3.8). One considers $U_-(t)\equiv U(t)|_{\Phi_-}\subset U(t)$, and its conjugate $U(t)^\dagger\subset U_-^\times(t)$, and proves mathematically that $U_-(t)$ is a $\mathcal{T}_{\Phi_-}$-continuous operator only for $-\infty<t\leqslant 0$. Therefore $U_-^\times(t)$ is defined and continuous for $-\infty<t\leqslant 0$ only and one has in $\Phi_-^\times$ the semigroup

$$U_-^\times(t)=(e^{iHt}|_{\Phi_-})^\times\equiv e_-^{-iH^\times t},\quad -\infty<t\leqslant 0. \tag{3.16}$$

Thus in the RHS (3.8) for the prepared states one has the semigroup (3.16) for times $t\leqslant t_0=0$, and in the RHS (3.9) for the registered observables one has the semigroup (3.15) for times $t\geqslant t_0=0$. Since $t=t_0=0$ is the time by which the state has been prepared and the registration of the observable can begin, this separation of the (mathematical) group (3.13) into the two semigroups (3.16) and (3.15) reflects the situation envisioned on empirical grounds in Sec. II. The scattering (e.g., resonance scattering) process is separated into two parts, the preparation part dealing with the preparation of the state $\phi^+\in\Phi$ and the registration part dealing with the registration of the observable (or detection of the out state) $\psi^-\in\Phi_+$. The time $t=0(t_0)$ is the time at which the preparation is completed and the registration can commence; the meaning of t_0 will be discussed in detail in Sec. IV.

In addition to the vectors ϕ^+ and ψ^- defined by the apparatuses, there also are the vectors in $\Phi_\pm^\times$ which are outside of $\mathcal{H}$:

$$|E,\theta_{\rm p},\varphi_{\rm p}^{\mp}\rangle\in\Phi_\pm^\times\quad\text{(Dirac's scattering states)}, \tag{3.17}$$

where (θ_p,φ_p) denotes the direction of momentum; and the

$$\psi^G=|E_R\mp i\Gamma/2,j,j_3^{\mp}\rangle\in\Phi_\pm^\times\quad\text{(Gamow's resonance states)}, \tag{3.18}$$

with the property

$$H^\times|E_R-i\Gamma/2^-\rangle=(E_R-i\Gamma/2)|E_R-i\Gamma/2^-\rangle. \tag{3.19}$$

In the RHS theory Dirac kets and Gamow vectors are mathematically very similar. Both are generalized eigenvectors of self-adjoint Hamiltonians in the sense of Eq. (3.2) and are equally well defined (though the choice of spaces Φ for which Gamow kets can be defined is smaller than for Dirac kets since the former also requires some analyticity properties as for Hardy class spaces Φ_+). Dirac kets and Gamow kets just differ in their eigenvalues; whereas Dirac's scattering state vectors in $\Phi_+^\times$ or $\Phi_-^\times$ have real (except for the $\pm i0$) eigenvalues corresponding to the scattering energies, and Gamow kets have complex eigenvalues corresponding to the resonance pole of the S matrix (see below).

The Gamow vectors $\psi^G=|E_R-i\Gamma/2\rangle\sqrt{2\pi\Gamma}\in\Phi_+^\times$ have a semigroup time evolution and obey an exponential law:

$$\psi^G(t)\equiv U_+^\times(t)\psi^G=e^{-iE_Rt}e^{-\Gamma t/2}|E_R-i\Gamma/2^-\rangle,\quad t\geqslant 0. \tag{3.20}$$

This is a formal consequence of applying the right hand side of Eq. (3.15) to ψ^G and using Eq. (3.19). But for the mathematical proof of Eq. (3.20), in particular of the semigroup character, the whole mathematical apparatus of the RHS of the Hardy class is needed [22].

There are other Gamow vectors $\tilde{\psi}^G=|E_R+i\Gamma/2^+\rangle\sqrt{2\pi\Gamma}\in\Phi_-^\times$, and there is another semigroup (3.16), $e_-^{-iH^\times t}$ for $t\leqslant 0$ in $\Phi_-\subset\mathcal{H}\subset\Phi_-^\times$ with the asymmetric evolution

$$\tilde{\psi}^G(t)=e^{-iH^\times t}|E_R+i\Gamma/2^+\rangle=e^{-iE_Rt}e^{\Gamma t/2}|E_R+i\Gamma/2^+\rangle,$$
$$t\leqslant 0. \tag{3.21}$$

Gamow vectors have the following features.

(1) They are derived as functionals of the resonance pole term at $z_R=E_R-i\Gamma/2$ (and at $z_R^*=E_R+i\Gamma/2$) in the second sheet of the analytically continued S matrix [9,27].

(2) They have a Breit-Wigner energy distribution $|\langle^-E|\psi^G\rangle|^2=(\Gamma/2\pi)1/[(E-E_R)^2+(\Gamma/2)^2]\rightarrow\delta(E-E_R)$ for

[4]Note that in $\mathcal{H}$ the right hand side of Eq. (3.13) is not defined by the exponential series which only converges with respect to $\mathcal{T}_{\mathcal{H}}$ on a dense subspace of analytic vectors in $\mathcal{H}$, but by the Stone–von Neumann calculus.

$\Gamma/E_R \to 0$ which extends to negative energy values on the second sheet indicated in the representation

$$|\psi^G\rangle = i\sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty_{\rm II}}^{+\infty} dE \frac{|E^-\rangle}{E-(E_R-i\Gamma/2)} \tag{3.22}$$

by $-\infty_{\rm II}$ [9].

(3) The decay probability $\mathcal{P}(t) = \mathrm{Tr}(\Lambda|\psi^G\rangle\langle\psi^G|)$ of $\psi^G(t)$, $t \geqslant 0$, into the final non-interacting decay products described by Λ can be calculated as a function of time, and from this the decay rate $R(t) = d\mathcal{P}(t)/dt$ is obtained by differentiation [24]. This leads to an exact golden rule (with the natural linewidth given by the Breit-Wigner distribution) and the exponential decay law

$$R(t) = e^{-i\Gamma t}\Gamma_\Lambda, \quad t \geqslant 0 \tag{3.23}$$

where Γ_Λ is the partial width for the decay products Λ (Γ_Λ equals the branching ratio times Γ). In the Born approximation ($\psi^G \to f^D$, an eigenvector of $H_0 = H - V$; $\Gamma/E_R \to 0$; $E_R \to E_0$) this exact golden rule goes into Fermi's golden rule No. 2 of Dirac.

(4) The Gamow vectors ψ_i^G are members of a ''complex'' basis vector expansion [27]. In place of the well-known Dirac basis system expansion (nuclear spectral theorem of the RHS) given by

$$\phi^+ = \sum_n |E_n)(E_n|\phi^+) + \int_0^{+\infty} dE|E^+\rangle\langle^+E|\phi^+\rangle \tag{3.24}$$

(where the discrete sum is over bound states, which we henceforth ignore), every prepared state vector $\phi^+ \in \Phi_-$ can be expanded as

$$\phi^+ = -\sum_{i=1}^N |\psi_i^G\rangle\langle\psi_i^G|\phi^+\rangle + \int_0^{-\infty_{II}} dE|E^+\rangle\langle^+E|\phi^+\rangle \tag{3.25}$$

(where $-\infty_{\rm II}$ indicates that the integration along the negative real axis or other contour including around cuts is in the second Riemann sheet of the S matrix). N is the number of resonances in the system (partial wave), each one occurring at the pole position $z_{R_i} = E_{R_i} - i\Gamma_i/2$. This allows us to mathematically isolate the exponentially decaying states ψ_i^G.

The ''complex'' basis vector expansion (3.25) is rigorous; it is sometimes also called ''complex spectral resolution,'' which is a misnomer. The Weisskopf-Wigner approximate methods are tantamount to omitting the background integral in Eq. (3.25), i.e., writing

$$\phi^+ \overset{\rm WW}{=} \sum_{i=1}^N |\psi_i^G\rangle c_i, \quad c_i = -\langle\psi_i^G|\phi^+\rangle. \tag{3.26}$$

For instance, for the $K_L - K_S$ system with $N=2$, Eq. (3.26) becomes

$$\phi^+ = \psi_S^G b_S + \psi_L^G b_L. \tag{3.27}$$

For the case of a single decaying state, $N=1$, Eq. (3.25) becomes

$$\phi^+ = -|\psi^G\rangle\langle\psi^G|\phi^+\rangle + \int_0^{-\infty} dE|E^+\rangle\langle^+E|\phi^+\rangle \tag{3.28}$$

and only approximately is a prepared state $\phi^+ \in \Phi_-$ represented by an exponentially decaying Gamow vector $\psi^G \in \Phi_+^\times$,

$$\phi^+ \overset{\rm WW}{=} \psi^G. \tag{3.29}$$

The time evolution of every state prepared by a macroscopic apparatus is thus

$$\begin{aligned}\phi^+(t) &= e^{-iHt}\phi^+\\ &= -e^{-iE_Rt}e^{-\Gamma/2t}|\psi^G\rangle\langle\psi^G|\phi^+\rangle\\ &\quad + \int_0^{-\infty} dE\, e^{-iEt}|E^+\rangle\langle^+E|\phi^+\rangle.\end{aligned} \tag{3.30}$$

It has in addition to the exponential time dependence due to Eq. (3.20) the time dependence given by the ''background integral'' which is nonexponential.

Theoretically such a background term is always present in the prepared state vector $\phi^+ \in \Phi_- \subset \mathcal{H}$ but it could be arbitrarily small (as in the Hilbert space theory). Its time dependence can be calculated from the energy wave function of the prepared state $\phi^{\rm in}(E) = \langle E|\phi^{\rm in}\rangle = \langle^+E|\phi^+\rangle$, $0 \leqslant E < \infty$ (using the Mellin transform [22,28]), if this experimental energy distribution of the prepared state is known. Since it depends upon the state preparation the effect of the background term will change with the experimental conditions and can become substantial. This is a familiar effect in resonance scattering experiments where deviations from the pure Breit-Wigner distribution of the Gamow state are commonly observed and attributed to the background phase shifts and apparatus resolution, see, e.g., [29]. Without knowing the specific form of the function $\phi^{\rm in}(E)$ all that one can infer from general mathematical theorems (Riemann-Lebesgue lemma) is that the background time dependence goes to zero in time and decreases slower than exponentially [28].

The properties (3.18)–(3.25) are not independently postulated conditions for the Gamow vectors but derived from each other in the mathematical theory of the RHS. One can start, for instance, with the most widely accepted definition of the resonance by the pair of poles of the S matrix (B3) of Appendix B at $z_R = E_R \mp i\Gamma/2$ and associate to it the Lippmann-Schwinger-Dirac ket $|z_R^\mp\rangle$ obtained from analytic continuation in Eq. (B3). Then one obtains the Breit-Wigner energy distribution (3.22) from the Hardy class property (3.11) and vice versa. From Eq. (3.22), using Eq. (3.11)—in particular, the property of the Schwartz space $\mathcal{S}$—one derives Eq. (3.19) as generalized eigenvalue equation $\langle\psi^-|H^{\times n}|z_R^-\rangle = z_R^n\langle\psi^-|z_R^-\rangle$, not only for $n=1$ but for all powers n. The generalized eigenvalue equation (3.20) is also derived from the representation (3.22) but only for $t \geqslant 0$ because of the Hardy class property (which in turn was needed to justify the Breit-Wigner energy distribution for the pole term of the S matrix). The Dirac basis vector expansion (3.24) is fulfilled for every RHS, e.g., when Φ is realized just by $\mathcal{S}$. The basis vector expansion (3.25) follows by analytic continuation and therefore requires the Hardy class property (3.10), (3.11). The derivation of the exact golden rule [24] uses in addition the Lippmann-Schwinger equation (B4).

IV. PHYSICAL INTERPRETATION AND THE MEANING OF THE INITIAL TIME

The semigroup time evolution introduces a new concept—the time $t_0(=0)$ at which the preparation of the state is completed and the registration of the observable can begin. This is the most difficult new concept, because one is unprepared for it by the school of thinking based on the old time-symmetric quantum mechanics. For the state of our universe as a whole, considered as a *closed* quantum mechanical system, there is no problem, because we deal only with one single system and the time t_0 is the time of the creation of this single universe (big bang time). Alternatively, we could consider this universe as a member of an ensemble of universes, of which we have access to only our universe. Then the probabilities (2.19a)–(2.26a) are the statistical probabilities (''relative frequencies'') of this ensemble and we have the usual interpretation of quantum mechanics, where the density operator ρ, W, the state vector ϕ^+, or wave function $\langle {}^+E|\phi^+\rangle$ is the mathematical representative of an ensemble of microsystems.

For an experiment performed on a quantum system in the laboratory, the states prepared by a macroscopic preparation apparatus, i.e., states described by $\phi^+\in\Phi_-$ or $W=\Sigma_i w_i|\phi_i^+\rangle\langle\phi_i^+|$, $\phi_i^+\in\Phi_-$, are best interpreted as ensembles (e.g., the proton or electron beam prepared by an accelerator). But there are other ''states'' which are prepared by a macroscopic apparatus in conjunction with a quantum scattering process (e.g., resonance scattering), which are best interpreted as states of a single microsystem. For their description the RHS offers, e.g., the Dirac kets (3.17) or the Gamow kets (3.18). From the basis vector expansion (3.25), we know that, mathematically, the apparatus-prepared state ϕ^+ can be represented as the sum of a Gamow vector ψ^G and a background integral. We shall now argue that the Gamow state can also be isolated experimentally and discuss its creation time t_0 and its asymmetric development in time.[5] This microphysical irreversibility is the analog of the arrow of time for the state of our universe.

The best example is the decaying state of the neutral kaon system because it is a wonderfully closed system, isolated from most external influences (including the electromagnetic field) whose (exact) evolution in time is probably entirely due to the Hamiltonian of the neutral kaon system and free of external influences like those mentioned in footnote 1 of Sec. II. Since we are here only interested in the fundamental concepts of decay, we discuss a simplified K^0 system for which the K_L^0 as well as the CP violation is ignored [30].

The process (idealized, because in the real experiment one does not use a π—but a proton beam) by which the neutral kaon state is prepared is

$$\pi^- p\Rightarrow\Lambda K^0,\quad K^0\Rightarrow\pi^+\pi^-. \tag{4.1}$$

K^0 is strongly produced with a time scale of 10^{-23}sec and it decays weakly, with a time scale of 10^{-10}sec, which is roughly the lifetime of the K_S^0, τ_{K_s}. Thus t_0, the time at which the preparation of the K^0 state, which we call W^{K^0}, is completed and the registration can begin, is very well defined. (Theoretical uncertainty is $10^{-13}\tau_K$.) A schematic diagram of a real experiment [31] is shown in Fig. 1. The state W^{K^0} is created instantly at the baryon target T [and the baryon B is excited from the ground state (proton) into the Λ state, with which we are no longer concerned]. We imagine that a single particle K^0 is moving into the forward beam direction, because somewhere at a distance, say at d_2 from T, we ''see'' a decay vertex for $\pi^+\pi^-$, i.e., a detector (registration apparatus) has been built such that it counts $\pi^+\pi^-$ pairs which are coming from the position d_2. The observable registered by the detector is the projection operator

$$\Lambda(t_2)=|\pi^+\pi^-,t_2\rangle\langle\pi^+\pi^-,t_2|=|\psi^{\text{out}}(t_2)\rangle\langle\psi^{\text{out}}(t_2)| \tag{4.2}$$

for those $\pi^+\pi^-$ which originate from the fairly well specified location d_2. From the position (in the laboratory frame) d_2^{lab}, the four-momentum p of the K^0(equal to the z component of the momentum of the $\pi^+\pi^-$ system) and the mass m_K of K^0, one obtains the time t_2^{rest} (in the K^0 rest frame) which the K^0 has taken to move from T to d_2^{lab}. This is given by the simple formula of relativity $d_2^{\text{lab}}=t_2^{\text{rest}}p/m_K$ which we write $d_2=t_2p/m_K$.

We do not have to focus at only one location d_2 but can count decay vertices at any distance d (of the right order of magnitude). The detector (described by the projection operator $\Lambda(t)\equiv|\pi^+\pi^-,t\rangle\langle\pi^+\pi^-,t|$) counts the $\pi^+\pi^-$ decays at different times $t=t_1,\ t_2,\ t_3,\ldots$ (in the rest frame of the K^0), and these correspond to the distances from the target $d_1=pt_1/m_K$, $d_2=pt_2/m_K,\ldots$ (in the laboratory frame).

One ''sees'' the decay vertex d_i for each single decay and imagines a single decaying K^0 microsystem that had been created on the target T at time $t_0=0$ and then traveled a time t_i until it decayed at the vertex d_i. We give the following interpretation to these observations: a single microphysical decaying system K^0 described by W^{K^0} has been produced by a macroscopic preparation apparatus and a quantum scattering process, at a time $t=0$. Each count of the detector is the result of the decay of such a single microsystem. This particular microsystem has lived for a time t_i—the time that it took the decaying system to travel from the scattering center T to the decay vertex d_i. The whole detector registers the counting rate $\Delta N(t)/\Delta t\approx NR(t)$ as a function of d_i, i.e., of $t_i=(m_K/p)d_i$, for $\cdots t_i>\cdots t_2>t_1>t_0=0$. ($N$ is the total number of counts.)

The counting rate $\Delta N(t_i)/\Delta t$ is plotted as a function of time t (in the K^0 rest frame), in Fig. 2.

No $\pi^+\pi^-$ are registered for $t<0$, i.e., clicks of the counter for $\pi^+\pi^-$ that would point to a decay vertex at the position d_{-1} in front of the target T are not obtained (if there were any, they would be discarded as noise). One finds for the counting rate

$$\frac{\Delta N(t_i)}{\Delta t}\approx 0,\quad t<0. \tag{4.3}$$

[5] For another discussion of the impossibility of time reversing the development of a decaying microphysical system, see Lee [17].

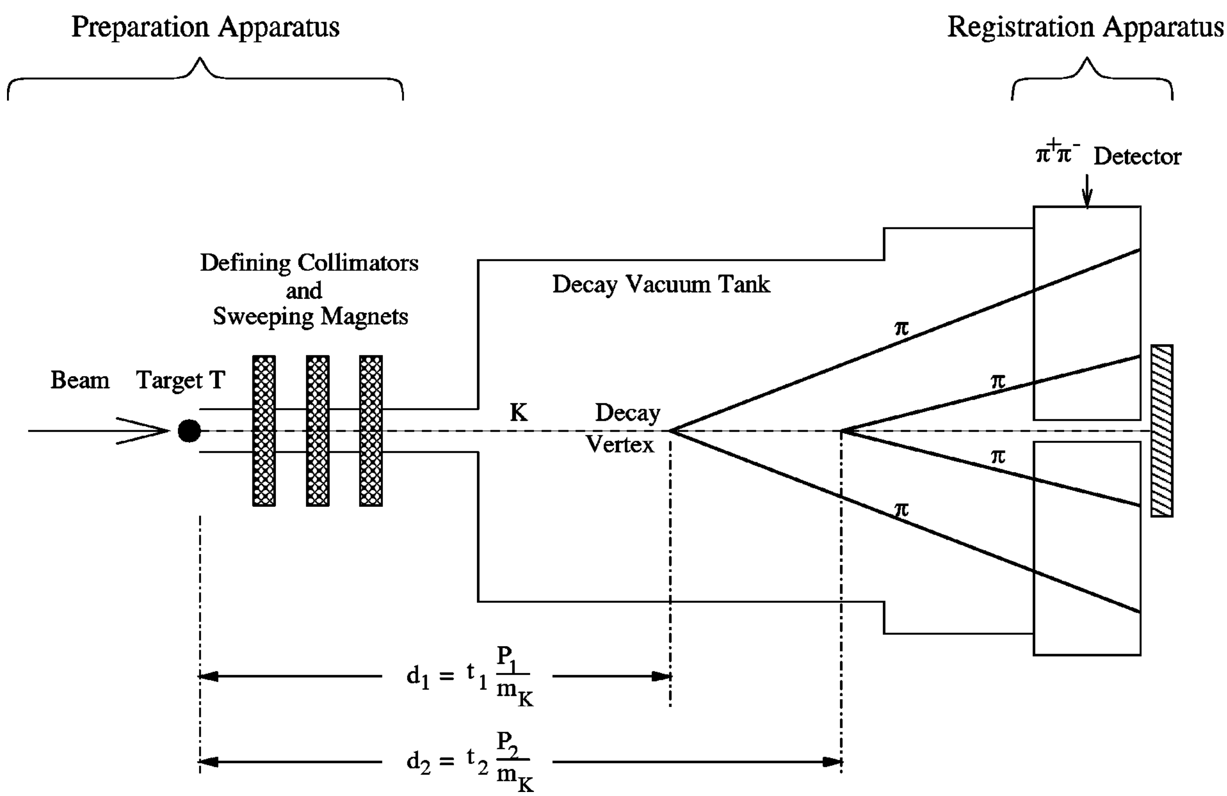

FIG. 1. Schematic diagram of the neutral K-meson decay experiment.

This is so obvious that one usually does not mention it. For $t>0$ one can fit the experimental counting rate with the exponential function to as good accuracy as one wants (by taking larger N and smaller Δt):

$$\frac{\Delta N(t_i)}{\Delta t} \approx N e^{-\Gamma t}, \quad t>0=t_0. \tag{4.4}$$

The $\approx$ in Eq. (4.4) means, as in Eq. (2.1), the equality between experimental numbers and the idealized, theoretical hypothesis $e^{-\Gamma t}$ [32,33].

Theoretically, the counting rate is given by the probability rate

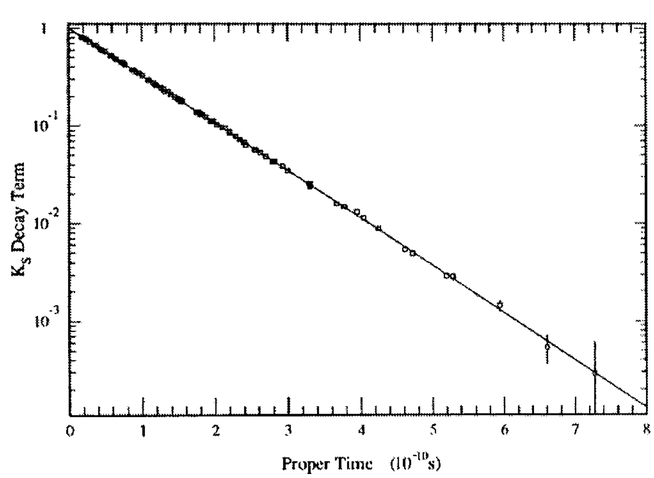

FIG. 2. K_S decay vs proper time.

$$R(t) = \frac{d\mathcal{P}(t)}{dt}, \tag{4.5}$$

where $\mathcal{P}(t)$ is the probability for the observable $\Lambda(t)$ of Eq. (4.2) (i.e., $\pi^+\pi^-$) in the state W^{K^0}.

According to the postulate (2.17), the probability should be given by

$$\mathcal{P}(t) = \mathrm{Tr}[\Lambda(t) W^{K^0}] = \mathrm{Tr}[\Lambda W^{K^0}(t)] \quad \text{for} \quad t \geqslant t_0 = 0, \tag{4.6}$$

where

$$W^{K^0}(t) = e^{-iHt} W^{K^0 e^{iHt}} \quad \text{for} \quad t \geqslant t_0. \tag{4.7}$$

For $t<t_0=0$, $W^{K^0}(t)$ is nonexistent because the K^0 had not been prepared by $t<t_0$.

To calculate theoretical results that agree with the observations (4.3) and (4.4), one has to choose the state operator $W^{K^0}(t)$ in Eq. (4.6) such that $W^{K^0}(t)$ is nonexistent for $t<t_0=0$, and such that for $t>t_0=0$, yields by Eqs. (4.5) and (4.6), a result that is in agreement with the right hand side of Eq. (4.4). The state operator which has this property is given by Eq. (2.16),

$$W^{K^0}(t) = |F(t)\rangle\langle F(t)|, \tag{4.8}$$

where

$$F(t) = U_+^{\times}(t) F_0 \tag{4.9}$$

is a semigroup solution (2.15) of the quantum mechanical Cauchy problem, and where the initial vector is given by the Gamow vector

$$F_0 = |E_R - i\Gamma/2^-\rangle \in \Phi_+^\times, \tag{4.10}$$

with $E_R = m_S$ and $\Gamma = 1/\tau_S$ for the K^0 at rest [34].

Then we obtain the time evolved state vector (3.20) by applying the semigroup (3.15). For this vector $|E_R - i\Gamma/2^-\rangle$ (and only for this Gamow vector $|E_R - i\Gamma/2^-\rangle \in \Phi_+^\times$, which is defined by the pole term of the S matrix) one derives the exact Golden Rule with the (exact) exponential decay law (3.23), thus reproducing the right hand side of Eq. (4.4) [33].

Therewith we see that the Gamow state vector $\psi^G = |E_R - i\Gamma/2^-\rangle$ or the operator $W^G = |\psi^G\rangle\langle\psi^G|$, whose time evolution is governed by the exact Hamiltonian H, describes the decaying neutral kaon system (4.1) in its rest frame if $E_R = m_s$ and $\Gamma = \Gamma_s = 1/\tau_s$, $W^G = W^{K^0}$.

For this Gamow state one can calculate the decay rate and decay probability as a function of time and obtain the exponential law for $t > t_0 = 0$. The decay probability is the *a priori* probability for a single decaying microsystem K^0 that has been created in the state W^{K^0} at the initial time $t=0$ (for the quantum mechanical Cauchy problem with semigroup evolution). This is the same point of view mentioned at the beginning of this section for the quantum state of our universe [2], except that its initial state $\rho(t_0)$ is probably not a ''pure'' Gamow state. Alternatively $W^{K^0}(t)$ can also be thought of as describing the state of an ensemble of single microsystems K^0 created at an ''ensemble'' of times t_0, all of which are chosen to be the initial time $t=0$ for the quantum mechanical Cauchy problem. Then the decay probabilities are the statistical probabilities for this ensemble of individual K^0 systems, but t in $W^{K^0}(t)$ is the time in the ''life'' of each single decaying K^0 system which had started at $t=0$. It is not the time in the experimentalists life or the time in the laboratory or the time of a ''wave packet'' of K^0's.

With this interpretation the single quasistable particle and the single quantum universe are perfectly analogous, and the time t_0, at which the preparation of the state is completed and at which the registration of the observables can begin, has been observationally defined.

V. SUMMARY AND CONCLUSION

If we want to have a quantum theory that applies to the (closed) universe as a whole then we would like this quantum theory to be time asymmetric, because of the cosmological arrow of time. By the same reasoning if a quantum theory is to apply to the electromagnetic field then it should be time asymmetric, because of the radiation arrow of time. Standard quantum theory is time symmetric. This is a mathematical consequence due to the property of the Hilbert space postulates.

There is a mathematical theory that describes time-symmetric as well as time-asymmetric quantum physics. It is an extension of the rigged Hilbert space formulation of quantum mechanics which in about 1965 gave a mathematical justification to Dirac's kets and his continuous basis vector expansion.

To incorporate causality, the RHS theory distinguishes meticulously between states and observables for which it uses two RHS's $\Phi_\mp \subset \mathcal{H} \subset \Phi_\mp^\times$ of Hardy class with complementary analyticity properties. The dual spaces in the RHS's contain, besides the Dirac kets $|E^\pm\rangle \in \Phi_\mp^\times$ [$\binom{\text{in}}{\text{out}}$ plane waves], also Gamow kets $|E_R - i\Gamma/2^\pm\rangle \in \Phi_\mp^\times$. The Gamow kets have all the properties attributed to eigenvectors of a complex finite-dimensional Hamiltonian in the phenomenological effective theories, in particular, an exponential time evolution and a Breit-Wigner energy wave function. Neither of these is possible in Hilbert space.[6] These effective finite-dimensional theories can therefore not be considered approximations [16] of standard quantum mechanics, but they go beyond it. The RHS formulation is the mathematical theory of which these finite-dimensional models, e.g., the two-dimensional Lee-Oehme-Yang theory for neutral kaons and the Weisskopf-Wigner method, are approximations.

Features of the exact theory which are not already features of these effective models and phenomenological methods are the Breit-Wigner wave function (3.22) of the Gamow ket which extends over $-\infty_{\mathrm{II}} < E < +\infty$ (rather than the values $0 \leqslant E < \infty$), the background integral in (3.25), and the exact Golden Rule [24] for the decay probability $P(t)$ from which the decay rate $R(t)$ with exponential time dependence (3.23) is obtained by differentiation. Dirac's Golden Rule for the initial decay rate is the Born approximation at $t=0$ of this exact rule for the decay rate $R(t)$. These are features which one may welcome or accept. The most surprising, unwanted, and mostly rejected feature of the exact RHS theory is the semigroup time evolution (3.20) and (3.21) of the Gamow state, which is a manifestation of a fundamental quantum mechanical arrow of time.

Decaying Gamow states can be experimentally isolated as quasistationary microphysical systems if their time of preparation can be accurately identified. The observed decay probabilities (4.3), (4.4) of the neutral kaon system have the same features as derived from a Gamow state, including the time ordering, Eq. (4.6). This time ordering is the same as the postulated time ordering in the probabilities of the histories of the universe considered as a quantum system, Eqs. (2.19)–(2.25). Under this hypothesis the fundamental quantum arrow of time—expressing the vague notion of causality—can be considered subsumed under the cosmological arrow of time.

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[6]The Breit-Wigner energy wave function would not be in the domain of the Hamiltonian.

APPENDIX A: THEOREMS OF THE HILBERT SPACE THEORY

In the Hilbert space formulation of quantum mechanics, the linear scalar product space $\Phi_{\rm alg}$ is completed with respect to the norm to obtain a Hilbert space $\mathcal{H}$. The Hamiltonian H in the Schrödinger–von Neumann equations (2.8) is self-adjoint and semibounded, and the initial data $\phi_0, \psi_0 \in \mathcal{H}$.

Then one has the following mathematical theorems

(1) (Gleason) For every probability $\mathcal{P}(\Lambda)$, there exists a positive trace class operator ρ such that $\mathcal{P}(\Lambda)=\mathrm{Tr}(\Lambda\rho)$ [35].

(2) (Stone–von Neumann) The solutions of the Schrödinger–von Neumann equations for this ρ are time symmetric and given by the group $U^{\dagger}(t)=e^{-iHt}$ of unitary operators [36].

(3) (Hegerfeldt) For every Hamiltonian H (self-adjoint, semibounded), either

$$\mathrm{Tr}[\Lambda(t)\rho]=\mathrm{Tr}[\Lambda\rho(t)]=0 \quad \text{for} \quad -\infty<t<\infty$$

or

$$\mathrm{Tr}[\Lambda(t)\rho]=\mathrm{Tr}[\Lambda\rho(t)]>0 \quad \text{for all} \quad t$$

(except on a set of Lebesgue measure zero).
Here, Λ can be any positive, self-adjoint operator such as $\Lambda=|\psi\rangle\langle\psi|$ or $\Lambda=C_\alpha$ of Eq. (2.20) and ρ any trace class operator like $\rho=|\phi\rangle\langle\phi|$ or $\rho=\rho(t_{\text{big bang}})$ [37].

Theorem 1 says that all probabilities must be given by the trace. From theorem 3 it then follows that there cannot be a state ρ in the Hilbert space $\mathcal{H}$ that has been created or prepared a finite time $t-t_0$ ago, and for which therefore $\mathrm{Tr}[\Lambda\rho(t)]=0$ for $t<t_0$, which for $t\geqslant t_0$ decays into decay products Λ with a decay probability that is different from zero. This means there exist no elements ϕ in $\mathcal{H}$ that can represent decaying states. Also absent are the states $\rho^{\rm eff}(t_i)$ that have been created at times $t=t_0=t_{\text{big bang}}$, $t=t_1>t_0$, $t=t_2>t_1$, etc., and whose probabilities $\mathrm{Tr}[P(t_n)\rho^{\rm eff}(t_{n-1})]$ are different from zero. Theorem 2 prohibits the asymmetric time evolution of a state in $\mathcal{H}$ and therewith the existence of a distinguished time t_0 of creation.

APPENDIX B: *S* MATRIX AND LIPPMANN-SCHWINGER EQUATION

Every experiment consists of a preparation stage and a registration stage. In the preparation stage of the scattering experiment, a (mixture of) initial states $\phi^{\rm in}$ is prepared before the interaction $V=H-K$ is effective (e.g., by an accelerator outside the interaction region of the target). The initial state vectors $\phi^{\rm in}$, describing the noninteracting beam and target, evolve in time according to the free Hamiltonian K: $\phi^{\rm in}(t)=e^{-iKt}\phi^{\rm in}$. When the beam reaches the interaction region, the free in state $\phi^{\rm in}$ turns into the exact state vector ϕ^{+} whose time evolution is governed by the exact Hamiltonian $H=K+V$,

$$\Omega^{+}\phi^{\rm in}(t)\equiv\phi^{+}(t)=e^{-iHt}\phi^{+}=\Omega^{-}\phi^{\rm out}. \tag{B1}$$

This vector leaves the interaction region and ends up as the well-determined state $\phi^{\rm out}$. The state vector $\phi^{\rm out}$ is determined from $\phi^{\rm in}$ by the dynamics of the scattering process:

$$\phi^{\rm out}=S\phi^{\rm in}, \quad S=\Omega^{-\dagger}\Omega^{+}. \tag{B2}$$

$\phi^{\rm in}$ *is controlled and determined by the preparation apparatus*. $\phi^{\rm out}$ is also controlled by the preparation apparatus and is in addition determined by the interaction V.

In the registration stage, the detector outside the interaction region does not detect $\phi^{\rm out}$, but rather it detects an observable $\psi^{\rm out}(t)=e^{iKt}\psi^{\rm out}$ (or a mixture thereof). $\psi^{\rm out}$ *is controlled by the registration apparatus* (trigger, energy efficiency, etc., of the detector). The detector counts are a measure of the probability to find the observable (property) $|\psi^{\rm out}\rangle\langle\psi^{\rm out}|$ in the state $\phi^{\rm out}$. This probability $|(\psi^{\rm out},\phi^{\rm out})|^2$ is calculated by the S matrix.

The S matrix is the probability amplitude $(\psi^{\rm out},\phi^{\rm out})$ which is calculated in the following way:

$$\begin{aligned}(\psi^{\rm out},\phi^{\rm out})&=(\psi^{\rm out},S\phi^{\rm in})=(\Omega^{-}\psi^{\rm out},\Omega^{+}\phi^{\rm in})=(\psi^{-},\phi^{+})\\ &=\int_0^\infty dE\langle\psi^{-}|E^{-}\rangle S(E+i0)\langle^{+}E|\phi^{+}\rangle.\end{aligned} \tag{B3}$$

$\phi^{+}(t)=e^{iHt/\hbar}\phi^{+}$ comes from the prepared in state $\phi^{\rm in}(t\to-\infty)=(\Omega^{+})^{-1}\phi^{+}(t\to-\infty)$. The free observable vector $\psi^{\rm out}$ emerges from the observable vector ψ^{-} whose time evolution is governed by the exact Hamiltonian H.

$\psi^{-}(t)=e^{iHt/\hbar}\psi^{-}$ goes into the measured out state $\psi^{\rm out}(t\to+\infty)=(\Omega^{-})^{-1}\psi^{-}(t\to+\infty)$. Ω^{+} and Ω^{-} are the Møller wave operators. The Lippmann-Schwinger equation relates the (known) eigenvectors of the free Hamiltonian K to two sets of eigenvectors of the exact Hamiltonian H,

$$\begin{aligned}|E^{\pm}\rangle&=|E\rangle+\frac{1}{E-K\pm i\epsilon}V|E^{\pm}\rangle=|E\rangle+\frac{1}{E-H\pm i\epsilon}V|E\rangle\\ &=\Omega^{\pm}|E\rangle,\end{aligned} \tag{B4}$$

where

$$K|E\rangle=E|E\rangle, \quad H|E^{\pm}\rangle=E|E^{\pm}\rangle. \tag{B5}$$

This defines the exact energy wave functions in terms of the in and out energy wave functions, whose modulus is given by the energy resolution of the experimental apparatuses:

$$\langle^{+}E|\phi^{+}\rangle=\langle E|\phi^{\rm in}\rangle \tag{B6}$$

describes the energy distribution of the incident beam, i.e., the energy distribution given by the accelerator (preparation apparatus),

$$\langle^{-}E|\phi^{-}\rangle=\langle E|\phi^{\rm out}\rangle \tag{B7}$$

is the energy distribution of the detected state; it is given by the energy resolution of the detector (registration apparatus). Since $\phi^{\rm in}$ is controlled by the preparation apparatus, so is ϕ^{+}. Likewise, since $\psi^{\rm out}$ is controlled by the registration apparatus, so is ψ^{-}. All this is quite standard, cf. [38], Chapter 7, except that of the two versions, mentioned on p. 188 of

[38] as equally valid descriptions, we allow only the first version, which is in agreement with our physical intuition of causality. In order to do this we distinguish between the set of in-state vectors $\{\phi^+\}\equiv\Phi_-$ and the set of out-observable vectors $\{\psi^-\}\equiv\Phi_+$. This hypothesis is quite natural since the state ϕ^+ (or $\phi^{\rm in}$) must be prepared before the observable $|\psi^-\rangle\langle\psi^-|$ (or $\psi^{\rm out}$) can be measured in it. As was discussed in Sec. III, Φ_- and Φ_+ are different dense subspaces of the same Hilbert space $\mathcal{H}$.

APPENDIX C: IRREVERSIBILITY IN CONVENTIONAL QUANTUM STATISTICAL MECHANICS

The irreversible time evolution described by the semigroup (2.15) and (2.16) is not the irreversible time evolution of open systems in quantum statistical mechanics which is described by a Liouvillian L not a Hamiltonian. Irreversibility in conventional quantum theory is always thought of as being due to external influences upon the nonisolated (''open'') quantum system. The irreversible time evolution of open quantum systems is described by the master equation [18]

$$\frac{\partial\rho(t)}{\partial t}=L\rho(t), \tag{C1}$$

where $\rho(t)$ represents the state of the open system S, and the Liouville operator L is given by

$$L\rho(t)=-\frac{i}{\hbar}[H,\rho(t)]+\mathcal{I}\rho(t), \tag{C2}$$

where H is the Hamiltonian of the open system and $\mathcal{I}$ is the interaction of the external reservoir upon the system, e.g.,

$$\mathcal{I}=\sum_{\alpha=1,2,\cdots}\{[V_\alpha\rho(t),V_\alpha^\dagger]+[V_\alpha,\rho(t)V_\alpha^\dagger]\}. \tag{C3}$$

For $\mathcal{I}\rho=0$, Eq. (C1) with Eq. (C2) is the von Neumann equation (2.8). Its Hilbert space solution is the reversible time evolution of the isolated quantum system given by Eq. (2.9). Equation (C1) with Eq. (C2) is the standard equation for extrinsic irreversibility under the influences of an external reservoir R (which could be, for example, a measuring apparatus) upon the system S. The term $\mathcal{I}\rho$ represents some complicated external effects of the reservoir R upon the quantum system S. Under particular assumptions about the term $\mathcal{I}\rho$ the irreversible time evolution of S can be shown to be described by a completely positive semigroup generated by a Liouvillian L:

$$\rho(t)=\Lambda(t)\rho(0),\quad \Lambda(t)=e^{Lt}\quad\text{for}\quad t\geqslant 0. \tag{C4}$$

This is the conventional semigroup evolution of open quantum systems. The semigroup (C4) can evolve a pure state $|\phi\rangle\langle\phi|$ into a mixture, which our semigroup (2.15) probably cannot. A special case of this irreversible time evolution is obtained if one chooses for the reservoir R the measuring apparatus. It has been argued [39] that the collapse axiom

$$\rho(t_0)\to\rho^{\rm after}(t_0)=\sum_i P_{\alpha_i}\rho(t_0)P_{\alpha_i} \tag{C5a}$$

or

$$\rho(t_0)\to\rho^{\rm after}(t_0)=P_{\alpha_1}(t_0)\rho(t_0)P_{\alpha_1}(t_0) \tag{C5b}$$

together with the Schrödinger equation (2.8) leads to semigroup evolutions (C4) generated by a Liouvillian L.

Our semigroup (2.15) or (2.16) is not the semigroup (C4) with Eq. (C3). The ''irreversibility'' described by our semigroup (2.15), (2.16) is also not a consequence of the increase of von Neumann entropy due to the collapse axiom (C5). On the contrary, it is our quantum mechanical arrow of time given by (2.15) that has a consequence for the von Neumann entropy increase.

The collapse axiom (C5) is an unrealistic mathematical idealization. A real measurement process which changes ρ to $\rho^{\rm after}$ is a scattering process of microsystem on a macrosystem (''measurement scattering'' [10]). Every measurement takes time and the collapse (C5b) cannot happen instantly, thus one is led to the time ordering as in Eq. (2.22). The measurement scattering, like every scattering experiment, possesses an arrow of time, because preparation of $\rho(t_0)$ must always precede registration of $P_{\alpha_i}(t_1)$. The change of state due to measurement, if it leads from a less mixed state $\rho(t_0)$ to a more mixed state and is given by something like Eq. (C5a), must therefore have the same time ordering as Eq. (2.22). This means in place of Eq. (C5a) one must have

$$\rho(t_0)\to\rho(t_1)=\sum_i P_{\alpha_i}(t_1)\rho(t_0)P_{\alpha_i}(t_1)\quad\text{with}\quad t_1>t_0\quad\text{only}. \tag{C6}$$

As a consequence the von Neumann entropy increases in the same time direction:

$$-\mathrm{Tr}[\rho^{\rm after}(t_1)\ln\rho^{\rm after}(t_1)]\geqslant-\mathrm{Tr}\,\rho\ln\rho,\quad\text{for}\quad t_1>t_0\quad\text{only}. \tag{C7}$$

This means the von Neumann entropy arrow of time is subsumed under the fundamental quantum mechanical arrow of time given by the time evolution semigroup (2.15).

A realistic quantum mechanical measurement is probably not precisely described by Eq. (C6), but one is probably right to assume that the state after a measurement $\rho^{\rm after}(t_1)$ is a more mixed state than the state $\rho(t_0)$ before a measurement so that Eq. (C7) will be valid also for a realistic quantum measurement. This means the von Neumann entropy increases in the direction of the quantum mechanical arrow of time. The above conclusion does not mean that we have derived the entropy increase from the quantum mechanical semigroup, because we also had to assume mixture increase, i.e., something like Eq. (C5a) or Eq. (C6) and this cannot be derived from the semigroup evolution alone. But we have obtained for the von Neumann entropy increase an arrow of time, which is the same as the direction of the quantum mechanical semigroup or the same as the preparation $\Rightarrow$ registration arrow of time, assuming state mixture increase (''decoherence'').

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[22] A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets*, Lecture Notes in Physics Vol. **348** (Springer-Verlag, Berlin, 1989).

[23] The Gamow vectors were envisioned by G. Gamow [Z. Phys. **51**, 204 (1928)] as far back as the inception of the Dirac kets. Mathematically, they were no more ill defined than the Dirac kets, though they never acquired the popularity of the latter. The reason was that, although for Dirac kets at least the probability density is finite everywhere, the probability density of Gamow's wave functions $|\langle r,\theta,\varphi|\psi^G(t)\rangle|^2$ increased exponentially for large values of the distance r and large negative values of time t (the ''exponential catastrophe''). The origin of this exponential catastrophe is that the emission of decay products had been assumed to go on for an arbitrarily long time, $t\rightarrow-\infty$, as dictated by the unitary group e^{-iHt}, $-\infty<t<\infty$. In reality, the emission of decay products must have begun at some finite time $t=t_0=0$ in the past, and the time evolution of the decaying state, as we now know, is described by the semigroup $e^{-iH^\times t}$, $0<t<\infty$. Therefore, in the time-asymmetric theory the probability density $|\langle r,\theta,\varphi|\psi^G(t)\rangle|^2$ is only defined for times $t>t_0(r)=r/v=mr/p\approx mr/\sqrt{2mE_R}$, and for these times one can show that there exists no exponential catastrophe for the probability densities. Thus the semigroup evolution of the Gamow vectors is not only a consequence of the mathematical theory, but also a necessity for the physical interpretation of the position probabilities $\int_{\Delta r}r^2drd\theta d\varphi|\langle r,\theta,\varphi|\psi^G(t)\rangle|^2$.

[24] A. Bohm and N. L. Harshmann, in *Irreversibility and Causality*, edited by A. Bohm, H. D. Doebner, P. Kielanowski (Springer, Berlin, 1998), p. 225, Sec. 7.4.

[25] The function $G_+(E)(=\langle {}^-E|\psi^-\rangle=\langle E|\psi^{\rm out}\rangle)$ is a very well-behaved function of the upper half plane $\mathbb{C}^+$ if it is well behaved, i.e., $G_+(E)\in\mathcal{S}$ (Schwartz space) and if it is the boundary value of an analytic function $G_+(z)$ in the upper half plane $\mathbb{C}^+$ which vanishes faster than any power at the infinite semicircle [i.e., $G_+(E)\in\mathcal{H}_+$]. Similarly the function $G_-(E)(=\langle {}^+E|\phi^+\rangle=\langle E|\phi^{\rm in}\rangle)$ is a very well-behaved function of the lower half plane $\mathbb{C}^-$ if it is well behaved $[G_-(E)\in\mathcal{S}]$ and if it is the boundary value of an analytic function in the lower half plane $\mathbb{C}^-$ which vanishes sufficiently fast at the lower semicircle $[G_-(E)\in\mathcal{H}_-]$. For the definition of Hardy class functions and their mathematical properties needed here see Appendix A.2 of Ref. [27] and P. L. Duren, $\mathcal{H}^p$ *Spaces*, (Academic Press, New York, 1970).

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[29] A. Bohm, *Quantum Mechanics*, 3rd ed. (Springer-Verlag, New York, 1994) Sec. XVIII.6-9, XX.3.

[30] For a detailed analysis of the K_L-K_S system in the framework of time-asymmetric quantum mechanics with and without CP violation, see A. Bohm, e-print hep-th/970542.

[31] This is the simplified schematic diagram of several generations of experiments measuring CP violation in the neutral kaon system; J. Christenson, J. Cronin, V. Fitch, and R. Turley, Phys. Rev. Lett. **13**, 138 (1964); K. Kleinknecht, in *CP Violation*, edited by C. Jarskog (World Scientific, 1989), p. 41, and references therein; G. D. Barr *et al.*, Experiment no. NA**31**, Phys. Lett. B **317**, 233 (1993), Experiment no. NA31; L. K. Gibbons *et al.*, E731, Phys. Rev. Lett. **70**, 1199 (1993), experiment no. E731; **70**, 1203 (1993).

[32] This approximate agreement between a sequence of rational numbers $\Delta N(t)/\Delta t$ on the experimental side and a continuous function of real numbers $e^{-\Gamma t}$ on the theoretical side is the fundamental limit to which the exponential law can be verified; experimental limitations given by the resolution, e.g., the finite size of Δt, are still more important. Therefore any *infinitesimal* (not given in terms of the scale $1/\Gamma$) deviations from the exponential law that are derived from a mathematical theory (e.g., the Hilbert space idealization, L. A. Khalfin, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 548 (1972) [JETP Lett. **15**, 388 (1972)], or our Φ-space idealizations), are physically meaningless.

[33] Observed deviations from the exponential law [S. R. Wilkinson *et al.*, Nature (London) **387**, 575 (1997)] are in the RHS theory explained by the basis vector expansion (3.25) and (3.28) for the experimantally prepared state ϕ^+. In addition to the exponential term, Eq. (3.30) predicts less than exponential time dependence for the background term.

[34] The K^0 is a relativistic decaying system and our discussion here is in terms of the non-relativistic Gamow vectors. Relativistic Gamow kets can be defined from the poles of the relativistic S matrix at the value of the invariant mass square $s_R=(m_R-i\Gamma/2)^2$. They have at rest the same semigroup time evolution as Eq. (3.20) with $E_R\rightarrow m$ and $\Gamma\rightarrow\hbar/\tau_R$. A. Bohm, H. Kaldass, *et al.*, e-print hetp-th/9905213; e-print hetp-th/9904053.

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