Entanglement of projection and a new class of quantum erasers

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We define a new measurement of entanglement, the entanglement of projection, and find that it is natural to write the entanglements of formation and assistance in terms of it. Our measure allows us to describe a new class of quantum erasers which restore entanglement rather than just interference. Such erasers can be implemented with simple quantum computer components. We propose realistic optical versions of these erasers. [S1050-2947(99)02508-1]

Entanglement is the degree to which the wave function does not factorize. For example, an $S=0$ two-particle system $|+-\rangle - |-+\rangle$ is maximally entangled: measurement of the spins reveals they are completely anticorrelated. The concept of entanglement goes to the very heart of quantum mechanics, and understanding its nature is a prerequisite to understanding quantum mechanics itself. Two-particle entanglement was used by Einstein, Podolsky, and Rosen [1] to argue that quantum mechanics could not be a complete description of reality; that there had to be an underlying local theory. But Bell used such entangled states to show that any local underlying theory would have to satisfy certain inequalities, which quantum mechanics explicitly violates [2]. Experiments on such entangled states have shown that these inequalities are violated just as quantum mechanics predicts [3]. Modern research on entanglement includes proposals for providing cleaner demonstrations of this nonlocality using three-particle entangled states [4], and on quantifying entanglement [5–8].

The goal of this paper is to define a new class of quantum erasers which restore entanglement of a multiparticle subsystem, rather than just interference, and to quantify that restoration with a new measure of entanglement [9]. Our measure highlights the importance of considering mixed states as higher dimensional pure states by showing that certain observables cannot be specified independent of information about the pure state, strongly suggesting that mixed states are not fundamental.

A quantum eraser [10] is a device in which coherence appears to be lost in a subset of the system, but in which that coherence can be restored by erasing the tagging information which originally ''destroyed'' it. Traditional erasers [11,12] need only two distinct subsystems. For example, if one sends particle A through two slits, and if one ''tags'' which slit A goes through via the interaction with a tagging particle, T, then the interference pattern will disappear. But if one makes the ''which slit'' information in T unobservable, even in principle, then one can restore the interference pattern for A. To avoid the use of a double-negative, one could refer to this as an *interference restorer*.

A simple way to erase this tagging information is to measure T in the $|0\rangle_T \pm |1\rangle_T$ basis. Here ''$|0\rangle_T$'' means ''T interacted with A at slit 0.'' The positions of A on the screen corresponding to T in the state $|0\rangle_T + |1\rangle_T$ display an interference pattern, and those corresponding to T in the state $|0\rangle_T - |1\rangle_T$ display a shifted interference pattern. While the overall pattern on the screen shows no interference, for the subsets of these events corresponding to $|0\rangle + |1\rangle$ or $|0\rangle - |1\rangle$, coherence is restored.

Our new class of erasers involves at least three subsystems, A, B, and T. Consider an entangled state $|00\rangle_{AB} + |11\rangle_{AB}$ of subsystem AB. If we tag the pieces of this with T so that the wave function of the whole system is $|00\rangle_{AB}|0\rangle_T + |11\rangle_{AB}|1\rangle_T$, then the entanglement of subsystem AB appears to be lost. But if one erases that tagging information, then the entanglement is restored. Thus we will refer to this object as a *disentanglement eraser*, or, equivalently, as an *entanglement restorer*. This leads to the profound point that the entanglement of any two particles that do not interact (directly or indirectly), never disappears but rather is encoded in the ancilla of the system. A projective measurement that seems to destroy such entanglement could always in *principle* (though rarely in practice) be erased by suitable manipulation of the ancilla.

In order to discuss these new erasers, we will need to define several measures of entanglement. For a *pure*, two-particle, two-state system that can be thought of as a pair of qubits (quantum bits), the entanglement is well-defined. One can always write such a pure ''2×2'' system in the Schmidt basis so that $|\psi_{AB}\rangle = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$, with α and β positive and real, and $\alpha^2 + \beta^2 = 1$. The AB system has a pure density matrix $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, while the subsystem for A alone has a mixed density matrix $\rho_A = \text{Tr}_B \rho_{AB}$. Then one can write entanglement of AB in terms of the von Neumann entropy [5],

$$E(\psi_{AB}) = -\text{Tr}[\rho_A \log_2 \rho_A] = e(\alpha^2), \quad (1)$$

where $e(x) = -[x \log_2 x + (1-x)\log_2(1-x)]$. Since $e_{min} = e(0) = e(1) = 0$ and $e_{max} = e(1/2) = 1$, $E(\psi)$ ranges from 0 for no entanglement, to 1 for a fully entangled state. This $E(\psi)$ remains constant with any unitary operation on A or B, and is changed only by operations where the effect on one subsystem (say A) depends upon another (either B or a third

subsystem). Such interactions can be implemented with controlled-NOT gates (c-NOTs). One can show that all logic gates of a quantum computer can be constructed solely in terms of unitary operations on individual qubits and on c-NOTs between the qubits. Thus, one role of multiqubit logic gates is to change the entanglement between pairs of qubits.

Mixed states, on the other hand, do not have a unique measure of entanglement. One can construct measures with different dependences on the possible decompositions of the density matrix ρ_{AB}. Let us write ρ_{AB} in terms of pure states $|\chi_i\rangle_{AB}$, with weights p_i,

$$\rho_{AB} = \sum_{i=0}^{m-1} p_i |\chi_i\rangle\langle\chi_i|. \tag{2}$$

Note that the χ_i's need not be orthogonal. Here $m \geq n$, with n being the number of nonzero eigenvalues of ρ_{AB}. If one naively tries to define the entanglement of Eq. (2) as $\sum_i p_i E(\chi_i)$, one finds that it depends on the χ_i's chosen [8]. Instead what has been done traditionally is to write the *entanglement of formation* [7],

$$E_f(\rho_{AB}) = \min \sum_{i=0}^{m-1} p_i E(\chi_i), \tag{3}$$

which is the *minimum* value of the naive measure over all decompositions of ρ_{AB}. Recently, a new measure has been defined called the *entanglement of assistance* [8], E_a, which is just the *maximum* value of the naive measure over all decompositions.

To see what the decomposition dependence of the naive measure really means, let us write the mixed state ρ_{AB} in terms of a higher dimensional pure state,

$$|\Psi_{ABT}\rangle = \sum_{i=0}^{d_T-1} \sqrt{p_i} |\psi_i\rangle_{AB} |i\rangle_T, \tag{4}$$

where $|i\rangle_T$ are d_T orthonormal pure states of a set of "tagging particles" or "taggants." If we trace over the d_T tagging states of the pure density matrix $|\Psi_{ABT}\rangle\langle\Psi_{ABT}|$, we obtain ρ_{AB} of Eq. (2) with d_T component pure states: $\{|\chi_i\rangle\} = \{|\psi_i\rangle\}$ with $m = d_T$. This decomposition depends upon the chosen taggant basis, $\{|i\rangle_T\}$. For a *given* taggant basis, the entanglement is well defined:

$$E_{p\{|i\rangle_T\}}(\Psi_{ABT}) = \sum_{i=0}^{d_T-1} p_i E(\psi_i), \tag{5}$$

where the entanglement of the component pure states, $E(\psi_i)$, is given by Eq. (1). We call this the *entanglement of projection* because it corresponds to the projection of the full pure state Ψ_{ABT} onto a given taggant basis to yield a mixed subsystem AB with an entanglement E_p. (Cohen [13] uses a similar quantity to bound the amount of classical information one would need to unlock entanglement hidden in a separable mixed state.) What this means practically is that if subsystem AB is entangled with a taggant T, and one measures the taggant in basis $\{|i\rangle_T\}$, the resulting projected pure states of AB have an average entanglement equal to $E_{p\{|i\rangle_T\}}$.

If one measures the taggant in a different basis $|i'\rangle_T = U|i\rangle_T$, then the entanglement of projection becomes the weighted average $\sum_i p_i' E(\psi_i')$, where we have rewritten Eq. (4) in the new taggant basis: $|\Psi_{ABT}\rangle = \sum_{i=0}^{d_T-1} \sqrt{p_i'} |\psi_i'\rangle_{AB} |i'\rangle_T$. This shows that for a given pure state $|\Psi_{ABT}\rangle$, E_p takes on different values for different choices of taggant basis; there is no unique measure of entanglement for a mixed subsystem AB. In fact, by taking the minimum and maximum values of $E_{p\{U\}}$ over all possible taggant bases $U|i\rangle_T$, one recovers the entanglements of formation and assistance,

$$E_f = \min_U E_{p\{U\}}, \quad E_a = \max_U E_{p\{U\}}, \tag{6}$$

so E_p is bounded by E_f and E_a: $E_f \leq E_{p\{U\}} \leq E_a$.

Our E_f in Eq. (6) is identical to that of Eq. (3) for all the cases we consider. For E_a, however, there are cases where the number of pure states m_a needed for an entanglement-maximizing decomposition of ρ is greater than the rank of ρ ($m_a > n$) [14]. Since we consider only projective measurements, this means that our E_a depends on d_T. For $d_T < m_a$, our E_a can be smaller than the E_a of Ref. [8]. Our E_a measures the amount of assistance a specific "friend" T can give to AB for a specific pure state Ψ_{ABT}, whereas their E_a measures how much assistance an arbitrary T leading to ρ_{AB} could give.

To quantify the entanglement in our erasers, we need to take into account whether or not the taggant has been measured. Let us define h to be the number of outcomes resulting from any measurements of T, and P_j as the projection operator for outcome j, which occurs with probability q_j and results in AB state ρ_j. Then we can define the *entanglement of projections' formation*,

$$E_{pf} = \sum_{j=0}^{h-1} q_j E_f(\rho_j). \tag{7}$$

If no measurement has been performed on T, then $h = 1$, $\rho_0 = \rho_{AB}$, and $E_{pf} = E_f$. If a nondegenerate measurement is performed on T, then $h = d_T$, the ρ_j are all pure, and E_{pf} is just E_p for the basis of T defined by the projectors $\{P_j\}$. For E_{pf} to increase after a measurement, there must have been entanglement between T and AB.

To illustrate the utility of E_p and E_{pf}, consider a pure system ABT whose 2×2 subsystem AB is a mixed state of only two pure states: $|\Psi_{ABT}\rangle = \alpha|00\rangle_{AB}|0\rangle_T + \beta|11\rangle_{AB}|1\rangle_T$. It is clear that $E_p = 0$ in the taggant basis $\{|0\rangle_T, |1\rangle_T\}$, and thus $E_f = 0$ for subsystem AB. Since T has not been measured, $E_{pf} = E_f = 0$. But if we project the taggant onto basis $|i'\rangle_T = U|i\rangle_T$, with

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{8}$$

the entanglement of projection of AB in that basis is

$$E_{p\{U\}} = p_0 e(a^2 \alpha^2 / p_0) + p_1 e(b^2 \alpha^2 / p_1), \tag{9}$$

with the probability of the taggant being projected into state $|0'\rangle$ being $p_0 = a^2 \alpha^2 + b^2 \beta^2$, and with $p_1 = 1 - p_0$. (For our choice of basis we can take a and b to be real.) Note that,
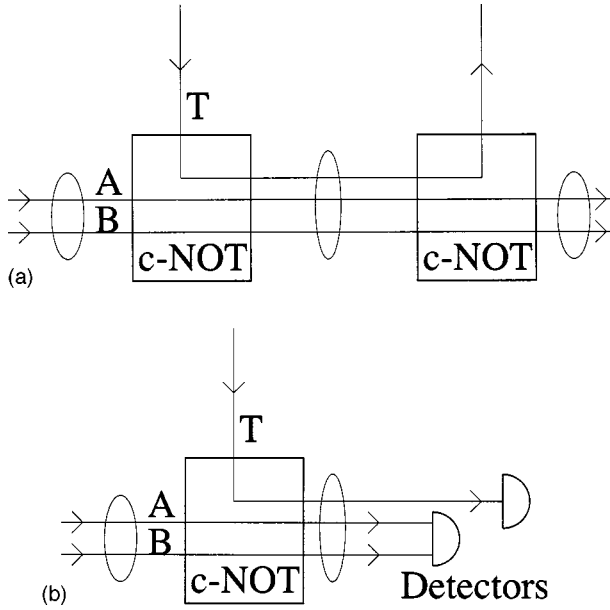
FIG. 1. (a) Entangled state AB enters the tagger, which dilutes the entanglement into the whole ABT system. Reversing this operation restores AB entanglement. (b) After tagging, entanglement is restored by measuring T.

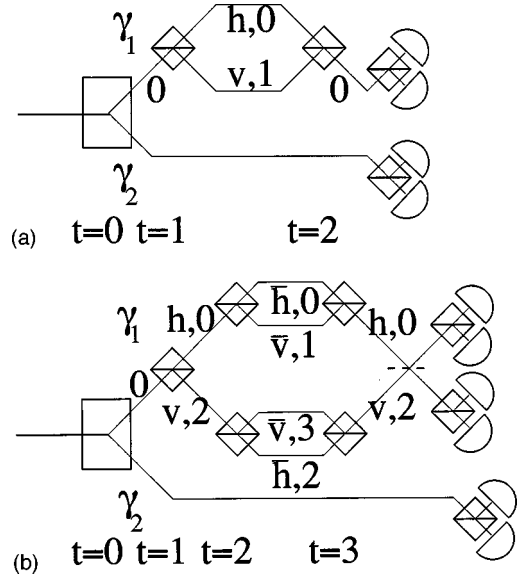FIG. 2. (a) At $t=1$, the s_1-s_2 entanglement of the two photons is tagged by path p_1 via a PBS (c-NOT). As in Fig. 1(a), the tagging operation is simply reversed. (b) The s_1-p_1 entangled state of $t=2$ is (re)tagged with s_2 at $t=3$. Next one measures s_1 and p_1 in a basis determined by the possible path BS and orientation of the s_1 PBS analyzers. Finally, as in Fig. 1(b), one can restore entanglement by measuring s_2.

after some algebra, Eq. (9) can be rewritten as $E_{p\{U\}}=e(\alpha^2)+e(a^2)-e(p_0)$. For $\alpha^2=\beta^2=\frac{1}{2}$, AB is in mixed state $\rho_{AB}=\frac{1}{2}(|00\rangle\langle00|+|11\rangle\langle11|)$ and the entanglement of projection depends on the choice of projection basis: $E_p=0$ ($E_p=1$) for $a^2=0$ ($a^2=1/2$). Thus $E_f=0$, $E_a=1$, and E_{pf} is between 0 to 1, depending on which basis one uses to measure T. For $\alpha^2=1$, AB is in the pure state $\rho_{AB}=|00\rangle\langle00|$, and $E_p=0$ in all bases, so that $E_a=E_f=E_{pf}=0$.

Before we use these definitions on our new erasers, we need to briefly address the entanglement of a 2×4 subsystem, where subsystem B has dimension 4 instead of 2. If the ''B'' part of Ψ_{ABT} can be written just using $|0\rangle_B$ and $|1\rangle_B$, and not $|2\rangle_B$ and $|3\rangle_B$, then the AB subsystem can simply be treated like the 2×2 case above. On the other hand, if Ψ_{ABT} can be written in the form

$$|\Psi_{ABT}\rangle=\frac{1}{2}\{[|00\rangle_{AB}+|11\rangle_{AB}]|0\rangle_T+[|02\rangle_{AB}+|13\rangle_{AB}]|1\rangle_T\},$$

(10)

then no rotation of the taggant basis will change the entanglement of the two component pure states, and thus the AB mixed state is unambiguously fully entangled ($E_{pf}=E_f=E_p=E_a=1$).

Disentanglement erasers can be divided into two kinds: reversible and irreversible. Reversible erasers restore entanglement by simply undoing the tagging operation that caused the apparent disentanglement. Consider Fig. 1(a), which starts with a fully entangled pure state

$$|\Psi_{ABT}\rangle=\frac{1}{\sqrt{2}}[|00\rangle_{AB}+|11\rangle_{AB}]|0\rangle_T.$$

(11)

By design, $E_{pf}=E_f=E_a=1$. Now let A (or B) act as the controller in a c-NOT on T in what we call the *tagger* (or,

alternatively, the *diluter*, since it dilutes the entanglement of AB into the full ABT state [5]). This puts ABT into GHZ state

$$|\Psi_{ABT}\rangle=\frac{1}{\sqrt{2}}[|000\rangle_{ABT}+|111\rangle_{ABT}],$$

(12)

whose entanglement of projection's formation is zero: $E_{pf}=E_f=0$. Note that E_a is still 1, which is the best possible E_{pf} achievable after erasure.

We accomplish the erasure in Fig. 1(a) simply by passing ABT through the same c-NOT. This *untagger* acts as a *concentrator* of entanglement into AB [5]. The wave function is left in the state of Eq. (11) with $E_{pf}=E_f=1$. Entanglement has thus been restored.

On the other hand, the eraser of Fig. 1(b) is irreversible. The entangled state of Eq. (11) again goes through a tagger, producing the state of Eq. (12) with $E_{pf}=E_f=0$. But now we erase the tagging information by measuring the taggant in some basis. Unlike the reversible eraser, this can be done as a delayed choice (i.e., after the measurement of A and B). If T is measured in basis $\{U\}$ defined above, then E_{pf} is just given by $E_{p\{U\}}$ in Eq. (9). In particular, if one measures T in basis $|0\rangle_T\pm|1\rangle_T$ (so that $a^2=1/2$), then $E_{pf}=E_p=1$, and thus entanglement is fully restored.

Two optical experiments in Fig. 2 illustrate the workings of entanglement restorers. Both use two-photon states produced from a parametric down-conversion crystal, and both are feasible with current technology. But an entanglement restorer needs three separate subsystems, so we need to use more than one quantum number on each particle, namely, their spin and path (i.e., position) [15,16]. In regions where no spin-path interactions occur, the states cannot interact,

even though they are on the same photon. So the states behave as if they were spatially separated.

The reversible eraser in Fig. 2(a) uses the two photon spins as the AB subsystem and one of their paths as the taggant. We can write the initial wave function as

$$|\Psi_{t=0}^{rev.}\rangle = \frac{1}{\sqrt{2}}[|hv\rangle_{s_1 s_2} - |vh\rangle_{s_1 s_2}]|0\rangle_{p_1}, \qquad (13)$$

which can be written as (11). Thus the s_1-s_2 subsystem of Eq. (13) has $E_{pf} = E_f = 1$. By passing photon 1 through a polarizing beam splitter (PBS), we create a spin-path interaction which is equivalent to a c-NOT on its path, giving

$$|\Psi_{t=1}^{rev.}\rangle = \frac{1}{\sqrt{2}}[|hv\rangle_{s_1 s_2}|0\rangle_{p_1} - |vh\rangle_{s_1 s_2}|1\rangle_{p_1}], \qquad (14)$$

which is the same as the tagged state in Eq. (12). Thus $E_{pf} = E_f = 0$. What this means is that if one were to measure the spins of the photons at this point (summing over paths 0 and 1), one would obtain a mixed state with $\rho = (|hv\rangle\langle hv| + |vh\rangle\langle vh|)/2$, which could just as well have been formed from states that were never entangled. And while Eq. (14) is technically a GHZ state, one cannot use it to perform an unambiguous test of nonlocality because there are only two distinct locations for the three states [15]. Still, any local effect mimicking GHZ correlations would involve some novel spin-position interaction and so an experimental test seems worthwhile.

To reversibly erase the tagging information at $t=2$, we simply perform the reverse of the operation of $t=1$. This PBS evolves Ψ back to Eq. (13), and thus entanglement is restored: $E_{pf} = E_f = 1$. Note that we could have instead constructed an irreversible s_1-s_2 eraser by removing the second PBS and measuring $p1$ in the $|0\rangle \pm |1\rangle$ basis. But this could not be done as a "delayed choice," since s_1 and p_1 are properties of the same photon.

The irreversible eraser in Fig. 2(b) treats the spin and path of photon 1 as its subsystem AB, and the spin of the other photon as the taggant. This allows us to restore the entanglement of AB *after* the properties of A and B have been measured. Since we start out with the wave function of Eq. (13), we need to create spin-path entanglement via s_1-p_1 interactions. First we pass photon 1 through a PBS oriented in the h/v direction to obtain

$$|\Psi_{t=1}^{irrev.}\rangle = \frac{1}{\sqrt{2}}[|h0\rangle_{s_1 p_1}|v\rangle_{s_2} - |v2\rangle_{s_1 p_1}|h\rangle_{s_2}]. \qquad (15)$$

This can be written as the tagged state $[|00\rangle_{AB}|0\rangle_T + |12\rangle_{AB}|1\rangle_T]/\sqrt{2}$, which is of the same form as Eq. (12).

Here we have made B a two-qubit subsystem since it encompasses four separate paths. So the operation at $t=1$ is a c-NOT on the first qubit of B by A. The reason we put ABT in a tagged state first is that the only ABT interaction that takes place in this eraser occurs in the down-conversion crystal at $t=0$. Thus we need to preserve the taggant connection to AB even in the fully entangled state.

To create the s_1-p_1 entangled state, we pass photon 1 through a pair of PBS's in the $\bar{h}/\bar{v} = (h+v)/(-h+v)$ direction, which act as a c-NOT on the second qubit of B by A (in the $\bar{0}/\bar{1}$ basis),

$$|\Psi_{t=2}^{irrev.}\rangle = \frac{1}{2}\{[|\bar{h}0\rangle_{s_1 p_1} - |\bar{v}1\rangle_{s_1 p_1}]|v\rangle_{s_2}$$
$$- [|\bar{h}2\rangle_{s_1 p_1} + |\bar{v}3\rangle_{s_1 p_1}]|h\rangle_{s_2}\}. \qquad (16)$$

This can be written as the 2×4 system in Eq. (10). As we stated before, no rotation of T for such a 2×4 system will change E_p from 1, so that $E_{pf} = E_f = 1$. The s_1-p_1 subsystem is fully entangled despite its connection to s_2.

To make $E_{pf} = 0$, we simply reverse the last step to obtain the tagged state of Eq. (15) again. Finally, we erase the tagging information by measuring the taggant s_2 in some basis. If we measure s_2 in the h/v basis, the AB subsystem is left in a mixed state $\rho_{AB} = (|00\rangle\langle 00| + |12\rangle\langle 12|)/2$, and $E_{pf} = E_p = 0$. But if we measure s_2 in the $\bar{h}/\bar{v}$ basis, AB is left in the mixed state $\rho_{AB} = ([|00\rangle + |12\rangle][\langle 00| + \langle 12|] + [|00\rangle - |12\rangle][\langle 00| - \langle 12|])/4$, whose component pure states each are fully entangled. Thus $E_{pf} = E_p = 1$ and we can restore s_1-p_1 entanglement even after photon 1 has been measured.

The entanglement of projection provides a new framework for quantifying the entanglement of mixed states by thinking of them as higher dimensional pure states. It allows us to describe a new class of quantum erasers, called entanglement restorers, which can be thought of as simple quantum computer components. They show how c-NOT operations can shift entanglement from one part of the computer to another. It is possible that understanding how entanglement changes in a quantum computer will aid in pinpointing the source of their exponential speedup over classical computers.

Recently there has been considerable progress in manipulating three- and four-photon states [17], although as of yet it has not been possible to implement a c-NOT on two photons. Once this technological hurdle has been cleared, it will be possible to construct three-particle disentanglement erasers. Until that time, the two-photon experiments described above should be used to test most of their interesting features.

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