Effect of the detector efficiency on the phase sensitivity in a Mach-Zehnder interferometer

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The effect of quantum efficiencies of detectors on the phase sensitivity of a Mach-Zehnder is analyzed for the Fock state input. The phase uncertainty of two different interference experiments, the second order and the fourth order, are compared with two practical detectors with less than unit quantum efficiencies. It is found out that in the second-order interference experiment we cannot beat the classical limit in the phase sensitivity, even with ideal detectors. We show that almost ideal detectors are required to get the Heisenberg limit in the phase sensitivity of the fourth-order interference for two Fock state light inputs. There exists an optimum angle given by the efficiency of the detectors. [S1050-2947(99)08207-4]

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A Mach-Zehnder interferometer (MZI) is one of the most widely used, exquisite apparatuses for measuring the small phase shift between the two paths. Its application ranges from small gas density changes in a combustion process to the detection of the gravitational wave. It is believed that it is possible to detect the graviton, provided that the phase sensitivity of an interferometer has the Heisenberg limit, i.e., $1/N$, where N is the total incident photon number in the measurement. At present, there are a number of experiments underway for the graviton with a Michelson-type interferometer employing a laser as an input light source. In this case, the sensitivity of the system is bounded by the classical limit, i.e., $1/\sqrt{N}$ [1,2]. Many efforts have been made to overcome the conventional limit by finding special kinds of input light for the ultimate sensitivity. Caves, for the first time, suggested a squeezed light instead of a laser light for the unused port of a Michelson interferometer [3]. Holland and Burnett came out with twin Fock state light inputs for MZI [4]. A correlated light as a possible candidate has been proposed for the quantum phase sensitivity [5,6].

A recent study showed that the phase sensitivity of a MZI can be different according to the measurement schemes of the interferometer in an ideal case, and that most measurement schemes are classified into two kinds of interference experiments, the second-order and the fourth-order experiment, in their sensitivity [7]. It is well known that imperfect detectors will degrade the sensitivity, but the details are not found yet, as far as we know. In the following, we derive the exact effects of practical detectors with less than unit quantum efficiencies on the phase sensitivity of the two kinds of measurements for the Fock state light inputs. The model of nonideal photodection has been studied in detail by Yuen and Shapiro [8,9] and Yurke [10]. They showed that for photodetection with quantum effucuency $0<\mu\leqslant 1$, the detected field mode is described by a photon annihilation operator, $\hat{a}'=\sqrt{\mu}\hat{a}+\sqrt{(1-\mu)}\hat{v}$, where $\hat{a}$ and $\hat{v}$ are the annihilation operators for the input mode and vacuum-state mode.

Let us assume that we have two photodetectors, D_1 and D_2, with quantum efficiencies of μ_1 and μ_2, respectively, as shown in Fig. 1. The annihilation operators for the detector can be described as

$$\begin{aligned}\hat{a}_5' &= \sqrt{\mu_1}\hat{a}_5+\sqrt{1-\mu_1}\hat{v}_5,\\ \hat{a}_6' &= \sqrt{\mu_2}\hat{a}_6+\sqrt{1-\mu_2}\hat{v}_6,\end{aligned} \tag{1}$$

where $\hat{a}_5,\hat{v}_5$ and $\hat{a}_6,\hat{v}_6$ represent the annihilation operators at output mode 5,6, respectively. The operators of the absorbed photon numbers by two practical detectors attached at the two output ports are given as follows:

$$\begin{aligned}\hat{n}_5' &= \hat{a}_5'^{\dagger}\hat{a}_5' = \mu_1\hat{n}_5+\sqrt{\mu_1(1-\mu_1)}\hat{U}_5+(1-\mu_1)\hat{V}_5,\\ \hat{n}_6' &= \hat{a}_6'^{\dagger}\hat{a}_6' = \mu_2\hat{n}_6+\sqrt{\mu_2(1-\mu_2)}\hat{U}_6+(1-\mu_2)\hat{V}_6,\end{aligned} \tag{2}$$

where the operators $\hat{U}_5,\hat{V}_5,\hat{U}_6,\hat{V}_6$ are defined by

$$\hat{U}_5 \equiv \hat{v}_5^{\dagger}\hat{a}_5+\hat{a}_5^{\dagger}\hat{v}_5,$$

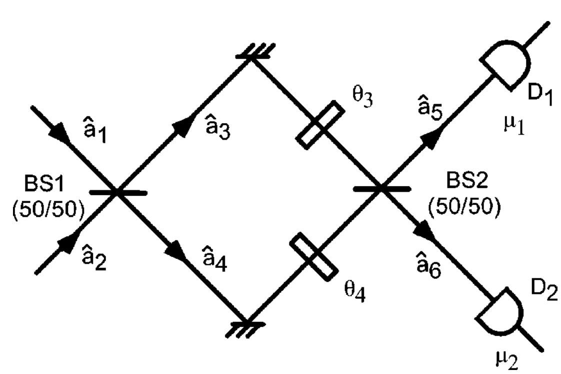

FIG. 1. A Mach-Zehnder interferometer with two detectors, D_1 and D_2, of quantum efficiencies of μ_1 and μ_2, respectively.

$$\hat{U}_6 \equiv \hat{v}_6^\dagger \hat{a}_5 + \hat{a}_6^\dagger \hat{v}_6,$$
$$\hat{V}_5 \equiv \hat{v}_5^\dagger \hat{v}_5, \tag{3}$$
$$\hat{V}_6 \equiv \hat{v}_6^\dagger \hat{v}_6.$$

The number operators at two output ports of the MZI of two lossless 50-50 beamsplitters have the relationship with annihilation operators $\hat{a}_1,\hat{a}_2$ for the two input field modes, 1,2,

$$\hat{n}_5 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) + \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1)\sin\theta - \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)\cos\theta,$$
$$\hat{n}_6 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) - \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1)\sin\theta + \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)\cos\theta, \tag{4}$$

where the creation and annihilation operators have the usual commutation relations $[\hat{a}_i,\hat{a}_j]=0$ and $[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij}$ for $(i,j)=(1,2)$ [11]. In this expression θ is the phase shift, $\theta_4 - \theta_3$, between the overall phase θ_3 in path 3, and the overall phase θ_4 in path 4 of the MZI.

Let us suppose that we are measuring the phase shift θ by the detection of photon number n_5 of output mode 5 by the detector D_1 with quantum efficiency of μ_1, which corresponds to the experiment of the second-order interference. When we perform the calculation of the operators, $\hat{v}_5$ and $\hat{v}_6$, for the vaccum states in the 5 and 6 mode, which cause the expectation value of all terms containing $\hat{V}_5,\hat{V}_6$ and the odd-order terms of $\hat{U}_5,\hat{U}_6$ to vanish, we are left with

$$\langle \hat{n}_5' \rangle = \mu_1 \langle \hat{n}_5 \rangle,$$
$$\langle \hat{n}_5'^2 \rangle = \mu_1^2 \langle \hat{n}_5^2 \rangle + \mu_1(1-\mu_1)\langle \hat{n}_5 \rangle. \tag{5}$$

These lead to a variance of photodetection at the output port 5,

$$(\Delta n_5')^2 = \langle \hat{n}_5'^2 \rangle - \langle \hat{n}_5' \rangle^2 = \mu_1^2(\langle \hat{n}_5^2 \rangle - \langle \hat{n}_5 \rangle^2) + \mu_1(1-\mu_1)\langle \hat{n}_5 \rangle. \tag{6}$$

This is merely but the Burgess variance theorem for photodetection with a detector of quantum efficiency μ_1.

If two Fock state lights of n_1 photons for input mode 1 and n_2 photons for input mode 2 are assumed to be incident on the MZI , namely $|n_1,n_2\rangle$, we can calculate the expectation values for the input state as follows:

$$\langle n_1,n_2|\hat{n}_5|n_1,n_2\rangle = \frac{1}{2}(n_1+n_2) - \frac{1}{2}(n_1-n_2)\cos\theta, \tag{7}$$

$$\langle n_1,n_2|\hat{n}_5^2|n_1,n_2\rangle = \frac{1}{4}[(n_1+n_2)^2 - 2(n_1^2-n_2^2)\cos\theta + (n_1-n_2)^2\cos^2\theta + (2n_1n_2+n_1+n_2)\sin^2\theta],$$

where we used the relation of $\hat{a}_i|n_i\rangle = \sqrt{n_i}|n_i-1\rangle$ for $i=1,2$. Then the variance of the detected photon numbers at the output mode 5 is obtained as

$$(\Delta n_5')^2 = \frac{1}{4}\mu_1^2(2n_2n_2+n_1+n_2)\sin^2\theta + \frac{1}{2}\mu_1(1-\mu_1) \times [(n_1+n_2)-(n_1-n_2)\cos\theta]. \tag{8}$$

Substituting the above equations into the expression for the square of the phase sensitivity (or phase uncertainty), $(\Delta\theta)^2$, one has

$$(\Delta\theta)^2 = \frac{(\Delta n_5')^2}{\left[\dfrac{\partial\langle \hat{n}_5' \rangle}{\partial\theta}\right]^2} = \frac{2n_1n_2+n_1+n_2}{(n_1-n_2)^2} + \frac{2(1-\mu_1)}{\mu_1}\frac{[(n_1+n_2)-(n_1-n_2)\cos\theta]}{(n_1-n_2)^2\sin\theta}, \tag{9}$$

which consists of two kinds of terms. The first term in the right side, independent of the quantum efficiency, represents the ideal case. On the other hand, the second term arises from quantum efficiency μ_1 of the detector, which goes to zero in the case of $\mu_1=1$.

Let us take two special input cases, a twin Fock state light $(n_2=n_2=n)$ and one Fock state light $(n_1=n,\ n_2=0)$. When a twin Fock state light is entering into MZI, the phase uncertainty diverges to infinity, $\Delta\theta\rightarrow\infty$, irrespective of the quantum efficiency of the detector. As we can see in Eq. (5), $\langle \hat{n}_5' \rangle$ is constant, independent of θ, which means we do not have an interference fringe in this case. For a latter case, the square of the phase uncertainty becomes

$$(\Delta\theta)^2 = \frac{1}{n} + \frac{2(1-\mu_1)}{\mu_1}\frac{1}{n}\frac{1}{1+\cos\theta}. \tag{10}$$

Figure 2 shows the dependence of the phase uncertainty, $\Delta\theta$, on the phase shift θ for a few different values of μ_1 in the case of $n=100$. The square of the phase uncertainty increases monotonically as θ goes to $\pi/2$ rad from 0 rad. No one can have a phase sensitivity greater than the classical limit in this second-order interference experiment.

Now, let us suppose that we are measuring the phase shift by the coincidence detection of two output photons, the fourth-order interference, by the two detectors of quantum efficiencies μ_1 and μ_2 as shown in Fig. 1. The straightforward calculation of the expectation values of the coincidence detection operators $\hat{N}_c' = :\hat{n}_5'\hat{n}_6':$ and $\hat{N}_c'^2$ for the vacuum states and for the general Fock state inputs $|n_1,n_2\rangle$ leads to

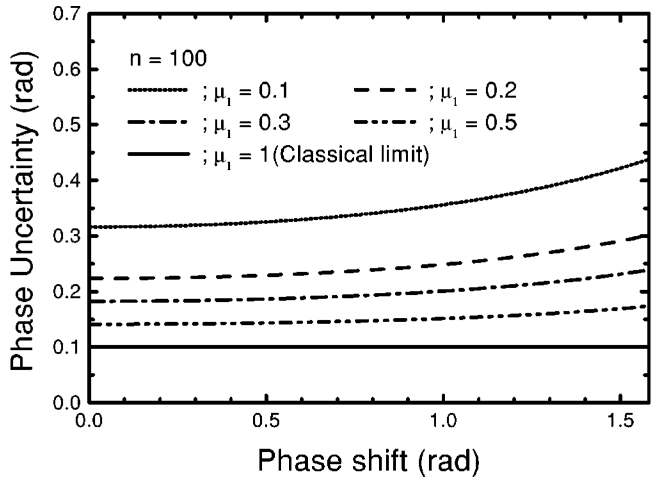

FIG. 2. The phase uncertainty for the various values of μ_1 in the second-order interference experiment with a single Fock state input ($n=100$).

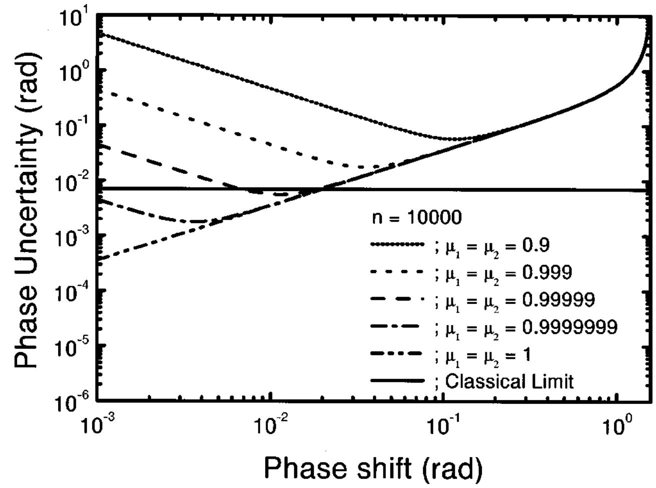

FIG. 3. The phase uncertainty for the various values of μ ($=\mu_1=\mu_2$) in the fourth-order interference experiment with a twin Fock state input ($n_1=n_2=10\,000$).

$$\langle \hat{N}_c' \rangle = \mu_1\mu_2\langle \hat{n}_5\hat{n}_6\rangle = \frac{1}{4}\mu_1\mu_2[(n_1+n_2)^2 - (2n_1n_2+n_1+n_2)\sin^2\theta - (n_1-n_2)^2\cos^2\theta] \tag{11}$$

and

$$\begin{aligned}\langle \hat{N}_c'^2 \rangle = {}& \mu_1^2\mu_2^2\langle n_1,n_2|\hat{n}_5^2\hat{n}_6^2|n_1,n_2\rangle + \mu_1\mu_2^2(1-\mu_1) \\ &\times\langle n_1,n_2|\hat{U}_5^2\hat{n}_6^2|n_1,n_2\rangle + \mu_1^2\mu_2(1-\mu_2) \\ &\times\langle n_1,n_2|\hat{n}_5^2\hat{U}_6^2|n_1,n_2\rangle + \mu_1\mu_2(1-\mu_1)(1-\mu_2) \\ &\times\langle n_1,n_2|\hat{U}_5^2\hat{U}_6^2|n_1,n_2\rangle,\end{aligned} \tag{12}$$

by the same logic for the operators $\hat{U}$'s and $\hat{V}$'s. From these expressions one can readily show the variance of the coincidence detection for the general Fock state input by the detectors as below,

$$\begin{aligned}(\Delta N_c')^2 = {}& \langle(\hat{N}_5'\hat{N}_6')^2\rangle - \langle \hat{N}_5'\hat{N}_6'\rangle^2 = \mu_1^2\mu_2^2[\langle(\hat{n}_5\hat{n}_6)^2\rangle \\ &- \langle \hat{n}_5\hat{n}_6\rangle^2] + \mu_1\mu_2^2(1-\mu_1)\langle \hat{n}_5\hat{n}_6^2\rangle + \mu_1^2\mu_2 \\ &\times(1-\mu_2)\langle \hat{n}_5^2\hat{n}_6\rangle + \mu_1\mu_2(1-\mu_1)(1-\mu_2)\langle \hat{n}_5\hat{n}_6\rangle.\end{aligned} \tag{13}$$

This is the relation between the variance $(\Delta N_c')^2$ of photoelectrons and the variance $(\Delta N_c)^2$ of photons in the fourth-order interference experiment. The square of the phase shift uncertainty is finally given by

$$\begin{aligned}(\Delta\theta)^2 = {}& \frac{(\Delta N_c)^2}{\left[\dfrac{\partial\langle \hat{N}_c\rangle}{\partial\theta}\right]^2} + \frac{(1-\mu_2)}{\mu_2}\frac{\langle \hat{n}_5^2\hat{n}_6\rangle}{\left[\dfrac{\partial\langle \hat{N}_c\rangle}{\partial\theta}\right]^2} + \frac{1-\mu_1}{\mu_1}\frac{\langle \hat{n}_5\hat{n}_6^2\rangle}{\left[\dfrac{\partial\langle \hat{N}_c\rangle}{\partial\theta}\right]^2} \\ &+ \frac{(1-\mu_1)(1-\mu_2)}{\mu_1\mu_2}\frac{\langle \hat{n}_5\hat{n}_6\rangle}{\left[\dfrac{\partial\langle \hat{N}_c\rangle}{\partial\theta}\right]^2},\end{aligned} \tag{14}$$

where $\langle \hat{N}_c\rangle$ and $(\Delta N_c)^2$ means $\langle \hat{n}_5\hat{n}_6\rangle$ and $\langle \hat{N}_c^2\rangle - \langle \hat{N}_c\rangle^2$, respectively. The first term in this expression is an inherent term, and the three remaining terms show the relations with the quantum efficiencies, on the other hand.

In order to gain some insight, let us take some special cases. For a single Fock state light input ($n_1=n\gg 1, n_2=0$), the square of the phase shift uncertainty becomes

$$\begin{aligned}(\Delta\theta)^2 \approx {}& \frac{1}{n} + \frac{1}{2n^2}\tan^2\theta + \frac{1}{2n}\left(\frac{1-\mu_1}{\mu_1} + \frac{1-\mu_2}{\mu_2}\right. \\ &\left. + \frac{2}{n}\frac{(1-\mu_1)(1-\mu_2)}{\mu_1\mu_2}\right)\frac{1}{\cos^2\theta} + \frac{1}{2n}\left(\frac{1-\mu_2}{\mu_2}\right. \\ &\left. - \frac{1-\mu_1}{\mu_1}\right)\frac{1}{\cos\theta}.\end{aligned} \tag{15}$$

We can easily see that we cannot get the quantum limit in the phase sensitivity in this case, even with the ideal detectors of $\mu_1=\mu_2=1$. If two Fock state lights with large numbers of photons ($n_1\approx n_2=n\gg 1$) are fed into both input ports of the interferometer, Eq. (14) can be approximated by

$$\begin{aligned}(\Delta\theta)^2 \approx {}& \frac{1}{2n^2} + \frac{1}{8}\tan^2\theta + \frac{1}{2n}\left(\frac{1-\mu_1}{\mu_1} + \frac{1-\mu_2}{\mu_2}\right. \\ &\left. + \frac{1}{n}\frac{(1-\mu_1)(1-\mu_2)}{\mu_1\mu_2}\right)\frac{1+\cos^2\theta}{\sin^2\theta\cos^2\theta}.\end{aligned} \tag{16}$$

The first two terms are for the ideal cases, and the last term is dependent on the quantum efficiencies of the detectors. Figure 3 shows the dependence of the phase uncertainty on the phase shift for the different values of μ (μ_1 and μ_2 are assumed to be equal to μ) in the input of $n_1=n_2=10\,000$. We can see that if we have the ideal detectors of $\mu_1=\mu_2=1$, we can have the Heisenberg limit in the phase sensitivity at $\theta=0$. However, unless the quantum efficiencies of the detectors are unity or very close to 1, it is difficult to beat the classical limit. It is interesting that as the quantum efficiency decreases, the uncertainty increases rapidly at a small angle. There exists an optimum angle θ for the values of μ.

We conclude that we can reach the Heisenberg limit in the fourth-order interference measurement with two Fock state light inputs of similar photon numbers only when it is performed with almost ideal detectors with quantum efficiencies of $\mu_1 \approx \mu_2 \approx 1$. Measurements should be performed at specific phase shifts according to the quantum efficiencies of the detectors to have the best results.

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