Sampling canonical phase distribution

Matteo G. A. Paris*
Theoretical Quantum Optics Group, INFM and Dipartimento di Fisica ''Alessandro Volta,'' Università di Pavia, via Bassi 6, I-27100 Pavia, Italy
and Optics Section, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 19 May 1999)

We suggest a method to measure the canonical (London) phase distribution of a single-mode radiation field by a heterodyne or a multiport homodyne detector. The technique is based on sampling the overlap between the signal mode and a phase coherent state. [S1050-2947(99)06212-5]

PACS number(s): 42.50.Dv, 03.65.Bz

The quantum description of the optical phase has been debated for a long time [1]. However, there is a general agreement that the probability density

$$P(\phi)=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}\varrho_{nm}\exp\{i(n-m)\phi\} \tag{1}$$

represents the canonical phase distribution for the single-mode radiation field described by the density operator $\hat{\varrho}$ [2]. The phase probability distribution of Eq. (1) was earlier introduced by London [3], and it also represents the limiting distribution of the truncation approach of Pegg and Barnett [4]. In addition, it has been independently derived by Helstrom [5] and Holevo [6] in the more general framework of quantum estimation theory. More recently, it has also been shown how to properly derive probability (1) starting only from the correspondence principle and the Born statistical rule [7]. Finally, we mention that the probability (1) represents the overlap between $\hat{\varrho}$, the state under examination, and the so-called Susskind-Glogower phase states [8]

$$|e^{i\phi}\rangle=\sum_{n=0}^{\infty}\exp\{in\phi\}|n\rangle. \tag{2}$$

This last characterization is at the basis of the measurement scheme that we are going to present in the following.

As a matter of fact, only a few schemes have been suggested for the reconstruction of the distribution (1), through homodyning the input signal with an unconventional reference state [9] or by tomographic reconstruction using approximate kernels [10,11]. On the other hand, a number of experiments have been performed [12], which lead to phase distributions different from the canonical one and strongly dependent on the adopted measurement scheme. This has provoked the diffuse conviction that the quantum optical phase can be defined only in connection with its measurement scheme (operational approach), whereas canonical distribution (1) plays no relevant role, since it does not correspond to an observable quantity. The first statement is certainly true, and it represents a general feature of the quantum mechanical description of measurements [13]. On the other hand, we will show how the canonical phase distribution may be directly sampled by means of realistic measurement schemes.

Our scheme is based on sampling the overlap $\langle\lambda|\hat{\varrho}|\lambda\rangle$ between the input signal $\hat{\varrho}$ and an excited phase coherent state (PCS) [16]

$$|\lambda\rangle=\sqrt{1-|\lambda|^2}\sum_{n=0}^{\infty}\lambda^n|n\rangle, \tag{3}$$

which, in turn, approaches a Susskind-Glogower phase state in the limit $|\lambda|\to 1$. Phase coherent states are defined as eigenstates of the lowering operator

$$\hat{E}_-=\sum_{n=0}^{\infty}|n\rangle\langle n+1|,\quad \hat{E}_-|\lambda\rangle=\lambda|\lambda\rangle,$$

with eigenvalues within the unit circle $|\lambda|<1$, and average number of photons given by

$$N_\lambda=\langle\lambda|a^\dagger a|\lambda\rangle=\frac{|\lambda|^2}{1-|\lambda|^2}. \tag{4}$$

In the following, we first describe how to sample the overlap between two quantum states by means of a two-photocurrent device such a heterodyne or a multiport homodyne detector. Then, we briefly resume how phase coherent states can be effectively synthesized, and finally, we describe the procedure for sampling the canonical phase distribution of Eq. (1).

A two-photocurrent device is a detector providing the *joint* measurements of two noncommuting field quadratures. Three examples of such kind of detector are available in quantum optics: heterodyne [17], eight-port homodyne [18,19], and six-port homodyne detectors [20]. The above quoted detectors are fully equivalent in probing the field, since they all measure the real and imaginary parts of the complex photocurrent [21]

$$\hat{Z}=a+b^\dagger, \tag{5}$$

$[a,a^\dagger]=1$ being the signal mode and $[b,b^\dagger]=1$ an idler mode of the detector. The complex random variable described by the photocurrent $\hat{Z}$ is the sum of two complex random variables pertaining to the two modes, respectively.

*Electronic address: matteo.paris@pv.infn.it

The random outcomes $z \in \mathbb{C}$ for $\hat{Z}$ are thus distributed in the phase space according to the convolution

$$K(z,\bar{z}) = \int_{\mathbb{C}} \frac{d^2\beta}{\pi^2} W_a(\beta,\bar{\beta}) W_b(z+\beta,\bar{z}+\bar{\beta}), \tag{6}$$

where $W_a(z,\bar{z})$ and $W_b(z,\bar{z})$ are the Wigner function of the signal and the idler, respectively. The convolution accounts for the action of the measuring apparatus, which plays the role of a quantum filter [14]. Different choices for the state of the idler mode lead to different phase space distributions, which provide a specific type of information concerning the signal under examination [15].

Wigner function is defined as

$$W(z,\bar{z}) = \int \frac{d^2\gamma}{\pi} \mathrm{Tr}\{\hat{\varrho}\hat{D}(\gamma)\} e^{\gamma\bar{z}-\bar{\gamma}z}, \tag{7}$$

$\hat{D}(\alpha) = \exp\{\alpha a^\dagger - \bar{\alpha} a\}$ being the displacement operator. By means of the operatorial identity [22]

$$\hat{\varrho} = \int_{\mathbb{C}} \frac{d^2\alpha}{\pi} \mathrm{Tr}\{\hat{\varrho}\hat{D}(\alpha)\}\hat{D}^\dagger(\alpha), \tag{8}$$

it is straightforward to express the phase-space distribution $K(z,\bar{z})$ of Eq. (6) in the following trace form:

$$K(z,\bar{z}) = \mathrm{Tr}_a\{\hat{\varrho}_a \hat{D}(z)\hat{\varrho}_b \hat{D}^\dagger(z)\} = \mathrm{Tr}_a\{\hat{\varrho}_a \hat{\Pi}(z,\bar{z})\}. \tag{9}$$

Equation (9) contains two crucial pieces of information.

(1) The joint measurement of $\mathrm{Re}(\hat{Z})$ and $\mathrm{Im}(\hat{Z})$ corresponds to the measurement of a generalized observable on the signal mode a. This generalized observable is described by the probability operator measure

$$\hat{\Pi}(z,\bar{z}) = \hat{D}(z)\hat{\varrho}_b \hat{D}^\dagger(z), \tag{10}$$

$\hat{\varrho}_b$ being the preparation of the idler mode.

(2) The probability of zero counts represents a direct sampling of the overlap between the signal and the idler state:

$$K(0,0) = \mathrm{Tr}_a\{\hat{\varrho}_a \hat{\varrho}_b\}. \tag{11}$$

In practice, each experimental event in a heterodyne or in a multiport homodyne detector consists of the simultaneous detection of two photocurrents which, in turn, trace a pair of conjugated field quadratures. Each event thus corresponds to a point in the complex plane representing the field amplitude. The experimental sample of the phase-space distribution $K(z,\bar{z})$ is obtained upon dividing the plane into small bins of equal area $\Delta Q = \Delta z \Delta\bar{z} = \Delta x \Delta y$ (with $z = x + iy$), and then building a histogram H_{ij} by counting the number of points which fall into each bin. For large enough samples of data one may use small bins, such that the probability of zero counts can be reliably estimated from the number of counts falling in the central bin, $H_{00} \simeq K(0,0)\Delta z \Delta\bar{z}$. Of course, this procedure, as any finite sampling of a quantum mechanical distribution, introduces a kind of coarse graining in the phase space. However, no systematic errors affect the estimation procedure itself. In fact, up to second order in $z,\bar{z}$, we may express the expected number of counts in the central bin by the linear relation [23]

$$\frac{1}{\Delta Q} H_{00} = K(0,0) + \Delta Q \times \mathrm{const}, \tag{12}$$

from which $K(0,0)$ can be easily extrapolated using bins of different areas. Therefore, sampling the overlap is statistically robust.

Since we have a method for sampling the overlap, the use of a phase coherent state as the idler beam provides a method to measure the canonical phase distribution. Actually, an interaction scheme involving nonlinear $\chi^{(2)}$ media has been suggested for the generation of a PCS [24]. The setup is based on parametric amplification of vacuum [25] followed by up-conversion of the resulting twin beam [26]. Remarkably, an experimentally achievable working regime to approximate the PCS with a high conversion rate has been individuated. In particular, the state with fidelity $F > 90\%$ to the nearest PCS can be obtained up to $N_\lambda = 15$. As we will see, this is enough for the present purposes.

By using a phase coherent state as the idler of a two-photocurrent detector we have that the probability of zero counts approaches the overlap $K_\lambda(0,0) = \langle\lambda|\hat{\varrho}|\lambda\rangle$ between the PCS and the signal $\hat{\varrho}$ under examination:

$$K_\lambda(0,0) = (1-|\lambda|^2) \sum_{nk=0}^{\infty} \varrho_{nk} |\lambda|^{n+k} e^{i(n-k)\phi}, \tag{13}$$

where $\varrho_{nk} = \langle n|\hat{\varrho}|k\rangle$ are the matrix elements of the signal in the Fock basis, and $\phi = \arg\lambda$ is the phase of the considered PCS. In the limit of an excited PCS, i.e., for $|\lambda| \to 1$, we have

$$K_\lambda(0,0) \overset{|\lambda|\to 1}{=} \langle e^{i\phi}|\hat{\varrho}|e^{i\phi}\rangle, \tag{14}$$

and therefore $K_\lambda(0,0)$ approaches the canonical phase probability at the phase value ϕ. By varying this phase we can explore the whole 2π window and, thus, sampling the canonical phase distribution. For realistic values of $|\lambda|$, lower than unity, the normalized distribution

$$P_{|\lambda|}(\phi) = \frac{1}{\mathcal{N}} K_\lambda(0,0), \quad \phi = \arg\lambda, \tag{15}$$

$$\mathcal{N} = (1-|\lambda|^2) \sum_{n=0}^{\infty} \varrho_{nn} |\lambda|^{2n}, \tag{16}$$

represents an approximation to the canonical phase distribution. Remarkably, for low excited states, i.e., in the relevant quantum regime, $P_{|\lambda|}(\phi)$ is a good approximation to $P(\phi)$ already for a phase coherent amplitude of about $|\lambda| \simeq 0.95$, corresponding to a relatively small number of photons $N_\lambda \simeq 10$.

In Fig. 1 we report some examples of phase distributions for nonclassical states, which can be obtained by sampling the overlap. As is apparent from the plots, also using a PCS with relatively low number of photons and a limited number

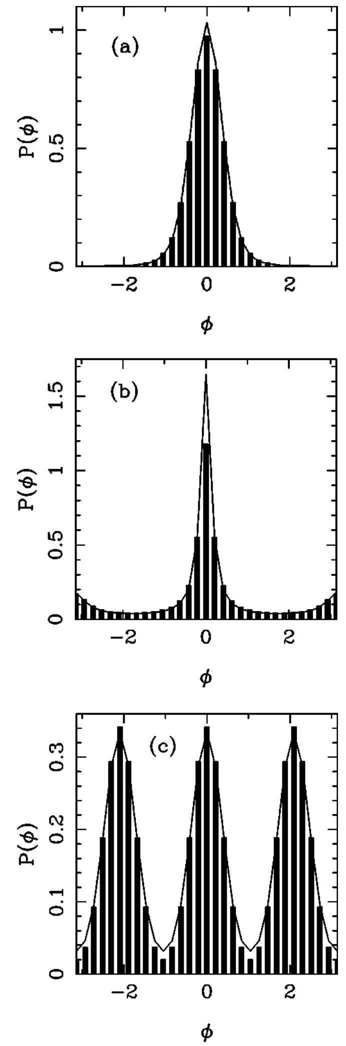

FIG. 1. Phase distribution (histogram) for different quantum states of radiation, as obtained by sampling the overlap with a set of phase coherent states. The solid line denotes the canonical distribution. In (a) the distribution for a coherent state $|\alpha\rangle$ with $|\alpha|^2=2$ average photons. In (b) for a squeezed state $|\alpha,r\rangle$ with $|\alpha| = \sinh^2 r=1$, and thus a total number of photons equal to $\langle a^\dagger a\rangle = 2$, and in (c) for a superposition of three coherent states $|\psi\rangle \propto |\alpha\rangle + |\alpha e^{i\pi/3}\rangle + |\alpha e^{i2\pi/3}\rangle$ with $\alpha=1.51$ (total average photons $\langle a^\dagger a\rangle = 1$). The distributions have been obtained by sampling the overlap with phase coherent states with an average number of photons equal to $\langle\lambda|a^\dagger a|\lambda\rangle = 10$ (i.e., $|\lambda|\simeq 0.95$); 30 PCS's with equally spaced phases in $[-\pi,\pi)$ have been used.

of scanning phases, we have a very good reconstruction of the relevant features of the distribution.

In order to measure how close the measured distribution $P_{|\lambda|}(\phi)$ is to the ideal one $P(\phi)$, we consider the Hilbert one-distance

$$D(|\lambda|) = \int_{-\pi}^{\pi} d\phi \, |P(\phi) - P_{|\lambda|}(\phi)|. \tag{17}$$

For values of $|\lambda|$ close to unity we may expand $P_{|\lambda|}(\phi)$ in terms of $(1-|\lambda|^2)$. Up to first order we have

$$P_{|\lambda|}(\phi) \simeq P(\phi) + \overline{n}(1-|\lambda|^2)P(\phi) - \frac{(1-|\lambda|^2)}{2\pi} \sum_{n,m=0}^{\infty} \varrho_{nm} \frac{n+m}{2} e^{i(n-m)\phi}, \tag{18}$$

where $\overline{n} = \mathrm{Tr}\{\hat{\varrho} a^\dagger a\}$ is the average number of photons in the signal under examination. Substituting Eq. (18) into Eq. (17) and using Eq. (4), we obtain

$$D(|\lambda|) \simeq \frac{\overline{n}}{N_\lambda} \int_{-\pi}^{\pi} \frac{d\phi}{4\pi^2} \left| \sum_{n,m=0}^{\infty} \varrho_{nm} e^{i(n-m)\phi} \left(1 - \frac{n+m}{2\overline{n}}\right) \right|,$$

which shows that the distance between the two distributions scales as the ratio between the intensity of the signal and that one of the ''probing'' PCS.

Let us now conclude the paper, by summarizing the procedure to sample the canonical phase distribution.

(1) First, one has to consider the states coming from the upconversion of twin-beam states, which in turn comes from parametric amplification of the vacuum. These are states that reliably approach the PCS in an experimentally achievable working regime and for a wide range of output intensities.

(2) By means of a two-photocurrent detector, the overlap between a phase coherent state and the input signal is measured. This represents a sample of the canonical phase probability at the value ϕ, where ϕ is the phase of the considered PCS, namely, the classical phase of the pump of the amplifier.

(3) Then, by varying the phase of the pump, the overlap with different phase coherent states can be measured, thus covering the whole 2π window.

The precision with which the phase distribution is measured mostly depends on two parameters. These are the number of measured data for each ''sampling the overlap'' experiment and the number of phases used in scanning the 2π phase window. In fact, the larger the sample of data is, the smaller can be the bins used to estimate $K(0,0)$. On the other hand, a large number of scanning values for the phase distribution offers the possibility of a detailed characterization of the phase properties of the signal under examination.

The present measurement scheme is feasible with currently avalailable quantum optical technology.

The author wishes to thank Mauro D'Ariano and Massimiliano Sacchi for many interesting discussions about phase measurements and phase-enhanced states, and Bernhard Böhmer, Ole Steuernagel, Alexei Chizhov, and Valentina De Renzi for interesting discussions about phase-space distributions. This work was supported by CNR and NATO, through Grant No. 215.31 of the Advanced Fellowship Program 1998.

[1] R. Lynch, Phys. Rep. **256**, 367 (1995).

[2] G. M. D'Ariano, and M. G. A. Paris, Phys. Rev. A **49**, 3022 (1994); U. Leonhardt, J. A. Vaccaro, B. Böhmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).

[3] F. London, Z. Phys. **40**, 193 (1927).

[4] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989); Europhys. Lett. **6**, 483 (1988).

[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

[6] A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982).

[7] M. G. A. Paris, Nuovo Cimento B **111**, 1151 (1996).

[8] L. Susskind and J. Glogower, Physics (Long Island City, N.Y.) **1**, 49 (1964).

[9] S. M. Barnett and D. T. Pegg, Phys. Rev. Lett. **76**, 4148 (1996).

[10] M. Dakna, L. Knöll, and D.-G. Welsch, Quantum Opt. **9**, 331 (1997); M. Dakna, G. Breitenbach, J. Mlynek, T. Opatrny, S. Schiller, and D.-G. Welsch, Opt. Commun. **152**, 289 (1998).

[11] M. Beck, D. T. Smithey, and M. G. Raymer, Phys. Rev. A **48**, R890 (1993).

[12] J. Noh, A. Fougeres, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); J .R. Torgeson and L. Mandel, Phys. Rev. Lett. **76**, 3939 (1996).

[13] P. Busch and P. J. Lahti, Riv. Nuovo Cimento **18**, 1 (1995).

[14] K. Wodkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); Phys. Lett. A **115**, 304 (1986).

[15] V. Buzek, C. H. Keitel, and P. L. Knight, Phys. Rev. A **51**, 2575 (1995); **51**, 2594 (1995).

[16] J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).

[17] J. H. Shapiro and S. S. Wagner, IEEE J. Quantum Electron. **QE-20**, 803 (1984); H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **IT-26**, 78 (1980).

[18] N. G. Walker and J. E. Carrol, Opt. Quantum Electron. **18**, 355 (1986); N. G. Walker, J. Mod. Opt. **34**, 15 (1987); Y. Lay and H. A. Haus, Quantum Opt. **1**, 99 (1989).

[19] M. Freyberger and W. Schleich, Phys. Rev. A **47**, 30 (1993); U. Leonhardt and H. Paul, Phys. Rev. A **47**, 2460 (1993).

[20] A. Zucchetti, W. Vogel, and D.-G. Welsch, Phys. Rev. A **54**, 856 (1996); M. G. A. Paris, A. Chizhov, and O. Steuernagel, Opt. Commun. **134**, 117 (1997).

[21] G. M. D'Ariano and M. F. Sacchi, Phys. Rev. A **52**, R4309 (1995).

[22] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).

[23] In Eq. (12) the proportionality constant is given by $\int d^2\beta[\partial_\beta W_a(\beta,\bar{\beta})\partial_\beta W_b(\beta,\bar{\beta})+\partial_{\bar{\beta}} W_a(\beta,\bar{\beta})\partial_\beta W_b(\beta,\bar{\beta}) + \partial_\beta W_a(\beta,\bar{\beta})\partial_{\bar{\beta}} W_b(\beta,\bar{\beta})+\partial_{\bar{\beta}} W_a(\beta,\bar{\beta})\partial_{\bar{\beta}} W_b(\beta,\bar{\beta})]$.

[24] G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, Phys. Rev. A **57**, 4894 (1998).

[25] M. Vasylìev, S-K. Choi, P. Kumar, and G. M. D'Ariano, Opt. Lett. **23**, 1393 (1998).

[26] F. De Martini, Phys. Lett. A **250**, 15 (1998).