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Relativistic dynamics of charges in electromagnetic fields: An eigenspinor approach

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(Received 30 October 1998)

Analytical solutions and fresh insights for the relativistic dynamics of charges in classical electromagnetic fields are made possible by an eigenspinor approach. The Lorentz-force equation takes a simple spinorial form when expressed in terms of an amplitude of the Lorentz transformation that describes the motion of the charge in Clifford's geometric algebra of physical space. Algebraic projectors allow explicit analytical solutions to be found for charges in arbitrary initial motion interacting with monochromatic plane waves, with directed plane-wave pulses, and with pulses superimposed on static axial fields. The treatment tests the classical effective mass of a dressed charge and leads to a refinement of the concept of ponderomotive momentum. Implications for a pulsed autoresonance laser accelerator are briefly discussed. [S1050-2947(99)07308-4]

PACS number(s): 12.20.Ds, 03.30.+p, 03.50.De, 41.75.Lx

I. INTRODUCTION

Matrix and Hamilton-Jacobi solutions for the motion of point charges in a linearly polarized monochromatic electromagnetic plane wave have been known for 50 years [1–3]. The advent of lasers and the possibility of particle acceleration in intense laser fields has renewed interest in such solutions, and extensions to more general electromagnetic fields [4] such as exponential pulses [5] and monochromatic plane waves of arbitrary polarization [6] have been published. Most of the known solutions are for charges initially at rest or with vanishing average velocity, and many are indirect or restricted to interactions approximately described by a ponderomotive potential. A more direct derivation of the motion of charges in monochromatic plane waves was given by Hestenes [7], using the Clifford algebra of Minkowski spacetime [8]. Although all of these treatments are classical and neglect radiation reaction, they give the behavior of quantum results at high field intensities [3] and are of interest for possible applications in high-energy particle accelerators, both for beam injectors and for high-gradient acceleration stages [9–12], for possible astrophysical mechanisms of particle acceleration, and for the motion of electrons photoionized in strong laser fields.

Solutions have established that in spite of the large electric fields present in intense laser beams, the net energy that can be transferred to a charge in a long sinusoidal beam is negligible [13–15]. However, it has recently been demonstrated theoretically [16] and experimentally [17] that significant energy can be transferred by short laser pulses of width on the order of a cycle or less. Other solutions have suggested accelerators driven by rectified laser pulses [11], looked at the acceleration of electrons driven outside the spot radius of a Gaussian laser beam [18], and demonstrated that when a circularly polarized monochromatic plane wave is superimposed on an axial magnetic field, a resonance can be achieved that permits large energy transfers [10,19,20].

Here, the classical eigenspinor approach [21,22], utilizing projector techniques in the paravector space of Clifford's geometric algebra of physical space [23–25], is used to study solutions to the Lorentz-force equation. As in Taub's solution [1], we use the Lorentz transformation between the laboratory frame and the inertial frame instantaneously comoving with the charge to describe the motion. However, unlike Taub, we follow Hestenes [7] in avoiding explicit matrices and in using the spinorial form of the Lorentz transformation that arises in Clifford-algebra treatments. In the spinorial form, Lorentz transformations are bilinear in the transformation elements. These elements, while closely associated with relativistic quantum amplitudes [21,26], are in fact a purely classical construction. Their use constitutes a classical approach that is much closer than traditional trajectory-based treatments to quantum formalism. Our approach differs from Hestenes' in its use of the covariant paravector algebra of physical space ($C\ell_3$) rather than the larger space-time algebra ($C\ell_{1,3}$).

The study is restricted to the interaction of isolated charges with electromagnetic fields. We justify the neglect of radiation reaction for most cases considered by calculations of the radiated power. The algebraic approach provides geometrical insight into previously known solutions and, in particular, explains the invariance of the space-time propagation vector of the plane wave in the comoving inertial frame of the accelerated charge. It also extends previously known explicit solutions to the dynamics of charges in arbitrary initial motion in plane-wave pulses superimposed on a constant axial electric or magnetic field. These analytical results help explain numerical and experimental results, including energy transfer from finite beams, and shed light on (and demonstrate limitations of) the approximations of the ponderomotive energy and the mass shift for electron motion in a laser beam. They also contribute to the analysis of the autoresonance laser accelerator (ALA) scheme [10] for accelerating charges by a pulsed circularly polarized laser beam superimposed on an axial static magnetic field.

We begin with a brief review of the paravector approach in the Clifford algebra of Euclidean space and its use in a covariant treatment of relativity. The role of bivectors as generators of rotations in Euclidean space is emphasized and generalized to that of biparavectors as generators of spacetime rotations (Lorentz transformations). Next, the classical eigenspinor is introduced, and the spinorial form of the Lorentz-force equation is shown to relate the coupling with the electromagnetic field to a spacetime rotation rate. An invariance of null-plane rotations is shown to lead to the conservation of the spacetime propagation vector in the frame of a charge accelerated by a directed plane wave, and this is used to solve the Lorentz-force equation for such charges. The ponderomotive momentum, its relativistic corrections and limitations, and the effective mass of a dressed charge are briefly addressed. Finally, analytical solutions are derived for the relativistic motion of charges in plane-wave pulses and in such pulses superimposed on axial fields.

II. THE APPROACH

A. Clifford's geometric algebra

Our approach is based on an algebra of vectors, developed by Clifford as an extension of Grassmann's exterior forms and Hamilton's quaternions [24,27]. Since a thorough introduction is available elsewhere [25], only a brief summary is given here. The algebra assumes an associative product of vectors, distributive over addition, that satisfies the following *fundamental axiom:* the square of any vector $\mathbf{v}$ is equal to its length squared:

$$\mathbf{v}^2=\mathbf{v}\cdot\mathbf{v}. \tag{1}$$

(The existence of such a quadratic form is assumed.) If $\mathbf{v}$ is written as the sum $\mathbf{v}=\mathbf{u}+\mathbf{w}$ of vectors $\mathbf{u}$ and $\mathbf{w}$, the axiom implies

$$\mathbf{uw}+\mathbf{wu}=2\mathbf{u}\cdot\mathbf{w}. \tag{2}$$

In particular, orthonormal basis vectors of a Euclidean basis obey

$$\mathbf{e}_j\mathbf{e}_k+\mathbf{e}_k\mathbf{e}_j=2\delta_{jk}. \tag{3}$$

For example, $\mathbf{e}_1^2=\mathbf{e}_2^2=1$ and $\mathbf{e}_1\mathbf{e}_2=-\mathbf{e}_2\mathbf{e}_1$. The vectors and all their products are elements of the algebra. It is clear from Eq. (3) that elements of the algebra generally do *not* commute. In fact, two vectors commute only if they are aligned. If they are perpendicular, they anticommute.

One may think of the elements as matrices and products as matrix products. There are many possible matrix representations of the algebra. The actual representation is immaterial; only the *algebra* of the representation is important. The Clifford algebra of n-dimensional Euclidean space $\mathbb{E}^n$ is denoted by $C\ell_n$.

B. Bivectors and rotations

Vectors are easier to picture than tensor elements, but the vector analysis common in physics uses vector (cross) products that are not useful in more than three dimensions. *Bivectors*, namely, the nonscalar part of the algebraic product of two vectors, provide an extension of the cross product to spaces of any finite dimension. (See Ref. [28] for an elementary discussion of the vector concept and Clifford algebra.) Clifford algebras also extend complex analysis to more than two dimensions.

An important application of bivectors is to rotations in an arbitrary plane in n-dimensional space. Let $\mathbf{e}_1,\mathbf{e}_2$ be orthonormal basis vectors in the plane. Then $\mathbf{e}_1\mathbf{e}_2$ is a bivector of the plane. It may be considered an operator that rotates any vector $\mathbf{v}$ in the plane by 90°, since by the basic relation (3),

$$\mathbf{e}_1\mathbf{e}_2\mathbf{v}=\mathbf{e}_1\mathbf{e}_2(v^1\mathbf{e}_1+v^2\mathbf{e}_2)=-v^1\mathbf{e}_2+v^2\mathbf{e}_1. \tag{4}$$

Rotations in the plane by an arbitrary angle θ are obtained by a linear combination of $\mathbf{e}_1\mathbf{e}_2$ and the unit operator 1:

$$(\cos\theta+\mathbf{e}_1\mathbf{e}_2\sin\theta)\mathbf{v}=\exp(\mathbf{e}_1\mathbf{e}_2\theta)\mathbf{v}, \tag{5}$$

where the exponential expression follows in the vector algebra from the property $(\mathbf{e}_1\mathbf{e}_2)^2=-1$. To rotate a vector $\mathbf{u}$ with components out of the $\mathbf{e}_1\mathbf{e}_2$ plane, one can use the fact that $\mathbf{e}_1\mathbf{e}_2$ anticommutes with vector components in the plane but commutes with components perpendicular to it. These properties lead to the so-called *spinorial form* of rotations:

$$\mathbf{u}\to R\mathbf{u}R^{-1},\quad R=\exp(-\mathbf{e}_1\mathbf{e}_2\theta/2). \tag{6}$$

This is the form for operator transformations in quantum theory. The similarity to quantum formalism appears to be more than coincidence [21,24]. It is made stronger by the wave-functionlike eigenspinor introduced below.

C. Higher-order products

Products of three or more vectors can also be important. Thanks to axiom (1), the total number of linearly independent elements in the algebra is finite. In an n-dimensional space, the most general algebraic element is a linear combination of 1 scalar, n vectors, $n(n-1)/2!$ bivectors, $n(n-1)(n-2)/3!$ trivectors, etc., up to the *volume element* of the space, $\mathbf{e}_T=\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$. The algebra $C\ell_n$ as a vector space is thus spanned by a *total* of 2^n basis elements. Many significant subspaces exist.

One defines the *Clifford dual* of any element x by

$$*x=x\mathbf{e}_T^{-1}. \tag{7}$$

It generalizes the Hodge dual that is generally defined only if x is a homogeneous k-vector and is then $\pm$ the Clifford dual. In $C\ell_3$, $\mathbf{e}_T=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ commutes with all elements and squares to -1; it may be identified with the unit imaginary i. With its help, rotation elements (6) in $\mathbb{E}^3$ can be expressed in terms of the axis of rotation

$$R=\exp(-\mathbf{e}_1\mathbf{e}_2\theta/2)=\exp(-i\mathbf{e}_3\theta/2). \tag{8}$$

D. Paravector space as spacetime

Paravectors are scalars plus vectors, where ''plus'' means addition as in the sum of real and imaginary numbers or of perpendicular vector components. Paravectors of an n-dimensional Euclidean space $\mathbb{E}^n$ are elements of an $(n+1)$-dimensional linear ''vector'' space. Thus if $\mathbf{p}$ is a vector in $\mathbb{E}^n$,

$$p=p^0+\mathbf{p}=p^\mu\mathbf{e}_\mu, \tag{9}$$

where p^0 is a scalar, is a *paravector*. The second form emphasizes its role as an $(n+1)$-dimensional vector, where repeated Greek letters are summed from 0 to n and $\mathbf{e}_0=1$. What makes paravector space interesting is its metric. We look for a scalar-valued quadratic form on paravector space. The square of a paravector can have both scalar and vector parts and is therefore not a candidate. We need a *conjugate* $\bar{p}$ of p such that $p\bar{p}$ is a scalar. The obvious choice is the *Clifford conjugate*

$$\bar{p}=p^0-\mathbf{p}, \tag{10}$$

with which the scalar (S) and vector (V) parts of a paravector can be isolated:

$$\langle p\rangle_S=\tfrac{1}{2}(p+\bar{p}), \quad \langle p\rangle_V=\tfrac{1}{2}(p-\bar{p}). \tag{11}$$

The quadratic form is then

$$p\bar{p}=p^\mu p^\nu \mathbf{e}_\mu \bar{\mathbf{e}}_\nu=p^\mu p^\nu \eta_{\mu\nu}, \tag{12}$$

where the *metric tensor of the paravector space* $\eta_{\mu\nu} \equiv\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_S$ has the *Minkowski* form

$$\eta_{\mu\nu}=\begin{cases} 1, & \mu=\nu=0 \\ -1, & \mu=\nu=1,\ldots,n \\ 0, & \mu\neq\nu. \end{cases} \tag{13}$$

Thus, *the paravectors of three-dimensional Euclidean space are four-dimensional vectors in a Minkowski space time.* Clifford conjugation is extended to general elements by the rule $\overline{qp}=\bar{p}\bar{q}$. The symmetric scalar product that follows from the quadratic form of $p+q$ is

$$\langle p\bar{q}\rangle_S=\tfrac{1}{2}(p\bar{q}+q\bar{p})=p^\mu q^\nu \eta_{\mu\nu}. \tag{14}$$

Another useful conjugation is *reversal*, which reverses the order of vector products and can be identified with Hermitian conjugation if we assume that the basis vectors are all real (equal to their Hermitian conjugates). The reversal (Hermitian conjugate) of pq is $(pq)^\dagger=q^\dagger p^\dagger$. The real ($\mathfrak{R}$) and imaginary ($\mathfrak{I}$) parts of an arbitrary element x are

$$\langle x\rangle_{\mathfrak{R}}=\tfrac{1}{2}(x+x^\dagger), \quad \langle x\rangle_{\mathfrak{I}}=\tfrac{1}{2}(x-x^\dagger). \tag{15}$$

Since $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3=-\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1=-(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^\dagger$, the volume element in $C\ell_3$ is purely imaginary.

The appearance of the Minkowski spacetime metric for the paravector space of $C\ell_3$ suggests the use of real paravectors to represent vectors in four-dimensional spacetime. The scalar parts of such paravectors are the time components of spacetime vectors. Examples include (i) dimensionless proper velocity $u=\gamma+\mathbf{u}=\gamma(1+\mathbf{v}/c)$, with $u\bar{u}=1$, (ii) vector potential $A=\phi/c+\mathbf{A}$, (iii) charge current $j=\rho c+\mathbf{j}$, and (iv) gradient operator $\partial=c^{-1}\partial/dt-\nabla$.

Linear transformations that leave the quadratic form $p\bar{p}$ invariant are homogeneous *Lorentz transformations.* Physical (restricted) Lorentz transformations are rotations in spacetime planes. They comprise rotations, boosts, and their products and can be written in a form analogous to rotations (6) in $\mathbb{E}^3$. Any spacetime vector p transforms as

$$p\rightarrow LpL^\dagger, \quad L=e^{\mathbf{W}/2}=\exp\left(\frac{1}{4}W^{\mu\nu}\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V\right). \tag{16}$$

The six basis *biparavectors* $\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V$ are generators of spacetime rotations and span the linear space of spacetime bivectors. They come in commuting pairs, such as $\mathbf{e}_3\bar{\mathbf{e}}_0, \mathbf{e}_1\bar{\mathbf{e}}_2$, representing *orthogonal* space-time planes, one timelike ($\mathbf{e}_3\bar{\mathbf{e}}_0$) and one spacelike ($\mathbf{e}_1\bar{\mathbf{e}}_2$). The two generators of any pair are duals of each other and anticommute with the other four unit biparavectors. Spacelike unit biparavectors square to -1, are purely imaginary, and generate spatial rotations as seen above. Timelike ones square to $+1$, are purely real, and generate *boosts* (rotations in hyperbolic planes). As we see below, linear combinations of timelike and spacelike biparavectors can be found that are *null.* All unit biparavectors are unitary and change sign under Clifford (bar) conjugation. They also obey characteristic commutation relations, and the symmetry group that relates them to each other is the direct product of SU(2) (spatial rotations) and U(1) (duality rotations).

Simple Lorentz transformations act in a single space-time plane. They mix paravector components in that plane but leave all paravectors in the orthogonal plane invariant. The biparavectors both of the plane in which the transformation acts and of its dual are invariant.

E. The electromagnetic field

The electromagnetic field (the *Faraday* [29]) is the *biparavector* [spacetime plane(s)],

$$\mathbf{F}=c\langle \partial\bar{A}\rangle_V=\tfrac{1}{2}c(\partial^\mu A^\nu-\partial^\nu A^\mu)\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V. \tag{17}$$

[*Sytème International* (SI) units are used. We can generally avoid the use of tensor components. The component expansions are given only for comparison to other treatments.] Expression (17) is covariant, but we may wish to expand $\mathbf{F}$ into the electric and magnetic field in a particular frame: $\mathbf{F}=\mathbf{E}+ic\mathbf{B}$, where $i=\mathbf{e}_T$ is the volume element in physical space.

Maxwell's equation relates the source current j to the field $\mathbf{F}$:

$$\bar{\partial}\mathbf{F}=Z\bar{j}, \tag{18}$$

where $Z=\mu c=(\varepsilon c)^{-1}$ is the impedance of the medium. In vacuum, $Z_0=4\pi\times\dot{3}0\ \Omega$, $\dot{3}=2.997\,924\,58$. By isolating real and imaginary parts, the inhomogeneous and homogeneous equations are obtained, and by further breaking these down into vector and scalar parts, Maxwell's four vector equations are retrieved, all from Eq. (18). Extensive applications of the paravector algebra of Cl(3) in electrodynamics are given elsewhere [25].

III. LORENTZ-FORCE EQUATION

In $C\ell_3$, we can combine the covariance of the tensor form $\dot{p}^\mu=eF^{\mu\nu}u_\nu$ of the Lorentz-force equation with the component-free simplicity of the common vector form:

$$\dot{p}=\langle e\mathbf{F}u\rangle_{\mathfrak{R}}, \tag{19}$$

where p is the space-time momentum of charge e interacting with the electromagnetic field $\mathbf{F}$, and a dot indicates differentiation with respect to the proper time τ.

A. Eigenspinors

The motion and orientation of a particle is determined by its *eigenspinor* Λ, which is just the Lorentz transformation L of the algebra that relates the particle frame (that is, the inertial frame comoving with the charge) to the laboratory frame. Properties such as a spacetime vector q_r known in the particle frame are transformed by Λ to the laboratory frame: $q=\Lambda q_r \Lambda^\dagger$. In particular, the time axis $\mathbf{e}_0=1$ in the particle frame is transformed to the proper velocity (in units of c) of the particle in the laboratory frame:

$$u=\Lambda\Lambda^\dagger. \tag{20}$$

More generally, elements of the rest-frame tetrad $\{\mathbf{e}_\mu\}$ are transformed to the particle paravectors

$$\mathbf{u}_\mu=\Lambda\mathbf{e}_\mu\Lambda^\dagger \tag{21}$$

in the laboratory frame, with $\mathbf{u}_0\equiv u$.

As the amplitude of a particular Lorentz transformation, the eigenspinor $\Lambda\in \mathrm{SL}(2,\mathbb{C})$ characterizes the motion of a particle in the laboratory. ''Eigen'' refers to the particle's *own* (proper) Lorentz transformation, and ''spinor'' indicates its behavior under a further Lorentz transformation L:

$$\Lambda\to L\Lambda. \tag{22}$$

In particular, the eigenspinor changes sign under any 360° rotation. The proper time rate of change of the eigenspinor can always be written

$$\dot{\Lambda}=\frac{1}{2}\mathbf{\Omega}\Lambda, \tag{23}$$

where $\mathbf{\Omega}\equiv 2\dot{\Lambda}\bar{\Lambda}$ is a biparavector identified with the *Darboux bivector* in spacetime [30]. Physically, $\mathbf{\Omega}$ is the spacetime rotation rate of the particle frame. If Eq. (23) can be solved, the time evolution of any space-time vector or bivector that is fixed in the particle frame can be found. For example, the momentum p has the fixed value mc in the rest frame, and in the laboratory,

$$\dot{p}=\dot{\Lambda}mc\Lambda^\dagger+\Lambda mc\dot{\Lambda}^\dagger=\tfrac{1}{2}(\mathbf{\Omega}p+p\mathbf{\Omega}^\dagger)=\langle\mathbf{\Omega}p\rangle_{\Re}. \tag{24}$$

B. Spinorial form of Lorentz-force equation

A comparison of Eq. (24) with the Lorentz-force equation (19) shows that the spacetime rotation rate $\mathbf{\Omega}$ of a charged particle may be identified with the electromagnetic field $\mathbf{F}$ at the position of the charge

$$\mathbf{\Omega}=\frac{e}{mc}\mathbf{F}, \tag{25}$$

and that the Lorentz-force equation itself follows from the *spinorial* form

$$\dot{\Lambda}=\frac{e}{2mc}\mathbf{F}\Lambda. \tag{26}$$

We look for solutions $\Lambda(\tau)$ in external fields $\mathbf{F}$. Contributions to $\mathbf{F}$ from the charge itself are omitted, and thus radiation reaction is neglected. Note that Λ gives orientation (spin) information that is lost when the proper velocity or momentum is calculated.

According to the form $\mathbf{\Omega}\to L\mathbf{\Omega}\bar{L}$ of the Lorentz transformation for spacetime bivectors, $\mathbf{\Omega}^2$ and $\mathbf{F}^2=(\mathbf{E}^2-c^2\mathbf{B}^2)+2ic\mathbf{E}\cdot\mathbf{B}$ are Lorentz invariants. If $\mathbf{E}\cdot\mathbf{B}=0$, $\mathbf{F}$ is *simple* and the spinorial Lorentz-force equation (26) has a straightforward geometrical interpretation: the electromagnetic field $\mathbf{F}$ induces a *rotation at the rate* $\mathbf{\Omega}$ *in the space-time plane of* $\mathbf{F}$. In any given frame, electric fields induce boosts, that is, rotations in timelike (hyperbolic) planes, and magnetic fields induce rotations in spacelike (elliptic) planes.

IV. DIRECTED PLANE WAVES

The paravector potential of a *directed plane wave* (not generally monochromatic) can be expressed as a function $A(s)$ that depends on the spacetime position x only through the Lorentz scalar

$$s=\langle k\bar{x}\rangle_S=\omega t-\mathbf{k}\cdot\mathbf{x}=\omega_r\tau, \tag{27}$$

where $k=\omega/c+\mathbf{k}=(\omega/c)(1+\hat{\mathbf{k}})$ is a constant null (lightlike) paravector whose vector part gives the propagation direction of the plane wave, and

$$\omega_r=c\langle k\bar{u}\rangle_S=\gamma\omega(1-\mathbf{v}\cdot\hat{\mathbf{k}}/c)=\dot{s}. \tag{28}$$

For oscillating plane waves of fixed $\hat{\mathbf{k}}$, k is usually taken to be the average spacetime propagation vector, and then ω_r is the corresponding frequency in the particle frame moving with proper velocity $cu=dx/d\tau$. The field $\mathbf{F}$ has the form

$$\mathbf{F}=c\langle\partial\bar{A}\rangle_V=c\langle k\bar{A}'\rangle_V=(1+\hat{\mathbf{k}})\mathbf{E}, \tag{29}$$

where $A'(s)=dA(s)/ds$. The energy density $\mathcal{E}$ and Poynting vector $\mathbf{S}$ are given by

$$\mathcal{E}+\mathbf{S}/c=\frac{\varepsilon_0}{2}\mathbf{F}\mathbf{F}^\dagger=\varepsilon_0\mathbf{E}^2(1+\hat{\mathbf{k}}). \tag{30}$$

A number of important properties follow from the *projector* factor $P_{\hat{\mathbf{k}}}=\frac{1}{2}(1+\hat{\mathbf{k}})=P_{\hat{\mathbf{k}}}^2$ in such waves. Since $k=(2\omega/c)P_{\hat{\mathbf{k}}}$ is *null*, $\bar{k}\mathbf{F}=0$. The scalar part of this gives the orthogonality of the fields with $\mathbf{k}$, $\mathbf{k}\cdot\mathbf{F}=0$, and the vector part $\mathbf{F}=\hat{\mathbf{k}}\mathbf{F}=i\hat{\mathbf{k}}\times\mathbf{F}$ relates electric and magnetic fields in the plane wave and implies that $\{\mathbf{E},\mathbf{B},\mathbf{k}\}$ is a right-handed orthogonal vector basis of three-dimensional space. In terms of the complementary projectors $P_{\hat{\mathbf{k}}}$ and $\bar{P}_{\hat{\mathbf{k}}}=1-P_{\hat{\mathbf{k}}}$,

$$\mathbf{F}=P_{\hat{\mathbf{k}}}\mathbf{F}=\mathbf{F}\bar{P}_{\hat{\mathbf{k}}}=P_{\hat{\mathbf{k}}}\mathbf{F}\bar{P}_{\hat{\mathbf{k}}}. \tag{31}$$

It follows that $\mathbf{F}$ is a *null biparavector* and hence simple, $\mathbf{F}^2=0$, which implies that $\mathbf{E}$ and $c\mathbf{B}$ are of equal magnitude and perpendicular to each other.

The propagating plane-wave field $\mathbf{F}=(1+\hat{\mathbf{k}})\mathbf{E}$ is what Penrose and Rindler [31] call a *flag*. It is tangent to the light cone on the lightlike paravector $(1+\hat{\mathbf{k}})$, known as the *flagpole.* The flagpole is orthogonal to itself and to every paravector in the flag plane spanned by $1+\hat{\mathbf{k}}$ and $\mathbf{E}$. The flag $\mathbf{F}=(1+\hat{\mathbf{k}})\mathbf{E}$ is rotated in the $i\mathbf{k}$ plane by multiplication by a scalar phase factor: $e^{-i\phi}\mathbf{F}=e^{-i\phi\hat{\mathbf{k}}/2}\mathbf{F}e^{i\phi\hat{\mathbf{k}}/2}$. Every vector in the dual plane $-i\mathbf{F}$, spanned by $1+\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}\times\mathbf{E}$, is orthogonal to the flag $\mathbf{F}$ and is invariant under rotations in the flag plane. In particular, the flagpole itself is unchanged by rotations in the flag plane. A further important property is that if $\mathbf{a}$ is any vector perpendicular to $\hat{\mathbf{k}}$,

$$\mathbf{F}\mathbf{a}=P_{\hat{\mathbf{k}}}\mathbf{F}\mathbf{a}P_{\hat{\mathbf{k}}}=\mathbf{F}\cdot\mathbf{a}(1+\hat{\mathbf{k}}). \tag{32}$$

Examples of directed plane waves include monochromatic circularly polarized waves with *real* electric fields $\mathbf{E}(s)=\mathbf{E}(0)\exp(\pm is\hat{\mathbf{k}})$ and linearly polarized Gaussian pulses with $\mathbf{E}(s)=\mathbf{E}(0)\exp(-\frac{1}{2}s^2/\sigma^2)\cos s$. Note that directed plane waves are more general than monochromatic plane waves. They may be any linear combination of monochromatic waves that share the same propagation direction $\hat{\mathbf{k}}$.

A. Charge motion in a plane wave

Although the Lorentz-force equation (26) is easily solved for $\mathbf{F}=\text{const.}$, solutions appear hopeless for fields $\mathbf{F}(s)$ that depend on the position of the charge and hence on the solution itself. Nevertheless, solutions for directed plane waves can be found by virtue of a surprising invariance, noted by Hestenes for monochromatic waves [30].

We assume the Lorenz-gauge condition

$$\langle\partial\bar{A}\rangle_S=\langle k\bar{A}'\rangle_S=0. \tag{33}$$

Then $\mathbf{F}=ck\bar{A}'(s)$ and $k\bar{A}'=-A'\bar{k}$. The relation $\bar{k}\mathbf{F}=0$ for directed plane waves, together with the spinorial Lorentz-force equation (26) imply

$$\bar{k}\dot{\Lambda}=0=\dot{\Lambda}^\dagger\bar{k}. \tag{34}$$

It follows that the propagation paravector k_r (both the frequency ω_r and the direction $\hat{\mathbf{k}}_r$) in the instantaneous rest frame of the accelerating charge is *constant*:

$$\bar{k}_r=\Lambda^\dagger\bar{k}\Lambda, \tag{35}$$

$$\frac{d}{d\tau}\bar{k}_r=\dot{\Lambda}^\dagger\bar{k}\Lambda+\Lambda^\dagger\bar{k}\dot{\Lambda}=0. \tag{36}$$

This seems counterintuitive because first-order Doppler shifts cannot be avoided. However, the acceleration of the charge is so contrived to make the *total* Doppler shift vanish. The result is understood in terms of our geometrical interpretation of the Lorentz-force equation: the field $\mathbf{F}$ induces a space-time rotation in the flag plane of $\mathbf{F}$, and the flagpole k (in any inertial frame) is invariant under such rotations.

The invariance of k_r permits solutions to the equation of motion (26). In terms of $s=\omega_r\tau$,

$$\frac{d}{ds}\Lambda=\frac{1}{\dot{s}}\frac{d}{d\tau}\Lambda=\frac{ek\bar{A}'(s)}{2m\omega_r}\Lambda(s)=-\frac{eA'(s)}{2m\omega_r}\bar{k}\Lambda, \tag{37}$$

where the gauge condition (33) was applied. Since by Eq. (34) $\bar{k}\Lambda(s)=\bar{k}\Lambda(s_0)$, Eq. (37) is immediately integrated to give the beautifully simple (but exact) solution

$$\Lambda(s)=\Lambda(s_0)-\frac{e[A(s)-A(s_0)]}{2m\omega_r}\bar{k}\Lambda(s_0). \tag{38}$$

(By superimposing solutions for various initial conditions, one can express the family of solutions in terms of an *eigenspinor field* that is closely associated with the Dirac wave function [21].)

In terms of the conjugate momentum $p+eA=mc\Lambda\Lambda^\dagger+eA$, a *lightlike* change

$$\Delta(p+eA)=\frac{k}{2m\omega_r}\langle e\Delta\bar{A}[2p(s_0)-e\Delta A]\rangle_S \tag{39}$$

is found, where $\Delta p=p(s)-p(s_0)$ and $\Delta A=A(s)-A(s_0)$. Note that the last factor $\langle\cdots\rangle_S$ is a Lorentz-invariant scalar, and from the gauge condition (33), $\Delta\bar{A}\Delta A=(\Delta\mathbf{A}\cdot\hat{\mathbf{k}})^2-(\Delta\mathbf{A})^2\equiv-\Delta\mathbf{A}_\perp^2$. For an oscillating paravector potential A, the squared dependence implies a second-harmonic contribution in the motion of the charge, and the component of Δp along $\hat{\mathbf{k}}$ implies a frequency shift in scattered radiation. The proper velocity $cu=p/m$ can be integrated to give the space-time position $x(s)$ and hence parametric equations for the trajectory. Immediate consequences are the invariance of the components of $p+eA$ along k [see also (33)],

$$\langle\Delta(p+eA)\bar{k}\rangle_S=\langle\Delta p\bar{k}\rangle_S=0, \tag{40}$$

and along any direction $\hat{\mathbf{b}}$ perpendicular to $\hat{\mathbf{k}}$:

$$\Delta(\mathbf{p}+e\mathbf{A})\cdot\hat{\mathbf{b}}=0. \tag{41}$$

The invariance of $p\bar{p}=m^2c^2$ can also be confirmed.

Remember that the plane wave A need not be monochromatic; it can be a pulse or string of pulses. The change in momentum depends only on the initial momentum and the net change in A. For a given ΔA between the beginning and end of the interaction, the variation of A during the pulse has *no* effect on the final momentum of the charge, although it can influence the particle trajectory. Laser pulses are usually represented by vector potentials that vanish before and after the pulse, and there is no transfer of energy or momentum to the charge from such pulses. However, physical pulses can create net changes in A. For example, the simple field pulse

$$\mathbf{F}(s)=(1+\hat{\mathbf{k}})\omega\mathbf{A}_0/\cosh^2 s \tag{42}$$

results from $\Delta A=-\mathbf{A}_0(1+\tanh s)\xrightarrow{s\to\infty}-2\mathbf{A}_0$. If the initial velocity lies along $\hat{\mathbf{k}}$, the proper velocity is

$$u=u_0+\mathbf{a}_0(1+\tanh s)+\frac{1}{2}\frac{kc}{\omega_r}\mathbf{a}_0^2(1+\tanh s)^2, \tag{43}$$

where $\mathbf{a}_0 = e\mathbf{A}_0/mc$ is the dimensionless vector-potential amplitude, $u_0 = \gamma_0(1+\mathbf{v}_0/c)$ is the initial proper velocity, and $kc = \omega(1+\hat{\mathbf{k}})$. The net energy gain

$$\Delta\gamma mc^2 = \langle \Delta u\rangle_S mc^2 = \frac{2\omega}{\omega_r}\mathbf{a}_0^2 mc^2 \tag{44}$$

can be large, particularly at high injection velocities where the *surfing factor* $\omega/\omega_r = \gamma_0^{-1}(1-\hat{\mathbf{k}}\cdot\mathbf{v}_0/c)^{-1}$ is large. Integration with respect to proper time gives the spacetime displacement,

$$\Delta r = u_0 c\Delta\tau + \frac{\mathbf{a}_0 c}{\omega_r}\ln(1+e^{2s}) + k\left(\frac{\mathbf{a}_0 c}{\omega_r}\right)^2 \times\left[\ln(1+e^{2s}) - \frac{1}{(1+e^{-2s})}\right]. \tag{45}$$

B. Ponderomotive potential

If ΔA oscillates rapidly about an average of zero, the momentum p oscillates rapidly as well. The *average* over such oscillations, see Eq. (38),

$$\Delta p_{av} \equiv p_{av} - p_0 = \frac{k}{2\omega_r}\mu^2 mc^2 \tag{46}$$

has been called the *ponderomotive momentum* [32], where $\mu^2 \equiv (e^2/m^2c^2)\langle(\Delta\mathbf{A}_\perp)^2\rangle_{av}$ is a dimensionless, Lorentz-invariant measure of the intensity of the electromagnetic field. The scalar part gives the average change in energy

$$\Delta E_{av} = \frac{\omega}{2\omega_r}\mu^2 mc^2 \geqslant 0, \tag{47}$$

and reduces at low initial velocities to the *ponderomotive potential* [4,33] $\frac{1}{2}\mu^2 mc^2$.

The *drift frame* is defined [6] as the frame in which $\mathbf{p}_{av} = 0$. It is the rest frame of the *dressed* electron. Let L_D be the transformation that boosts a particle from rest to the velocity v_D of the drift frame,

$$p_{av} = p_0 + \frac{k}{2\omega_r}\mu^2 mc^2 = L_D m^* c L_D^\dagger = m^* c\gamma_D(1+\mathbf{v}_D/c), \tag{48}$$

where $m^* = \sqrt{p_{av}\bar{p}_{av}}/c = m\sqrt{1+\mu^2}$ is the effective mass. The drift velocity is thus

$$\mathbf{v}_D = c\frac{\langle p_{av}\rangle_V}{\langle p_{av}\rangle_S} = c\frac{2\omega_r\gamma_0\mathbf{v}_0/c + \hat{\mathbf{k}}\omega\mu^2}{2\omega_r\gamma_0 + \omega\mu^2}, \tag{49}$$

and $\gamma_D = (2\omega_r\gamma_0 + \omega\mu^2)/(2\omega_r m^*)$. The ratio of the rest-frame frequency ω_r to the frequency ω_D in the drift frame is

$$\frac{\omega_r}{\omega_D} = \frac{\gamma_0(1-\mathbf{v}_0\cdot\hat{\mathbf{k}}/c)}{\gamma_D(1-\mathbf{v}_D\cdot\hat{\mathbf{k}}/c)} = \frac{m^*}{m}. \tag{50}$$

Note important corrections to the usual applications of the ponderomotive potential to plane waves: (a) The surfing factor ω/ω_r is missing from most treatments, which assume nonrelativistic average velocities, but it can be large for a charge moving at high velocity along the propagation direction. (The factor is included in Ref. [32], which is restricted, however, to interactions with monochromatic plane waves.) (b) The ponderomotive potential is isotropic, but the surfing factor ω/ω_r implies that ΔE_{av} depends strongly on the initial velocity of the charge relative to the propagation direction of the wave. (c) The oscillations may not average to zero, particularly for short pulses of radiation. If the pulse is a plane wave, it is safer to use the more general result (39), with terms that depend on the polarization of the wave.

The significance of the effective mass is explored further below, by determining the proper acceleration of charges dressed by plane waves in the presence of an axial electric field.

C. Axial electric field

To solve the eigenspinor equation (26) for the case of a plane wave plus a constant axial field, we employ the complementary projectors $P_{\hat{\mathbf{k}}} = (1+\hat{\mathbf{k}})/2$ and $\bar{P}_{\hat{\mathbf{k}}} = 1 - P_{\hat{\mathbf{k}}}$ to separate the equation into parts belonging to distinct ideals of the algebra. The parts can be solved independently and recombined. With this approach we find explicit solutions that have eluded more traditional methods. In particular, we know of no other method that has given analytical solutions for charge motion in plane waves plus axial electric field, and as far as we know the analytical solutions for plane waves plus axial magnetic field, which led to proposals for the autoresonance laser accelerator (ALA), have been indirect and limited to monochromatic plane waves [10,19,20].

Consider the electric field

$$\mathbf{F} = E_0\hat{\mathbf{k}} + (1+\hat{\mathbf{k}})\omega\bar{A}'(s) \tag{51}$$

of a plane wave plus a constant axial electric field of amplitude E_0. Applying the projectors $P_{\hat{\mathbf{k}}}$ and $\bar{P}_{\hat{\mathbf{k}}}$ to the Lorentz-force equation (26) and noting that $\hat{\mathbf{k}}P_{\hat{\mathbf{k}}} = P_{\hat{\mathbf{k}}}$ and $\hat{\mathbf{k}}\bar{P}_{\hat{\mathbf{k}}} = -\bar{P}_{\hat{\mathbf{k}}}$, we obtain

$$P_{\hat{\mathbf{k}}}\dot{\Lambda} = \frac{1}{2}\alpha P_{\hat{\mathbf{k}}}\Lambda - \frac{e\omega}{mc}A'(s)\bar{P}_{\hat{\mathbf{k}}}\Lambda, \tag{52}$$

$$\bar{P}_{\hat{\mathbf{k}}}\dot{\Lambda} = -\frac{1}{2}\alpha\bar{P}_{\hat{\mathbf{k}}}\Lambda, \tag{53}$$

where $\alpha = eE_0/(mc)$. The solution to Eq. (53), namely,

$$\bar{P}_{\hat{\mathbf{k}}}\Lambda(\tau) = \exp(-\alpha\tau/2)\bar{P}_{\hat{\mathbf{k}}}\Lambda(0), \tag{54}$$

allows us to relate s and τ. The frequency of the plane wave in the charge frame,

$$\omega_r(\tau) = \dot{s}(\tau) = c\langle\overline{k u(\tau)}\rangle_S = 2\omega\langle\Lambda^\dagger(\tau)\bar{P}_{\hat{\mathbf{k}}}\Lambda(\tau)\rangle_S = e^{-\alpha\tau}\omega_r(0), \tag{55}$$

is immediately integrated to give

$$s(\tau)=\frac{\omega_r(0)}{\alpha}(1-e^{-\alpha\tau}), \qquad (56)$$

where we have taken $\tau=0$ at the intersection of the particle world line with the light cone $s=0$. The Lorentz-invariant phase s of the oscillating field as sampled by the charge is thus limited to $0\leqslant s<\omega_r(0)/\alpha$ as τ increases from 0 to infinity.

Equation (52) becomes

$$P_{\hat{\mathbf{k}}}\dot{\Lambda}=\frac{1}{2}\alpha P_{\hat{\mathbf{k}}}\Lambda-e^{-\alpha\tau/2}\frac{e\omega}{mc}A'(s)\overline{P}_{\hat{\mathbf{k}}}\Lambda(0). \qquad (57)$$

To solve, we put

$$\Lambda(\tau)\equiv e^{\alpha\tau\hat{\mathbf{k}}/2}K(\tau). \qquad (58)$$

Then

$$P_{\hat{\mathbf{k}}}\dot{\Lambda}=e^{\alpha\tau/2}P_{\hat{\mathbf{k}}}[\tfrac{1}{2}\alpha K(\tau)+\dot{K}(\tau)], \qquad (59)$$

and by comparison with Eq. (57),

$$P_{\hat{\mathbf{k}}}\dot{K}(\tau)=-e^{-\alpha\tau}\frac{e\omega}{mc}A'(s)\overline{P}_{\hat{\mathbf{k}}}K(0). \qquad (60)$$

Since $ds=\omega_r(0)e^{-\alpha\tau}d\tau$, integration gives

$$P_{\hat{\mathbf{k}}}K(\tau)=P_{\hat{\mathbf{k}}}K(0)-\frac{a\omega}{\omega_r(0)}\overline{P}_{\hat{\mathbf{k}}}K(0), \qquad (61)$$

where $a\equiv e\Delta A/(mc)$. Adding $P_{\hat{\mathbf{k}}}\Lambda+\overline{P}_{\hat{\mathbf{k}}}\Lambda$ and applying the gauge condition (33), we find

$$\Lambda(\tau)=\left\{e^{-\alpha\tau/2}\overline{P}_{\hat{\mathbf{k}}}+e^{\alpha\tau/2}P_{\hat{\mathbf{k}}}\left[1+\frac{\bar{a}\omega}{\omega_r(0)}\right]\right\}\Lambda(0)$$

$$=e^{\alpha\tau\hat{\mathbf{k}}/2}\left[1+P_{\hat{\mathbf{k}}}\frac{\bar{a}\omega}{\omega_r(0)}\right]\Lambda(0). \qquad (62)$$

When the plane-wave field vanishes, $a=0$, and the solution reduces to the well-known case of hyperbolic motion. On the other hand, if the constant electric field vanishes, then $\alpha=0$ and solution (38) is regained. More generally, the momentum is

$$p(\tau)=mc\Lambda\Lambda^{\dagger}$$

$$=e^{\alpha\tau\hat{\mathbf{k}}/2}\left[p(0)+2\left\langle P_{\hat{\mathbf{k}}}\frac{\bar{a}\omega}{\omega_r(0)}p(0)\right\rangle_{\Re}\right.$$

$$\left.-mc\frac{a\bar{a}\omega}{\omega_r(0)}P_{\hat{\mathbf{k}}}\right]e^{\alpha\tau\hat{\mathbf{k}}/2}. \qquad (63)$$

It may be verified that $p(\tau)\overline{p(\tau)}=m^2c^2$.

If the plane wave oscillates rapidly compared to the acceleration by the axial field, $\omega_r(0)\gg\alpha$, we can average over the oscillations to obtain

$$p_{av}(\tau)=e^{\alpha\tau\hat{\mathbf{k}}/2}\left[p(0)+\mu^2mc\frac{\omega}{\omega_r(0)}P_{\hat{\mathbf{k}}}\right]e^{\alpha\tau\hat{\mathbf{k}}/2}, \qquad (64)$$

where as above, $\mu=\langle -a\bar{a}\rangle_{av}^{1/2}$. Result (64) describes the uniform acceleration of a dressed charge of initial momentum $p_{av}(0)$ and mass $m^*=\sqrt{p_{av}\overline{p}_{av}}/c=m\sqrt{1+\mu^2}$. The proper acceleration rate is not αc because τ is the proper time of the rapidly oscillating frame rather than of the drift frame of the dressed charge. The proper acceleration of the dressed charge,

$$\alpha_D c=\frac{d\tau}{d\tau_D}\alpha c=\frac{\gamma_D}{\gamma}\alpha c, \qquad (65)$$

is less than αc by the factor γ_D/γ, where

$$\gamma_D=\frac{\langle p_{av}\rangle_S}{m^*c}=\frac{m}{m^*}\left[\langle e^{\alpha\tau\hat{\mathbf{k}}}u(0)\rangle_S+\frac{1}{2}\mu^2e^{\alpha\tau}\frac{\omega}{\omega_r(0)}\right]. \qquad (66)$$

The effective mass of the dressed charge in the drift frame is defined operationally as the force exerted by the constant field E_0 divided by the proper acceleration,

$$m_D=\frac{eE_0}{\alpha_D c}=\frac{\gamma}{\gamma_D}m \qquad (67)$$

and after averaging over rapid oscillations this is identical with m^*, since

$$\gamma_{av}=\frac{\langle p\rangle_{S,av}}{mc}=\frac{m^*}{m}\gamma_D. \qquad (68)$$

Note that μ is inversely proportional to m, so that in the limit $m\to 0$, the effective mass $m^*=m\sqrt{1+\mu^2}$ approaches $e\langle -\Delta A\Delta\bar{A}\rangle_{av}^{1/2}/c$.

D. Axial magnetic field

Consider the field

$$\mathbf{F}=icB_0\hat{\mathbf{k}}+(1+\hat{\mathbf{k}})\omega\bar{A}'(s) \qquad (69)$$

of a plane wave plus an axial magnetic field. Application of $\overline{P}_{\hat{\mathbf{k}}}$ to both sides of Eq. (26) yields

$$\overline{P}_{\hat{\mathbf{k}}}\dot{\Lambda}=-\frac{i}{2}\omega_c\overline{P}_{\hat{\mathbf{k}}}\Lambda, \qquad (70)$$

where $\omega_c=eB_0/m$ is the proper cyclotron frequency. Its solution is

$$\overline{P}_{\hat{\mathbf{k}}}\Lambda(\tau)=\exp(-i\omega_c\tau/2)\overline{P}_{\hat{\mathbf{k}}}\Lambda(0), \qquad (71)$$

which we use to find

$$\dot{s}(\tau)=c\langle\Lambda^{\dagger}(\tau)\bar{k}\Lambda(\tau)\rangle_S=c\langle\Lambda^{\dagger}(0)\bar{k}\Lambda(0)\rangle_S$$

$$=c\langle\overline{k}u(0)\rangle_S=\omega_r. \qquad (72)$$

Thus $s(\tau)=s_0+\omega_r\tau$, where $s_0=s(0)$. Although the rest-frame paravector k_r is no longer invariant, the frequency ω_r is, and this suffices to determine $s(\tau)$.

Applying $P_{\hat{\mathbf{k}}}$ to the Lorentz-force equation (26) and using Eq. (71) with the gauge condition (33), we find

$$\begin{aligned} P_{\hat{\mathbf{k}}}\dot{\Lambda} &= \frac{i}{2}\omega_c P_{\hat{\mathbf{k}}}\Lambda - \omega a' \bar{P}_{\hat{\mathbf{k}}}\Lambda(\tau) \\ &= \frac{i}{2}\omega_c P_{\hat{\mathbf{k}}}\Lambda - e^{-i\omega_c\tau/2}\omega a' \bar{P}_{\hat{\mathbf{k}}}\Lambda(0), \end{aligned} \tag{73}$$

where $a'(s) = eA'(s)/(mc)$. To find $P_{\hat{\mathbf{k}}}\Lambda$, we transform to a rotating frame and define K to be the eigenspinor

$$K(s(\tau)) = \exp(-i\hat{\mathbf{k}}\omega_c\tau/2)\Lambda(\tau). \tag{74}$$

In particular, at $\tau=0$, $K(s_0)=\Lambda(0)$, and with the help of Eq. (71),

$$\bar{P}_{\hat{\mathbf{k}}}K(s) = e^{i\omega_c\tau/2}\bar{P}_{\hat{\mathbf{k}}}\Lambda(\tau) = \bar{P}_{\hat{\mathbf{k}}}K(s_0). \tag{75}$$

Applying $P_{\hat{\mathbf{k}}}$ to Λ and differentiating, we get

$$P_{\hat{\mathbf{k}}}\dot{\Lambda} = P_{\hat{\mathbf{k}}}\frac{d}{d\tau}(e^{i\omega_c\tau/2}K) = \frac{i}{2}\omega_c P_{\hat{\mathbf{k}}}\Lambda + e^{i\omega_c\tau/2}P_{\hat{\mathbf{k}}}\dot{K}.$$

Comparison with Eq. (73) gives

$$P_{\hat{\mathbf{k}}}\dot{K} = \exp(-i\omega_c\tau)\omega P_{\hat{\mathbf{k}}}\bar{a}'\Lambda(0), \tag{76}$$

and in terms of s,

$$P_{\hat{\mathbf{k}}}K'(s) = \exp[-i\omega_c(s-s_0)/\omega_r]P_{\hat{\mathbf{k}}}\frac{\omega\bar{a}'}{\omega_r}K(s_0). \tag{77}$$

Integrating Eq. (77) and adding $\bar{P}_{\hat{\mathbf{k}}}K(s) = \bar{P}_{\hat{\mathbf{k}}}K(s_0)$ we obtain

$$K(s) = (1+\boldsymbol{\Theta})K(s_0) \tag{78}$$

and thus

$$\Lambda(\tau) = e^{i\hat{\mathbf{k}}\omega_c\tau/2}(1+\boldsymbol{\Theta})\Lambda(0), \tag{79}$$

$$u(\tau) = e^{i\hat{\mathbf{k}}\omega_c\tau/2}[u(0) + 2\langle\boldsymbol{\Theta}u(0)\rangle_{\Re}]e^{-i\hat{\mathbf{k}}\omega_c\tau/2} + \boldsymbol{\Theta}u(0)\boldsymbol{\Theta}^\dagger, \tag{80}$$

where $\boldsymbol{\Theta}$ is the dimensionless null biparavector,

$$\boldsymbol{\Theta} = \frac{\omega}{\omega_r}P_{\hat{\mathbf{k}}}\int_{s_0}^{s} ds' \exp[-i\omega_c(s'-s_0)/\omega_r]\bar{a}'(s') = P_{\hat{\mathbf{k}}}\hat{\mathbf{a}}\vartheta(s), \tag{81}$$

with $\hat{\mathbf{a}}$, any real unit vector perpendicular to $\hat{\mathbf{k}}$, and ϑ, the complex scalar-valued function

$$\vartheta(s) \equiv \int_{s_0}^{s} ds' \exp[-i\omega_c(s'-s_0)/\omega_r]\frac{e\mathbf{F}(s')}{mc\omega_r}\cdot\hat{\mathbf{a}}. \tag{82}$$

Note that because of relation (72) and the projector properties of $\bar{P}_{\hat{\mathbf{k}}}$,

$$\begin{aligned} \boldsymbol{\Theta}u(0)\boldsymbol{\Theta}^\dagger &= \boldsymbol{\Theta}\bar{P}_{\hat{\mathbf{k}}}u(0)\bar{P}_{\hat{\mathbf{k}}}\boldsymbol{\Theta}^\dagger = 2\boldsymbol{\Theta}\langle\bar{P}_{\hat{\mathbf{k}}}u(0)\rangle_S\bar{P}_{\hat{\mathbf{k}}}\boldsymbol{\Theta}^\dagger \\ &= \frac{\omega_r}{\omega}\boldsymbol{\Theta}\boldsymbol{\Theta}^\dagger = \frac{\omega_r}{\omega}|\vartheta|^2 P_{\hat{\mathbf{k}}}. \end{aligned} \tag{83}$$

The energy gain of the charge is mc^2 times the change in $\gamma = \langle u\rangle_S$. From Eq. (80) and the relation $\langle\boldsymbol{\Theta}u(0)\rangle_{\Re S} = \langle\boldsymbol{\Theta}\rangle_{\Re}\cdot\mathbf{u}(0)$,

$$\Delta\gamma \equiv \gamma(\tau) - \gamma(0) = 2\langle\boldsymbol{\Theta}\rangle_{\Re}\cdot\mathbf{u}(0) + \frac{\omega_r}{2\omega}|\vartheta|^2. \tag{84}$$

While all components of u are given directly by Eq. (80), the longitudinal component of the proper velocity is also related by Eq. (72):

$$\mathbf{u}_\parallel \equiv (\mathbf{u}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} = \left(\gamma - \frac{\omega_r}{\omega}\right)\hat{\mathbf{k}}, \tag{85}$$

and the magnitude of the transverse component is given by unimodularity $u\bar{u}=1$:

$$\mathbf{u}_\perp^2(\tau) = \mathbf{u}_\perp^2(0) + 2\frac{\omega_r}{\omega}\Delta\gamma. \tag{86}$$

Large gains in energy give rise to velocities that are increasingly collimated along $\mathbf{k}$. Note that if the initial velocity is longitudinal, then $\langle\boldsymbol{\Theta}u(0)\rangle_{\Re} = (\omega_r/\omega)\langle\boldsymbol{\Theta}\rangle_{\Re}$, the scalar part of which vanishes.

Consider the case of a circularly polarized monochromatic wave:

$$a(s) = e^{i(s-s_0)\hat{\mathbf{k}}}\mathbf{a} \tag{87}$$

with $\mathbf{a} = e\mathbf{A}_\perp(s_0)/(mc)$, a constant vector perpendicular to $\mathbf{k}$. Integration gives

$$\vartheta = |\mathbf{a}|\omega\left\{\frac{1-\exp[i(\omega_r-\omega_c)\tau]}{\omega_r-\omega_c}\right\}. \tag{88}$$

If the rest-frame frequency ω_r is close to the cyclotron frequency ω_c, then

$$\vartheta \simeq -i|\mathbf{a}|\omega\tau. \tag{89}$$

More generally,

$$|\vartheta|^2 = \omega^2\mathbf{a}^2\left(\frac{\sin[(\omega_r-\omega_c)\tau/2]}{(\omega_r-\omega_c)/2}\right)^2. \tag{90}$$

There is a strong resonance in the interaction when the proper cyclotron frequency of the relativistic charge matches the Doppler-shifted frequency of the wave. When $\omega_r = \omega_c$, relation (84) reduces to

$$\Delta\gamma = \omega\tau(\hat{\mathbf{k}}\times\mathbf{a})\cdot\mathbf{u}(0) + \tfrac{1}{2}\omega\omega_r\tau^2\mathbf{a}^2. \tag{91}$$

This is the basis of the ALA. Of course, the resonance condition will be difficult to hold for more than, say, 10^4 cycles, because it is difficult to make a magnetic field B_0 more homogeneous than about a part in 10^4 over macroscopic distances. On the other hand, short-range inhomogeneities, over distances on the order of $\gamma^2 c/\omega$ at relativistic velocities, are well tolerated. Nevertheless, with $\omega_r \tau$ on the order of 10^4 and dimensionless amplitudes **a** on the order of unity, large energy gains by factors on the order of 10^8 appear plausible.

E. Plane-wave pulse

To investigate the ALA case further, consider a plane-wave pulse centered at $s=0$, with the derivative of the real paravector field $a(s)=e(mc)^{-1}\Delta A(s)$ taken to be the circularly polarized Gaussian wave packet

$$a'(s)=e^{is\hat{\mathbf{k}}-\frac{1}{2}s^2/\sigma^2}\mathbf{a}'(0), \tag{92}$$

which is related to the electric field $\mathbf{E}$ of the pulse by $a'=-e\mathbf{E}/(mc\omega)$. After the pulse has passed, Eq. (81) gives

$$\vartheta=|\mathbf{a}'(0)|e^{i\varphi}\sqrt{2\pi}\sigma\frac{\omega}{\omega_r}\exp\left[-\frac{\sigma^2}{2}\left(\frac{\omega_c}{\omega_r}-1\right)^2\right], \tag{93}$$

where the constant phase angle φ depends on the initial value s_0 (assumed $\ll -\sigma$). If $u(0)$ is longitudinal, the energy increase $mc^2\Delta\gamma$ is quadratic in the field strength:

$$\Delta\gamma=\frac{\omega}{\omega_r}[\mathbf{a}'(0)]^2\pi\sigma^2\exp\left[-\sigma^2\left(\frac{\omega_c}{\omega_r}-1\right)^2\right]. \tag{94}$$

To maximize the energy transfer, the pulse width σ can be set to $|\omega_c/\omega_r-1|^{-1}$, which can be quite large when ω_c and ω_r are well matched. The enhancement factor arising from the resonance $\omega_r \simeq \omega_c$ is limited only by the homogeneity of the large magnetic field required. In the absence of the axial magnetic field, $\omega_c=0$. The presence of the magnetic field is seen to enhance the energy gain by a factor of σ^2.

Two practical concerns for the application of such a scheme are the potential loss of energy to radiation and the required length of the accelerator.

F. Larmor power

Loeb and Friedland [10] have established that the radiated energy by the ALA is negligible. We confirm this result with simple analytical relations. The Larmor power lost to radiation can be written [25]

$$P=-\tfrac{2}{3}mcr_e\dot{u}\bar{\dot{u}}, \tag{95}$$

where $r_e=e^2/(4\pi\varepsilon_0 mc^2)\simeq 2.82\times 10^{-15}$ m is the classical electron radius. For a plane wave plus an axial magnetic field

$$\mathbf{F}=icB_0\hat{\mathbf{k}}+(1+\hat{\mathbf{k}})\mathbf{E}, \tag{96}$$

the Lorentz-force equation (19) of the charge leads to

$$P=\frac{2}{3}mcr_e\left(\omega_c\mathbf{u}\times\hat{\mathbf{k}}+\frac{\omega_r}{\omega}\frac{e}{mc}\mathbf{E}\right)^2. \tag{97}$$

The fraction of particle energy γmc^2 radiated by the cyclotron component per period $2\pi\gamma/\omega_c$ in the laboratory frame after the driving laser pulse has passed is

$$(\gamma mc^2)^{-1}P\frac{2\pi\gamma}{\omega_c}=\frac{4\pi}{3}\frac{r_e}{r_c}|\mathbf{u}_\perp|^3, \quad \mathbf{E}=0, \tag{98}$$

with $r_c=|c\mathbf{u}_\perp/\omega_c|$ the orbital radius of the cyclotron. The cyclotron is a very efficient radiator at high transverse velocities $|\mathbf{u}_\perp|\sim 10^6$. The situation at first appears bad for the ALA because the cyclotron radius is quite small:

$$r_c=\frac{c}{\omega_c}\left(2\frac{\omega_c}{\omega}\Delta\gamma\right)^{1/2}\sim 0.1 \text{ m} \tag{99}$$

for $\omega_c/\omega \sim 10^{-3}$ and 1-TeV electrons with $\mathbf{u}_\perp(0)=0$. What rescues the ALA is the high pitch of the spiral:

$$\left|\frac{\mathbf{u}_\parallel}{\mathbf{u}_\perp}\right|=\frac{\gamma_0 v_0/c+\Delta\gamma}{(2\omega_r\Delta\gamma/\omega)^{1/2}}\simeq\frac{c}{|\mathbf{v}_\perp|}\sim 3\times 10^4 \tag{100}$$

under the circumstances above. Thus, the transverse velocity component is nonrelativistic. With $\mathbf{u}_\perp^2=2\Delta\gamma\omega_r/\omega$ [see Eq. (86)] and at 1 tesla $\omega_c=1.8\times 10^{11}$ $\text{s}^{-1}\approx\omega_r$, one finds

$$(\gamma mc^2)^{-1}P\frac{2\pi\gamma}{\omega_c}\simeq 1.4\times 10^{-14}\Delta\gamma, \quad \mathbf{E}=0. \tag{101}$$

One can similarly investigate the radiation from the reaction to the laser pulse in the case of vanishing $\mathbf{B}_0$ by putting $\omega_c=0$ in Eq. (97).

Of more interest, however, is the total power loss in an ALA. In the presence of both an axial magnetic field and a laser pulse, there can be interference in P [Eq. (97)] between the cyclotron motion and the term linear in the field $\mathbf{E}$ of the laser. Consider a circularly polarized plane wave of the form (87). The proper velocity is given by Eq. (80), and in the resonant case $\omega_r=\omega_c$, for which $\vartheta=-i\omega\tau$,

$$2\langle\mathbf{\Theta}u\rangle_{\Re}=\langle i\omega\tau\mathbf{a}(1-\hat{\mathbf{k}})u\rangle_{\Re}=\langle i\omega\tau\mathbf{a}(1-\hat{\mathbf{k}})(\gamma+\mathbf{u}_\parallel+\mathbf{u}_\perp)\rangle_{\Re}. \tag{102}$$

Since

$$\omega_r(1-\hat{\mathbf{k}})=\omega(1-\hat{\mathbf{k}})(\gamma-\mathbf{u}\cdot\hat{\mathbf{k}})=\omega(1-\hat{\mathbf{k}})(\gamma+\mathbf{u}_\parallel)$$

and

$$\langle i\omega\tau\mathbf{a}(1-\hat{\mathbf{k}})\mathbf{u}_\perp\rangle_{\Re}=-\omega\tau\mathbf{a}\times\mathbf{u}_\perp(1+\hat{\mathbf{k}}),$$

Eq. (80) gives

$$\mathbf{u}_\perp(\tau)=e^{i\hat{\mathbf{k}}\omega_c\tau}[\mathbf{u}_\perp(0)+\omega_r\tau\mathbf{a}\times\hat{\mathbf{k}}]. \tag{103}$$

Substitution into Eq. (97) with $\omega_r=\omega_c$ yields the radiated power

$$P=\tfrac{2}{3}mcr_e\omega_c^2[\mathbf{u}(0)\times\hat{\mathbf{k}}-\omega_c\tau\mathbf{a}+\hat{\mathbf{k}}\times\mathbf{a}]^2. \tag{104}$$

In principle, one can choose $\mathbf{u}_\perp(0)$ to minimize energy loss and/or to maximize energy gain, but in practice it is difficult to synchronize injection with the phase of the laser wave. If instead we choose axial injection, $\mathbf{u}_\perp(0)=0$,

$$P=\tfrac{2}{3}mcr_e\omega_c^2\mathbf{a}^2(1+\omega_c^2\tau^2). \tag{105}$$

The ratio of the total energy radiated,

$$\int P\,dt=\int P\gamma\,d\tau=\int P[\gamma(0)+\Delta\gamma]d\tau, \tag{106}$$

to the energy gained $\Delta\gamma mc^2=\frac{1}{2}\omega\omega_c\tau^2\mathbf{a}^2mc^2$ is then

$$\begin{aligned}\frac{\int P\,dt}{\Delta\gamma mc^2}&=\frac{4}{3}\frac{r_e\omega_c}{c\tau^2\omega}\int(1+\omega_c^2\tau^2)\left[\gamma(0)+\frac{1}{2}\omega\omega_c\tau^2\mathbf{a}^2\right]d\tau\\&=\frac{4}{9}\frac{r_e\omega_c}{c\tau\omega}\left[(3+\omega_c^2\tau^2)\gamma(0)+\left(1+\frac{3}{5}\omega_c^2\tau^2\right)\Delta\gamma\right]\\&\sim\frac{4}{15}\frac{r_e\omega_c^3\tau}{c\omega}\Delta\gamma,\end{aligned} \tag{107}$$

where the last line is the limiting case when $\omega_c\tau\gg1$ and $\Delta\gamma\gg\gamma(0)=\frac{1}{2}(\omega/\omega_c+\omega_c/\omega)$. The restriction that this ratio be small places an upper limit on the energy gain that can be realized from an ALA for a resonance pulse of $\omega_c\tau$ radians:

$$\Delta\gamma\lesssim\frac{c\omega}{r_e\omega_c^3\tau}. \tag{108}$$

The limit can be quite large: for an electron in an axial magnetic field of 10 T ($\omega_c\simeq1.8\times10^{12}\ \mathrm{s}^{-1}$) and a pulse length of $\omega_c\tau=10^3$ from a Ti:sapphire laser ($\omega\simeq2.4\times10^{15}\ \mathrm{s}^{-1}$), it is $\Delta\gamma\lesssim10^{11}$.

G. Size of accelerator

The acceleration is limited by practical considerations of size. Particles moving at constant proper velocity u for proper time interval τ travel a longitudinal distance

$$\Delta\mathbf{x}\cdot\hat{\mathbf{k}}=c\tau\mathbf{u}\cdot\hat{\mathbf{k}}, \tag{109}$$

so that 1-TeV electrons ($\gamma\simeq2\times10^6$) stretch a 1-μs proper-time interval (5×10^5 rad of cyclotron motion in a magnetic field of 1 T) into 2 s and cover a distance of 6×10^5 km, almost a round trip to the moon and rather large for an accelerator! Of course, the electrons do not enter the accelerator with 1 TeV of energy, so the actual size requirements are less severe.

To determine the conditions more precisely, consider a square-wave pulse of a circularly polarized plane wave tuned to resonance with the cyclotron frequency in the frame of the charge moving in an axial magnetic field, with coaxial injection. Label the beginning of the pulse at the particle by proper time $\tau=0$. During the interaction of the charge in the pulse, Eqs. (80), (81), and (89) give

$$u(\tau)=u(0)+[\omega_c\tau e^{i\hat{\mathbf{k}}\omega_c\tau}\hat{\mathbf{k}}\times\mathbf{a}+\omega\omega_c\tau^2\mathbf{a}^2P_{\hat{\mathbf{k}}}] \tag{110}$$

in the resonance limit $\omega_r\to\omega_c$. Integration yields

$$\begin{aligned}\Delta x=u(0)c\tau+\frac{c}{\omega_c}\bigg[&e^{i\hat{\mathbf{k}}\omega_c\tau}(\hat{\mathbf{k}}\times\mathbf{a}-\mathbf{a}\omega_c\tau)\\&+\frac{\omega}{3\omega_c}(\omega_c\tau)^3\mathbf{a}^2P_{\hat{\mathbf{k}}}\bigg].\end{aligned} \tag{111}$$

Thus, the cyclotron radius is

$$r_c=\frac{c|\mathbf{a}|}{\omega_c}\sqrt{1+(\omega_c\tau)^2} \tag{112}$$

and the longitudinal distance traveled is

$$\Delta\mathbf{x}\cdot\hat{\mathbf{k}}=c\tau\left[\mathbf{u}(0)\cdot\hat{\mathbf{k}}+\frac{\Delta\gamma}{3}\right]. \tag{113}$$

It is advantageous to use high magnetic fields and modest pulse widths $\omega_c\tau$ in order to limit the acceleration length $\Delta\mathbf{x}\cdot\hat{\mathbf{k}}$. Of course, short pulses must be balanced by higher laser intensities. For example, with a 10-T magnetic field and a laser of wavelength 0.8 μm (Ti:sapphire) with a dimensionless vector-potential amplitude $\mathbf{a}^2=10$, one could boost 342 MeV electrons to 1 TeV in under 1.9 km with a 17-rad pulse, and to 10 TeV in 59 km with a 54-rad pulse. Although intensities of $4.3\times10^{19}\ \mathrm{W/cm^2}$ corresponding to $\mathbf{a}^2=10$ [see Eq. (30) and recall use of circular polarization] are feasible in focused beams from tabletop lasers, much more massive devices are required to produce such intensities over a spot that encompasses the cyclotron orbit of radius $r_c\simeq1$ cm at 1 TeV along the length of the accelerator. To achieve a given energy change, if the amplitude of the plane wave is scaled by a factor f, the required pulse length and hence the length of the accelerator are scaled by f^{-1}.

V. DISCUSSION

With the help of projectors in an eigenspinor approach to the classical Lorentz-force equation, we have obtained relatively simple analytical solutions for the motion of charges in some electromagnetic fields. The covariant approach, which employs Clifford's geometric algebra of physical space, is valuable in identifying and explaining conserved quantities, and although basically classical, it bears an intriguing resemblance to quantum methods. We have concentrated here on simple cases of plane-wave fields, both monochromatic and pulsed, plus constant axial electric or magnetic fields. Although we have used plane waves rather than focused Gaussian beams that would more realistically describe practical lasers, it is only important that the field at the charge be well represented by a plane wave, and the simplicity of the results may provide useful insights in situations of more complex geometry. Since the plane wave can be a pulse, the approach may even simulate qualitative results from charges entering finite laser beams. We have looked at applications to the ALA scheme, in which electrons are accelerated to TeV energies by circularly polarized lasers resonant with the cyclotron motion in a constant magnetic field. One could similarly use microwaves to accelerate ions or protons in axial magnetic fields. The results may also be useful in analyzing possible astrophysical acceleration mechanisms.

The applications made here demonstrate the potential of the eigenspinor approach as a powerful tool of analysis in problems of relativistic dynamics.

ACKNOWLEDGMENT

The authors thank the Natural Sciences and Engineering Research Council of Canada for support of this work.

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