PHYSICAL REVIEW A VOLUME 60, NUMBER 1 JULY 1999

Quantum logic gate operation between different ions in a trap

Li-Xiang Li* and Guang-Can Guo†
Department of Physics, University of Science and Technology of China, HeFei 230026, People's Republic of China
(Received 28 December 1998)

We suggest a scheme for realizing a universal two-quantum-bit (qubit) operation. The scheme uses two laser beams with different intensities to illuminate two different ions in a harmonic trap. Both beams are tuned to the red motional sideband. We find that, under certain conditions, the interaction will realize a universal two-qubit operation, the controlled-rotation operation, which is to rotate the phase of the state by π only when the two states are both in the $|1\rangle$ state. The scheme requires careful control of laser intensity and Lamb-Dicke parameters. But after we reach the required parameters, the logic operation is very simple. [S1050-2947(99)01807-7]

PACS number(s): 32.80.Pj, 03.65.Bz

I. INTRODUCTION

Since a quantum computer can provide more powerful computational ability than a classical one [1], much of people's attention has been attracted to this field. To build a real quantum computer, we need to find quantum systems, which have little decoherence and easy ways for logic operations. Although many systems have been proposed, few of them have succeeded, including the cavity QED model [2], the trapped ion system [3], and the NMR system [4]. A promising candidate for realizing a small scale of quantum computation is a trapped ion system, which is first proposed by Cirac and Zoller [5]. Recently, King *et al.* has reported that collective motion of two $^9\mathrm{Be}^+$ ions has been cooled to ground state [6]. This is a further step towards realizing the Cirac-Zoller quantum computer.

The original idea of Cirac and Zoller involves the center-of-mass (c.m.) mode of vibration as a "bus quantum bit (qubit)." Each two-level ion is treated as an individual qubit. Any two-qubit operation is completed by interacting the "bus qubit" with two ions one by one. Also, a third auxiliary level is needed during the interaction. Monroe *et al.* has demonstrated this scheme in their experiment [3]. Also, a simplified scheme without the auxiliary level for realizing quantum logic operation between the "bus qubit" and the internal level of ion has been suggested [7]. Recently, quantum computation with hot vibrational state is under consideration [8]. Although these efforts have been made, we still need to investigate a much easier scheme in order to realize logic operations between different ions in practice.

In this paper, we consider the situation of two beams of laser, both tuned to red vibrational sideband, interacting with two ions simultaneously. We find under certain conditions, it can realize the controlled rotation (CROT) operation:

$$\hat{R}_{12}=|0,0\rangle\langle 0,0|+|0,1\rangle\langle 0,1|+|1,0\rangle\langle 1,0|-|1,1\rangle\langle 1,1|. \quad (1)$$

The widely discussed operation, controlled NOT,

$$\hat{C}_{12}=|0,0\rangle\langle 0,0|+|0,1\rangle\langle 0,1|+|1,0\rangle\langle 1,1|+|1,1\rangle\langle 1,0|, \quad (2)$$

can be realized from this operation easily by the following way,

$$\hat{C}_{12}=\hat{R}_2\left(\frac{\pi}{4},\frac{3\pi}{2}\right)\cdot\hat{R}_{12}\cdot\hat{R}_2\left(\frac{\pi}{4},\frac{\pi}{2}\right), \quad (3)$$

where $\hat{R}_i$ denotes single qubit rotation. In the computational basis $\{|0\rangle, |1\rangle\}$, it is described by the matrix,

$$\hat{R}_i(\theta,\varphi)=\begin{pmatrix}\cos\theta & ie^{-i\varphi}\sin\theta\\ ie^{i\varphi}\sin\theta & \cos\theta\end{pmatrix}. \quad (4)$$

The subscripts in the equations above denote which ion is being addressed. In the following, we show that single interaction can give the operation in Eq. (1).

II. INTERACTION MODEL AND SOLUTION

In an interaction picture, the Hamiltonian describing two running wave lasers interact with two different ions (ion 1 and 2) in a trap has the form

$$H_I=(\Omega_1 e^{i\phi_1}\sigma_1^- e^{i\Delta_1 t}e^{-ik_1x_1}+\mathrm{H.c.})+(\Omega_2 e^{i\phi_2}\sigma_2^- e^{i\Delta_2 t}e^{-ik_2x_2}+\mathrm{H.c.}), \quad (5)$$

where Ω_i is Rabi frequency, and Δ_i denotes detuning of each laser with the ions. We set $\Delta_1=\Delta_2=-\nu$, where ν is the frequency of the external potential. We neglect all the other normal modes in x_i, except the c.m. motion. Using the rotating wave approximation, we obtain

$$H_I=\left(\Omega_1 e^{i\phi_1}\sigma_1^- e^{-\frac{\eta_1^2}{2}}\sum_{m=0}^{\infty}\frac{(i\eta_1)^{2m+1}(a^\dagger)^{m+1}a^m}{(m+1)!m!}+\mathrm{H.c.}\right)+\left(\Omega_2 e^{i\phi_2}\sigma_2^- e^{-\frac{\eta_2^2}{2}}\sum_{m=0}^{\infty}\frac{(i\eta_2)^{2m+1}(a^\dagger)^{m+1}a^m}{(m+1)!m!}+\mathrm{H.c.}\right), \quad (6)$$

where $a^\dagger$ and a are the creation and annihilation operators for the vibrational c.m. mode, respectively, and σ_i^- denotes the lowering operator for ion *i*. η_i is the Lamb-Dicke parameter. The invariant subspaces of this Hamiltonian are $\{|0,g_1,g_2\rangle\}$, $\{|1,g_1,g_2\rangle,|0,g_1,e_2\rangle,|0,e_1,g_2\rangle\}$, and $\{|n,g_1,g_2\rangle,|n-1,g_1,e_2\rangle,|n-1,e_1,g_2\rangle,|n-2,e_1,e_2\rangle\}$, $n\geqslant 2$.

*Electronic address: lxli@mail.ustc.edu.cn
†Electronic address: gcguo@ustc.edu.cn

For state $|0,g_1,g_2\rangle$, the corresponding eigenvalue is zero, and it remains unchanged under the Hamiltonian (6). The remaining problem is to solve the eigenvalue and eigenvector in other invariant subspaces.

First, we define a notation Ω_{in} by the equation

$$\Omega_{in}=\Omega_i e^{-\eta_i^2/2}\left|\left\langle n\left|\sum_{m=0}^{\infty}\frac{(i\eta_i)^{2m+1}(a^\dagger)^{m+1}a^m}{(m+1)!m!}\right|n-1\right\rangle\right|$$

$$=\frac{\Omega_i\eta_i e^{-\eta_i^2/2}L_{n-1}^1(\eta_i^2)}{\sqrt{n}} \tag{7}$$

for $n\geqslant 2$. L_{n-1}^1 is Laguerre polynomial. For $n=1$, if we let $L_0^1(x)=1$, the above notation can also be used. The Hamiltonian matrix in the invariant space $\{|1,g_1,g_2\rangle,|0,g_1,e_2\rangle,|0,e_1,g_2\rangle\}$ is given by

$$\begin{pmatrix} 0 & i\Omega_{21}e^{i\phi_2} & i\Omega_{11}e^{i\phi_1} \\ -i\Omega_{21}e^{-i\phi_2} & 0 & 0 \\ -i\Omega_{11}e^{-i\phi_1} & 0 & 0 \end{pmatrix}. \tag{8}$$

The eigenvalues of this matrix are $0,\pm\sqrt{\Omega_{11}^2+\Omega_{21}^2}$. We write out the eigenvalues and corresponding eigenvectors. They are, respectively,

$$E_{11}=-\sqrt{\Omega_{11}^2+\Omega_{21}^2},$$

$$|1,1\rangle=\frac{1}{\sqrt{2}}\left(|1,g_1,g_2\rangle+\frac{i\Omega_{21}e^{-i\phi_2}}{E_{13}}|0,g_1,e_2\rangle+\frac{i\Omega_{11}e^{-i\phi_1}}{E_{13}}\left|0,e_1,g_2\right\rangle\right),$$

$$E_{12}=0,$$

$$|1,2\rangle=\frac{1}{E_{13}}(-i\Omega_{11}e^{-i\phi_2}|0,g_1,e_2\rangle+i\Omega_{21}e^{-i\phi_1}|0,e_1,g_2\rangle),$$

$$E_{13}=\sqrt{\Omega_{11}^2+\Omega_{21}^2},$$

$$|1,3\rangle=\frac{1}{\sqrt{2}}\left(|1,g_1,g_2\rangle-\frac{i\Omega_{21}e^{-i\phi_2}}{E_{13}}|0,g_1,e_2\rangle-\frac{i\Omega_{11}e^{-i\phi_1}}{E_{13}}|0,e_1,g_2\rangle\right). \tag{9}$$

We consider the evolution of the initial state $|0,g_1,e_2\rangle$. At time t, the state evolves into

$$|\psi_1(t)\rangle=e^{-iH_It}|0,g_1,e_2\rangle=\frac{\Omega_{21}e^{i\phi_2}\sin E_{13}t}{E_{13}}|1,g_1,g_2\rangle+\frac{\Omega_{21}^2\cos E_{13}t+\Omega_{11}^2}{E_{13}^2}|0,g_1,e_2\rangle+\frac{\Omega_{21}\Omega_{11}e^{i(\phi_2-\phi_1)}}{E_{13}^2}(\cos E_{13}t-1)|0,e_1,g_2\rangle. \tag{10}$$

For another initial state $|0,e_1,g_2\rangle$, the state after time t becomes

$$|\psi_2(t)\rangle=e^{-iH_It}|0,e_1,g_2\rangle=\frac{\Omega_{11}e^{i\phi_1}\sin E_{13}t}{E_{13}}|1,g_1,g_2\rangle+\frac{\Omega_{21}\Omega_{11}e^{i(\phi_1-\phi_2)}}{E_{13}^2}(\cos E_{13}t-1)|0,g_1,e_2\rangle+\frac{\Omega_{11}^2\cos E_{13}t+\Omega_{21}^2}{E_{13}^2}|0,e_1,g_2\rangle. \tag{11}$$

If the time duration satisfies the condition $E_{13}t=2m\pi$, where m is an integer, we see that both state, $|0,g_1,e_2\rangle$ and $|0,e_1,g_2\rangle$, returns to the original ones, respectively.

Next, we need to solve this problem in the subspace spanned by $\{|n,g_1,g_2\rangle,|n-1,g_1,e_2\rangle,|n-1,e_1,g_2\rangle,|n-2,e_1,e_2\rangle\}$. The Hamiltonian matrix is

$$\begin{pmatrix} 0 & i\Omega_{2n}e^{i\phi_2} & i\Omega_{1n}e^{i\phi_1} & 0 \\ -i\Omega_{2n}e^{-i\phi_2} & 0 & 0 & i\Omega_{1,n-1}e^{i\phi_1} \\ -i\Omega_{1n}e^{-i\phi_1} & 0 & 0 & i\Omega_{2,n-1}e^{i\phi_2} \\ 0 & -i\Omega_{1,n-1}e^{-i\phi_1} & -i\Omega_{2,n-1}e^{-i\phi_2} & 0 \end{pmatrix}. \tag{12}$$

The eigenvalues of this matrix are

$$E_{ni}=\pm\sqrt{\frac{\Omega_A^2\pm\sqrt{(\Omega_A^2)^2-4(\Omega_B^2)^2}}{2}},$$

$$\Omega_A^2=\Omega_{1n}^2+\Omega_{2n}^2+\Omega_{1,n-1}^2+\Omega_{2,n-1}^2,$$

$$\Omega_B^2=\Omega_{1n}\Omega_{1,n-1}-\Omega_{2n}\Omega_{2,n-1}. \tag{13}$$

We use E_{n1}, E_{n2}, E_{n3}, and E_{n4} to denote them from low to high, respectively. The eigenvector corresponding to eigenvalue E_{ni} is given by

TABLE I. Some results for the case of $\eta_1=\eta_2=\eta$.

	$m=1,\ l=0,\ k=2$	$m=2,\ l=0,\ k=2$	$m=2,\ l=1,\ k=2$
η	0.51764	0.93913	0.70711
Ω_2/Ω_1 or Ω_1/Ω_2	2.03951	1.51899	4.02791

$$\begin{aligned}|n,i\rangle = N_{ni}\Bigg\{ & \frac{-e^{-i(\phi_1+\phi_2)}(E_{ni}^2-\Omega_{1,n-1}^2-\Omega_{2,n-1}^2)}{\Omega_{1,n-1}\Omega_{2n}+\Omega_{2,n-1}\Omega_{1n}}|n,g_1,g_2\rangle \\ & +\frac{1}{E_{ni}}\Bigg[i\Omega_{1,n-1}e^{i\phi_1} \\ & + i\Omega_{2n}e^{i\phi_1}\frac{E^2-\Omega_{1,n-1}^2-\Omega_{2,n-1}^2}{\Omega_{1,n-1}\Omega_{2n}+\Omega_{2,n-1}\Omega_{1n}}\Bigg]|n-1,g_1,e_2\rangle \\ & +\frac{1}{E_{ni}}\Bigg[i\Omega_{2,n-1}e^{i\phi_2} \\ & + i\Omega_{1n}e^{i\phi_2}\frac{E^2-\Omega_{1,n-1}^2-\Omega_{2,n-1}^2}{\Omega_{1,n-1}\Omega_{2n}+\Omega_{2,n-1}\Omega_{1n}}\Bigg]|n-1,e_1,g_2\rangle \\ & +|n-2,e_1,e_2\rangle\Bigg\}, \end{aligned} \tag{14}$$

where N_{ni} is a normalization factor. We need to find out the time evolution of state $|n-2,e_1,e_2\rangle$. From Eq.(14), it follows that $|n-2,e_1,e_2\rangle$ can be expressed as

$$|n-2,e_1,e_2\rangle = N_{n1}|n,1\rangle + N_{n2}|n,2\rangle + N_{n3}|n,3\rangle + N_{n4}|n,4\rangle. \tag{15}$$

At time t, the state evolves into

$$\begin{aligned}|\psi(t)\rangle &= e^{-iH_I t}|n-2,e_1,e_2\rangle \\ &= N_{n1}e^{-iE_{n1}t}|n,1\rangle + N_{n2}e^{-iE_{n2}t}|n,2\rangle \\ &\quad + N_{n3}e^{-iE_{n3}t}|n,3\rangle + N_{n4}e^{-iE_{n4}t}|n,4\rangle. \end{aligned}\tag{16}$$

We can easily get the following result:

$$\begin{aligned}\langle n-2,e_1,e_2|\psi(t)\rangle &= N_{n1}^2e^{-iE_{n1}t} + N_{n2}^2e^{-iE_{n2}t} + N_{n3}^2e^{-iE_{n3}t} \\ &\quad + N_{n4}^2e^{-iE_{n4}t}. \end{aligned}\tag{17}$$

Because $E_{n1}=-E_{n4}$, and $E_{n2}=-E_{n3}$, if we let $E_{n4}t=(2k+1)\pi$ and $E_{n3}t=(2l+1)\pi$, the state $|n-2,e_1,e_2\rangle$ will be changed to $-|n-2,e_1,e_2\rangle$. This result in the special case of $n=2$, combined with the time evolution of $|0,g_1,e_2\rangle$ and $|0,e_1,g_2\rangle$ discussed above, yields a CROT quantum operation between two different ions, as long as the initial state of the c.m. mode is in a vacuum state. The c.m. mode of ions remains in the vacuum state after the operation.

Now we consider the details for the operation to be valid. It requires

$$\begin{aligned} E_{23}t &= (2l+1)\pi, \\ E_{24}t &= (2k+1)\pi, \\ E_{13}t &= 2m\pi. \end{aligned}\tag{18}$$

Using the eigenvalues in Eq. (9) and Eq. (13), finally we get

$$\begin{aligned} \Omega_{21}^2+\Omega_{22}^2 &= (2l+1)^2+(2k+1)^2-(2m)^2, \\ |\Omega_{21}\Omega_{11}-\Omega_{22}\Omega_{12}| &= (2k+1)(2l+1), \\ \Omega_{11}^2+\Omega_{12}^2 &= (2m)^2, \end{aligned}\tag{19}$$

where the unit of Ω is set to be π/t. Use the definition of Ω in Eq. (7), and solve the equations for selected $\{k,l,m\}$; we can choose a particular set of $\{\Omega_1,\Omega_2,\eta_1,\eta_2\}$ to be used for real experiments. We have presented some solutions of Eq. (19) in Table I.

III. CONCLUSION

The easy way for realizing quantum computation in small scale is of particular interest for experimentalists, and it is very important for developing a real quantum computer. In our scheme of quantum computation with trapped ions, the requirement of an auxiliary level is eliminated. As pointed out by Monroe *et al.* [7], this simplification may be useful in experiments. Recently, some schemes for coherent operation with hot vibrational states have been proposed [8]. But our scheme is still worth some consideration because of its simplicity. Also, if we use the c.m. motion as a ‘‘bus qubit,’’ perhaps we will need both single bit rotation for the c.m. motion and internal state, while in this scheme, single bit rotations for the c.m. motion are not needed.

ACKNOWLEDGMENTS

This project was supported by the National Natural Science Foundation of China and the Doctoral Education Fund of the State Education Commission of China.

[1] P. Shor, in *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, edited by Shafi Goldwasser (IEEE Computer Society Press, New York, 1994), p. 124; L. K. Grover, Phys. Rev. Lett **79**, 325 (1997); for a review, see D. Aharonov, e-print quant-ph/9812037.

[2] Q. A. Truchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995).

[3] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).

[4] I. Chuang, N. Gershenfeld, and M. Kubinec, Phys. Rev. Lett. **80**, 3408 (1998).
[5] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[6] B. E. King *et al.*, Phys. Rev. Lett. **81**, 1525 (1998).
[7] C. Monroe *et al.*, Phys. Rev. A **55**, R2489 (1997).
[8] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 1322 (1998); A. Sørensen and K. Mølmer, e-print quant-ph/9810039.